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EXTENSIONS OF A THEOREM OF MARCINKIEWICZ-ZYGMUND AND OF ROGOSINSKI'S FORMULA AND AN APPLICATION TO UNIVERSAL TAYLOR SERIES

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Abstract. This paper extends Rogosinski's formula and the Marcinkiewicz-Zygmund Theorem about circular structure of the limit points of the partial sums of (C,1) summable Taylor series. Also a result about summability of H^p Taylor series is proved and an application on Universal Taylor series is given.

1. Introduction

Let $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be a power series, convergent for $|z| < 1$. A classical theorem of Marcinkiewicz-Zygmund (see [2], [5], [9], Vol. II, p. 178) says that, if this series is (C,1) summable at every point z of a subset E of the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$, then, for almost every z in E, the set of limit points of the partial sums of the series has circular structure with center the (C,1) sum.

One of the results of this paper is an extension of the just mentioned theorem to (C,k) summability with $k \geq 1$. This is Theorem 1 in section 2. This result came as an immediate consequence of an extention of the main ingredient in the proof of the theorem of Marcinkiewicz-Zygmund, namely the formula of Rogosinski (see [2], [9], Theorem 12.16, Ch. III). We extend this formula in Theorem 2 of section 2.

Our work on the formula of Rogosinski was motivated by our desire to answer certain questions on the subject of Universal Taylor series (see [7]); more precisely, whether such a series can be (C,k) summable on its circle of convergence and whether it can belong to any of the Hardy spaces H^p. It was J. -P. Kahane who suggested the above extension of Rogosinski's formula in order to answer these questions. The results related to this subject are contained in section 3.

Section 4 contains remarks and some further comments.

2. Main results

Let $S(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series. We denote by $S_N(z) = \sum_{n=0}^{N} a_n z^n$ the partial sums of this series, and, generally, by $S_N^{(k)}(z)$ its (C,k) sums. These are

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defined inductively (for integer k) by

$$S_N^{(0)}(z) = S_N(z),$$
$$S_N^{(k+1)}(z) = S_0^{(k)}(z) + \cdots + S_N^{(k)}(z).$$

In the particular case of the constant power series 1 (which means $a_0 = 1,\ a_1 = a_2 = \cdots = 0$) the corresponding sums are denoted by $A_N^{(k)}$. Hence

$$A_N^{(0)} = 1,$$
$$A_N^{(k+1)} = A_0^{(k)} + \cdots + A_N^{(k)}.$$

It is easy to see that $A_N^{(k)} = \binom{N+k}{N} \sim \frac{N^k}{\Gamma(k+1)}$, as $N \to \infty$.

By $\sigma_N^{(k)}(z)$ we denote the (C,k) means of the series, defined by

$$\sigma_N^{(k)}(z) = \frac{S_N^{(k)}(z)}{A_N^{(k)}}.$$

We say that $S(z)$ is (C,k) summable at the point z and that it has $\sigma^{(k)}(z)$ as its (C,k) sum, if $\sigma_N^{(k)}(z) \to \sigma^{(k)}(z)$, as $N \to \infty$.

All this is classical and the basic terminology and facts concerning (C,k) summability are described in [1] and [9]. For simplicity we restrict ourselves to the case of integral k. Our first main result is the following:

Theorem 1. *Let the power series $S(z)$ converge for $|z| < 1$. Also let it be (C,k) summable for every z in a certain subset E of the unit circle T, with (C,k) sum $\sigma^{(k)}(z)$. Then, for almost every z of E, the set $L(z)$ of limit points of the sequence*

$$\frac{S_N(z) - \sigma^{(k)}(z)}{N^{k-1}}, \quad N = 1, 2, 3, \dots ,$$

has circular structure with center 0.

A set L in $\mathbb{C}$ has circular structure with center α if, for every z in L, the whole circle $\{\zeta : |\zeta - \alpha| = |z - \alpha|\}$ belongs to L.

The theorem of Marcinkiewicz-Zygmund is the special case $k = 1$ of Theorem 1. Observe that, for $k \geq 2$, the actual value of $\sigma^{(k)}(z)$ plays no role in the structure of $L(z)$.

