

## UNIFORM DISTRIBUTION MODULO ONE ON SUBSEQUENCES

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ABSTRACT. Let  $\mathcal{P}$  be a set of primes with a divergent series of reciprocals and let  $\mathcal{K} = \mathcal{K}(\mathcal{P})$  denote the set of squarefree integers greater than one that are divisible only by primes in  $\mathcal{P}$ . G. Myerson and A. D. Pollington proved that  $(u_n)_{n \geq 1} \subset [0, 1)$  is uniformly distributed (mod 1) whenever the subsequence  $(u_{kn})_{n \geq 1}$  is uniformly distributed (mod 1) for every  $k$  in  $\mathcal{K}$ . We show that in fact  $(u_n)_{n \geq 1}$  is uniformly distributed (mod 1) whenever the subsequence  $(u_{pn})_{n \geq 1}$  is uniformly distributed (mod 1) for every  $p \in \mathcal{P}$ .

### 1. INTRODUCTION

In [4], G. Myerson and A. D. Pollington proved several intriguing results about uniform distribution modulo one, including the following theorem.

**Theorem A** (Myerson-Pollington). *Let  $\mathcal{P}$  be a set of primes such that  $\sum\{1/p : p \in \mathcal{P}\}$  diverges. Let  $\mathcal{K} = \mathcal{K}(\mathcal{P})$  denote the set of squarefree integers greater than one divisible only by primes in  $\mathcal{P}$ . If  $(u_{kn})_{n \geq 1}$  is uniformly distributed (mod 1) for every  $k$  in  $\mathcal{K}$ , then  $(u_n)_{n \geq 1}$  is uniformly distributed (mod 1).*

The set  $\mathcal{K}$  in Theorem A functions as a kind of “test set,” used to probe the distribution of the sequence  $(u_n)$ . The following definition formalizes this notion.

**Definition.** A set  $S$  of integers greater than 1 is called a *u.d. test set* if the sequence  $(u_n)_{n \geq 1} \subset [0, 1)$  is uniformly distributed whenever the subsequence  $(u_{sn})_{n \geq 1}$  is uniformly distributed for all  $s \in S$ .

Note that if  $S$  is a set of positive integers greater than 1 and  $S$  contains a u.d. test set, then *a fortiori*  $S$  is a u.d. test set.

It is natural to look for other u.d. test sets besides the sets  $\mathcal{K}(\mathcal{P})$  and their supersets. Our main result, Theorem 1 below, shows that the set of primes  $\mathcal{P}$  in Theorem A is itself a u.d. test set. Note that Theorem A is a consequence of Theorem 1, because in the former result  $\mathcal{P} \subset \mathcal{K}$ .

**Theorem 1.** *Let  $\mathcal{P}$  be a set of primes. Then  $\mathcal{P}$  is a u.d. test set if and only if  $\sum\{1/p : p \in \mathcal{P}\}$  diverges.*

Myerson and Pollington proved Theorem A by using the Weyl criterion for uniform distribution and properties of the Möbius  $\mu$ -function. The path that we take

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to Theorem 1 is quite different. We introduce a second type of integer set, called a *density test set*, whose definition makes no reference to uniform distribution. In Theorem 2, stated below, the property of being a u.d. test set is shown to be equivalent to that of being a density test set. After proving Theorem 2, we provide a sufficient condition for being a density test set. Finally, we prove that a set of primes is a density test set if and only if the series of reciprocals of its elements diverges.

We require some notation to state the definition of a density test set.

*Notation.* Let  $A \subseteq \mathbb{N}$ . We let  $\mathbf{d}A = \lim_{N \rightarrow \infty} \frac{1}{N} \#\{n \leq N : n \in A\}$  denote the natural density of  $A$ , provided the limit exists. For each  $s \in \mathbb{N}$ , we define  $A_s = \{n \in \mathbb{N} : sn \in A\}$ . One may think of  $\mathbf{d}A_s$  (if it exists) as being the density of the set  $\{n \in A : s|n\}$  relative to the set  $\{s, 2s, 3s, \dots\}$ .