The proof of Theorem 1, as we mentioned in the Introduction, depends heavily on the following extension of the formula of Rogosinski:

Theorem 2. *Let $S(z)$ be convergent for $|z| < 1$ and be (C,k) summable at z_0, with $|z_0| = 1$. Let $\{z_N\}$ be a sequence with $z_N - z_0 = O(\frac{1}{N})$. Then,*

$$S_N(z_N) - \sigma^{(k)}(z_0)$$
$$= (\frac{z_N}{z_0})^N \sum_{\mu=0}^{k} (1 - \frac{z_0}{z_N})^{\mu} \sum_{m=\mu}^{k} \binom{k-\mu}{m-\mu} (-1)^m A_{N-m}^{(k)} (\sigma_{N-m}^{(k)}(z_0) - \sigma^{(k)}(z_0)) + o(1),$$

as $N \to \infty$.

One trivially sees that, when $k = 1$, the formula of Theorem 2 becomes

$$S_N(z) - \sigma^{(1)}(z) = (\frac{z_N}{z_0})^N (S_N(z_0) - \sigma^{(1)}(z_0)) + o(1),$$

which is identical to Theorem 12.16, Ch. III in [9] (Rogosinski's formula) with a slight difference. In Rogosinski's formula $z_0 = e^{ix},\ z_N = e^{i(x+a_N)}$ with $a_N = O(\frac{1}{N})$, i.e. z_N is confined on the unit circle T. Here we allow z_N to go out of T. The possibility of doing this was initially observed by V. Nestoridis.

Proof of Theorem 2. Without loss of generality we assume that $z_0 = 1$ and we supress it from all occurrences, i.e. $\sigma^{(k)} = \sigma^{(k)}(1), S_N^{(k)} = S_N^{(k)}(1)$ etc. Hence $a_n = S_n^{(0)} - S_{n-1}^{(0)}$ (where of course $S_{-1}^{(0)} = 0$), and with repeated summations by parts we find:

$$S_N(z_N) = \sum_{n=0}^{N} a_n z_N^n = S_N^{(0)} z_N^N + (1 - z_N) \sum_{n=0}^{N-1} S_n^{(0)} z_N^n$$

$$= S_N^{(0)} z_N^N + S_{N-1}^{(1)} z_N^{N-1}(1 - z_N) + (1 - z_N)^2 \sum_{n=0}^{N-2} S_n^{(1)} z_N^n,$$

and finally:

$$\begin{aligned} S_N(z_N) &= S_N^{(0)} z_N^N + S_{N-1}^{(1)} z_N^{N-1}(1 - z_N) + \cdots + S_{N-k}^{(k)} z_N^{N-k}(1 - z_N)^k \\ &+ (1 - z_N)^{k+1} \sum_{n=0}^{N-k-1} S_n^{(k)} z_N^n. \end{aligned} \tag{1}$$

The same formula applied to the constant series 1 implies:

$$\begin{aligned} 1 &= A_N^{(0)} z_N^N + A_{N-1}^{(1)} z_N^{N-1}(1 - z_N) + \cdots + A_{N-k}^{(k)} z_N^{N-k}(1 - z_N)^k \\ &+ (1 - z_N)^{k+1} \sum_{n=0}^{N-k-1} A_n^{(k)} z_N^n. \end{aligned} \tag{2}$$

Multiplying (2) by $\sigma^{(k)}$ and substracting from (1) we get:

$$\begin{aligned} S_N(z_N) - \sigma^{(k)} &= \sum_{\mu=0}^{k} (S_{N-\mu}^{(\mu)} - \sigma^{(k)} A_{N-\mu}^{(\mu)}) z_N^{N-\mu}(1 - z_N)^\mu \\ &+ (1 - z_N)^{k+1} \sum_{n=0}^{N-k-1} (S_n^{(k)} - \sigma^{(k)} A_n^{(k)}) z_N^n. \end{aligned} \tag{3}$$

Now consider the last sum in (3) i.e.

$$(1 - z_N)^{k+1} \sum_{n=0}^{N-k-1} (S_n^{(k)} - \sigma^{(k)} A_n^{(k)}) z_N^n = (1 - z_N)^{k+1} \sum_{n=0}^{N-k-1} A_n^{(k)} z_N^n (\sigma_n^{(k)} - \sigma^{(k)}).$$