**Definition.** Let  $S$  be a set of integers greater than 1. We say that  $S$  is a *density test set* if whenever a set  $A$  of positive integers has the property that there exists  $\alpha \in [0, 1]$  such that  $\mathbf{d}A_s = \alpha$  for each  $s \in S$ , then  $\mathbf{d}A = \alpha$ .

**Theorem 2.**  $S$  is a u.d. test set if and only if  $S$  is a density test set.

## 2. PROOF OF THEOREM 2

We need a well-known result from the theory of uniform distribution. A proof is given in [3, Theorem I.4.1].

**Lemma 3.** Let  $(a_n)_{n \geq 1}$  be a given sequence of distinct integers. Then the sequence  $(a_n x)_{n \geq 1}$  is uniformly distributed (mod 1) for almost all real numbers  $x$ .

*Proof of Theorem 2.* Assume that  $S$  is a density test set. Let  $(u_n)_{n \geq 1} \subset [0, 1)$ , and suppose that  $(u_{sn})_{n \geq 1}$  is uniformly distributed for each  $s \in S$ . Let  $x \in (0, 1]$  be fixed, and let  $A = A(x) = \{n : 0 \leq u_n < x\}$ . For each  $s \in S$  we have

$$\frac{1}{N} \#\{n \leq N : n \in A_s\} = \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} \rightarrow x \quad \text{as } N \rightarrow \infty,$$

that is,  $\mathbf{d}A_s = x$ . Since  $S$  is a density test set, we may conclude that  $\mathbf{d}A = x$ . Then

$$\frac{1}{N} \#\{n \leq N : 0 \leq u_n < x\} = \frac{1}{N} \#\{n \leq N : n \in A\} \rightarrow x \quad \text{as } N \rightarrow \infty.$$

It follows that  $(u_n)$  is uniformly distributed. Thus,  $S$  is a u.d. test set.

Next, we assume that  $S$  is a u.d. test set. Let  $A$  be a set of positive integers, and suppose that there is an  $\alpha \in [0, 1]$  such that  $\mathbf{d}A_s = \alpha$  for each  $s \in S$ . It is convenient to consider the cases  $0 < \alpha < 1$ ,  $\alpha = 1$ , and  $\alpha = 0$  separately.

We assume first that  $0 < \alpha < 1$ . Let  $A'_s = \mathbb{N} \setminus A_s$  for each  $s \in S$ . The sets  $A_s$  and  $A'_s$  have positive density, and hence are infinite. For each  $s \in S$ , let  $\Gamma_s$  denote the set of all real  $x \in [0, 1]$  for which the sequence  $(snx : n \in A_s)$  is uniformly distributed (mod 1), and let  $\Gamma'_s$  denote the set of all real  $x \in [0, 1]$  for which  $(snx : n \in A'_s)$  is uniformly distributed (mod 1). Let  $\Gamma = (\bigcap_{s \in S} \Gamma_s) \cap (\bigcap_{s \in S} \Gamma'_s)$ . By Lemma 3, each of the sets  $[0, 1] \setminus \Gamma_s$  and  $[0, 1] \setminus \Gamma'_s$  has measure 0, whence  $[0, 1] \setminus \Gamma$  has measure 0. In particular,  $\Gamma$  is not empty. Select any  $\gamma \in \Gamma$ . Put  $w_n = \{n\gamma\}$  for  $n \geq 1$ , that is,  $w_n$  is the fractional part of  $n\gamma$ . By construction, the sequences  $(w_{sn} : n \in A_s)$  and  $(w_{sn} : n \in A'_s)$  are uniformly distributed for each  $s \in S$ . We now define a sequence

$(u_n)_{n \geq 1} \subset [0, 1)$  by

$$u_n = \begin{cases} \alpha w_n, & \text{if } n \in A, \\ \alpha + (1 - \alpha)w_n, & \text{otherwise.} \end{cases}$$

Let  $s \in S$ . We claim that  $(u_{sn})_{n \geq 1}$  is uniformly distributed. Take  $0 < x \leq 1$ . If  $x \leq \alpha$ , then

$$\begin{aligned} \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} &= \frac{1}{N} \#\{n \leq N : sn \in A, 0 \leq \alpha w_{sn} < x\} \\ &= \frac{1}{N} \#\{n \leq N : n \in A_s, 0 \leq w_{sn} < x/\alpha\} \\ &\rightarrow \mathbf{d}A_s \cdot \frac{x}{\alpha} = x \quad \text{as } N \rightarrow \infty, \end{aligned}$$

since  $(w_{sn} : n \in A_s)$  is uniformly distributed. On the other hand, if  $x > \alpha$ , then

$$\begin{aligned} \frac{1}{N} \#\{n \leq N : 0 \leq u_{sn} < x\} &= \frac{1}{N} \#\{n \leq N : sn \in A\} \\ &\quad + \frac{1}{N} \#\{n \leq N : sn \notin A, 0 \leq \alpha + (1 - \alpha)w_{sn} < x\} \\ &= \frac{1}{N} \#\{n \leq N : n \in A_s\} \\ &\quad + \frac{1}{N} \#\{n \leq N : n \in A'_s, 0 \leq w_{sn} < (x - \alpha)/(1 - \alpha)\} \\ &\rightarrow \mathbf{d}A_s + \mathbf{d}A'_s \cdot \frac{x - \alpha}{1 - \alpha} = x \quad \text{as } N \rightarrow \infty \end{aligned}$$

because  $(w_{sn} : n \in A'_s)$  is uniformly distributed. Hence,  $(u_{sn})_{n \geq 1}$  is uniformly distributed, as claimed. We deduce that  $(u_n)_{n \geq 1}$  is uniformly distributed, by virtue of  $S$  being a u.d. test set. Noting that  $A = \{n : 0 \leq u_n < \alpha\}$ , we see that  $\mathbf{d}A$  exists and equals  $\alpha$ .

Suppose now that  $\alpha = 1$ . Define the sets  $\Gamma_s$  for  $s \in S$  as above, and let  $\Gamma = \bigcap_{s \in S} \Gamma_s$ . Select  $\gamma \in \Gamma$ , and let  $w_n = \{n\gamma\}$  for  $n \geq 1$ . We define a sequence  $(u_n)_{n \geq 1} \subset [0, 1)$  by

$$u_n = \begin{cases} w_n, & \text{if } n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Arguing as in the last paragraph, we show that the subsequences  $(u_{sn})_{n \geq 1}$  are uniformly distributed for each  $s \in S$ . Then  $(u_n)$  is itself uniformly distributed because  $S$  is a u.d. test set. In particular,  $1 - \mathbf{d}A = \mathbf{d}\{n : u_n = 0\} = 0$ , and hence,  $\mathbf{d}A = \alpha$ .

Finally, we assume that  $\alpha = 0$ . Define the sets  $\Gamma'_s$  for  $s \in S$  as before, and let  $\Gamma = \bigcap_{s \in S} \Gamma'_s$ . Select  $\gamma \in \Gamma$ , and let  $w_n = \{n\gamma\}$  for  $n \geq 1$ . We define  $(u_n)_{n \geq 1} \subset [0, 1)$  by

$$u_n = \begin{cases} 0, & \text{if } n \in A, \\ w_n, & \text{otherwise.} \end{cases}$$

As above, we show that the subsequences  $(u_{sn})_{n \geq 1}$  are uniformly distributed for each  $s \in S$ , which implies that  $(u_n)$  is itself uniformly distributed because  $S$  is a u.d. test set. Consequently,  $\mathbf{d}A = \mathbf{d}\{n : u_n = 0\} = 0 = \alpha$ .

With all cases checked, we conclude that  $S$  is a density test set. □

3. PROOF OF THEOREM 1

We will deduce the principal implication in Theorem 1 from the following proposition.

**Proposition 4.** *Let  $S$  be a set of integers greater than 1 such that  $\sum\{1/s : s \in S\} = \infty$ , and let  $f(y) := \sum\{1/s : s \in S, s \leq y\}$ . Let  $[s, t]$  denote the least common multiple of positive integers  $s$  and  $t$ . Suppose that*

$$(1) \quad \sum_{\substack{s,t \in S \\ s,t \leq y}} \frac{1}{[s,t]} \sim f(y)^2 \quad (y \rightarrow \infty).$$

Then  $S$  is a density test set.

*Proof.* Let  $\tau_y(n, S)$  denote the number of  $s \in S$  such that  $s \leq y$  and  $s|n$ . In the course of the proof, we use Turán’s variance method (see [1, Chapter 4], for example) to show that  $\tau_y(n, S)$  is close to  $f(y)$  for “most”  $n$ .

Let  $A$  be a set of positive integers, and suppose that there exists  $\alpha \in [0, 1]$  such that  $\mathbf{d}A_s = \alpha$  for each  $s \in S$ .

Let  $1 \leq y \leq x$ . Observe that

$$\begin{aligned} \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} \tau_y(n, S) &= \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} \sum_{\substack{s \leq y \\ s \in S \\ s|n}} 1 = \sum_{\substack{s \leq y \\ s \in S}} \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A \\ s|n}} 1 = \sum_{\substack{s \leq y \\ s \in S}} \left( \frac{\alpha}{s} + o_s(1) \right) \\ &= \alpha f(y) + o_y(1) \quad (x \rightarrow \infty). \end{aligned}$$

It follows that

$$\frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1 + \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} \left( \frac{\tau_y(n, S) - f(y)}{f(y)} \right) = \alpha + o_y(1) \quad (x \rightarrow \infty),$$

so that

$$(2) \quad \limsup_{x \rightarrow \infty} \left| \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1 - \alpha \right| \leq \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right|$$

for all  $y$  large enough that  $f(y) > 0$ .

We now further restrict  $y$  so that  $1 \leq y \leq \sqrt{x}$ . We have

$$\sum_{n \leq x} \tau_y(n, S)^2 = \sum_{n \leq x} \sum_{\substack{[s,t]|n \\ s,t \in S \\ s,t \leq y}} 1 = \sum_{\substack{[s,t] \leq x \\ s,t \in S \\ s,t \leq y}} \left[ \frac{x}{[s,t]} \right] = x \sum_{\substack{s,t \in S \\ s,t \leq y}} \frac{1}{[s,t]} + O(x).$$

A similar but simpler argument shows that

$$\sum_{n \leq x} \tau_y(n, S) = x f(y) + O(y).$$

We combine our last two results to obtain

$$\begin{aligned} \sum_{n \leq x} (\tau_y(n, S) - f(y))^2 &= \sum_{n \leq x} \tau_y(n, S)^2 - 2f(y) \sum_{n \leq x} \tau_y(n, S) + [x]f(y)^2 \\ &= x \left( \sum_{\substack{s,t \in S \\ s,t \leq y}} \frac{1}{[s,t]} - f(y)^2 \right) + O(x). \end{aligned}$$

Using the hypothesis (1) we deduce that

$$(3) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \left( \frac{\tau_y(n, S) - f(y)}{f(y)} \right)^2 = o(1) \quad (y \rightarrow \infty).$$

From (2), the Cauchy-Schwarz inequality, and (3), we have

$$\begin{aligned} \limsup_{x \rightarrow \infty} \left| \frac{1}{x} \sum_{\substack{n \leq x \\ n \in A}} 1 - \alpha \right| &\leq \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right| \cdot 1 \\ &\leq \limsup_{x \rightarrow \infty} \frac{1}{x} \left\{ \sum_{n \leq x} \left| \frac{\tau_y(n, S) - f(y)}{f(y)} \right|^2 \right\}^{1/2} \cdot x^{1/2} \\ &= o(1) \quad (y \rightarrow \infty). \end{aligned}$$

We let  $y \rightarrow \infty$  to conclude that  $\mathbf{d}A = \alpha$ . Thus,  $S$  is a density test set. □

While Proposition 4 provides us with a sufficient condition for being a density test set, the next lemma offers a necessary condition. To state the result concisely, we use some terminology from the theory of sets of multiples. For  $S \subseteq \mathbb{N}$ , let  $\mathcal{M}(S) = \{ms : m \geq 1, s \in S\}$  be the set of multiples of numbers in  $S$ . A set  $S \subseteq \mathbb{N} \setminus \{1\}$  is said to be *Behrend* if  $\mathcal{M}(S)$  has natural density 1. The interested reader can find a detailed discussion of Behrend sets in [2].

**Lemma 5.** *If  $S$  is a density test set, then  $S$  is Behrend.*

*Proof.* Let  $S$  be a density test set, and set  $A = \mathcal{M}(S)$ . Since  $\mathbf{d}A_s = 1$  for each  $s \in S$ , we have  $\mathbf{d}A = 1$ . Thus,  $S$  is Behrend. □

The proof of Theorem 1 uses a basic property of Behrend sets, which we state in Lemma 6. A proof is given in [2, Corollary 0.10].

**Lemma 6.** *The series of reciprocals of elements from a Behrend set diverges.*

*Proof of Theorem 1.* In view of Theorem 2, it suffices to show that  $\mathcal{P}$  is a density test set if and only if  $\sum\{1/p : p \in \mathcal{P}\}$  diverges.

If  $\mathcal{P}$  is a density test set, then  $\sum\{1/p : p \in \mathcal{P}\}$  diverges by Lemmas 5 and 6.

Now assume that  $\sum\{1/p : p \in \mathcal{P}\} = \infty$ . Let  $f(y) = \sum\{1/p : p \in \mathcal{P}, p \leq y\}$ . We have

$$\sum_{\substack{p, q \in \mathcal{P} \\ p, q \leq y}} \frac{1}{[p, q]} = f(y)^2 + f(y) + O(1) \sim f(y)^2 \quad (y \rightarrow \infty).$$

Then  $\mathcal{P}$  is a density test set by Proposition 4. □

#### 4. CONCLUDING REMARKS

A problem that is complementary to the one considered in this paper is the following. If  $(u_n) \subset [0, 1)$  is uniformly distributed, need the subsequence  $(u_{kn})_{n \geq 1}$  be uniformly distributed for some  $k > 1$ ? Myerson and Pollington [4] have shown that the answer is a resounding “no.” They constructed a uniformly distributed sequence  $(u_n)$  such that the subsequence  $(u_{kn+j})_{n \geq 1}$  is not uniformly distributed for *all* integers  $k \geq 2$  and  $j \geq 0$ .

It would be of interest to find a simple characterization of density test sets (or equivalently, of u.d. test sets). One might conjecture that a set  $S$  is a density test

set if and only if  $S$  is Behrend. Indeed, Theorem 1 and Lemmas 5 and 6 establish this equivalence when  $S$  consists only of primes. We can prove that the equivalence also holds when the elements of  $S$  have at most two prime factors, but at this point we are unable to settle the general case.

## REFERENCES

1. P. D. T. A. Elliott, *Probabilistic number theory: mean value theorems*, Grundlehren der Math. Wiss. 239, Springer-Verlag, 1979. MR **82h**:10002a
2. R. R. Hall, *Sets of multiples*, Cambridge University Press, 1996. MR **98d**:11012
3. L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley, 1974. MR **54**:7415
4. G. Myerson and A. D. Pollington, *Notes on uniform distribution modulo one*, J. Austral. Math. Soc. (Series A) **49** (1990), 264–272. MR **92c**:11075

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