It can be considered as a "Toeplitz mean" of the sequence $\{\sigma_n^{(k)} - \sigma^{(k)}\}$. This sequence tends to 0 and the two properties of the "coefficients":

- $(1 - z_N)^{k+1} A_n^{(k)} z_N^n \to 0$, as $N \to \infty$, for fixed n
- $|1 - z_N|^{k+1} \sum_{n=0}^{N-k-1} |A_n^{(k)}||z_N^n| \le (\frac{M}{N})^{k+1} \sum_{n=0}^{N-k-1} cn^k (1 + \frac{M}{N})^n \le cM^{k+1} e^M$ (c, M are absolute constants),

guarantee that the last sum of (3) is $o(1)$, as $N \to \infty$.

Next, if $\mu < k$, we get:

$$\begin{aligned} S_n^{(\mu)} &= S_n^{(\mu+1)} - S_{n-1}^{(\mu+1)} = S_n^{(\mu+2)} - 2S_{n-1}^{(\mu+2)} + S_{n-2}^{(\mu+2)} \\ &= \cdots = \sum_{m=0}^{k-\mu} \binom{k-\mu}{m} (-1)^m S_{n-m}^{(k)} \end{aligned} \tag{4}$$

and the same formula for $A_n^{(\mu)}$.

Replacing (4) and the similar formula for $A_n^{(\mu)}$ in the first sum of (3) and taking into account that the last sum of (3) is $o(1)$ we get:

$$S_N(z_N) - \sigma^{(k)}$$

$$= \sum_{\mu=0}^{k} \sum_{m=0}^{k-\mu} \binom{k-\mu}{m} (-1)^m (S_{N-\mu-m}^{(k)} - \sigma^{(k)} A_{N-\mu-m}^{(k)}) z_N^{N-\mu} (1 - z_N)^\mu + o(1)$$

$$= z_N^N \sum_{\mu=0}^{k} (1 - \frac{1}{z_N})^\mu \sum_{m=\mu}^{k} \binom{k-\mu}{m-\mu} (-1)^m A_{N-m}^{(k)} (\sigma_{N-m}^{(k)} - \sigma^{(k)}) + o(1)$$

and this proves Theorem 2.

Proof of Theorem 1. Using $z_N = z_0, \ \ N = 0, 1, 2, \ldots$, in the formula of Theorem 2 one finds:

$$S_N(z_0) - \sigma^{(k)}(z_0) = \sum_{m=0}^{k} \binom{k}{m} (-1)^m A_{N-m}^{(k)} (\sigma_{N-m}^{(k)}(z_0) - \sigma^{(k)}(z_0)) + o(1).$$

This is the term $\mu = 0$ of the sum in the same formula. Therefore

$$\frac{S_N(z_N) - \sigma^{(k)}(z_0)}{N^{k-1}} = (\frac{z_N}{z_0})^N \frac{S_N(z_0) - \sigma^{(k)}(z_0)}{N^{k-1}}$$

$$+(\frac{z_N}{z_0})^N \sum_{\mu=1}^{k} (1 - \frac{z_0}{z_N})^\mu \frac{1}{N^{k-1}} \sum_{m=\mu}^{k} \binom{k-\mu}{m-\mu} (-1)^m A_{N-m}^{(k)} (\sigma_{N-m}^{(k)}(z_0) - \sigma^{(k)}(z_0))$$

$$+o(\frac{1}{N^{k-1}}).$$

The last sum is, in absolute value, less than or equal to

$$c(1 + \frac{M}{N})^N \sum_{\mu=1}^{k} (\frac{M}{N})^\mu \frac{1}{N^{k-1}} \sum_{m=\mu}^{k} (N-m)^k |\sigma_{N-m}^{(k)}(z_0) - \sigma^{(k)}(z_0)| = \sum_{\mu=1}^{k} o(\frac{1}{N^{\mu-1}}),$$

which is $o(1)$, as $N \to \infty$. Therefore,

$$\frac{S_N(z_N) - \sigma^{(k)}(z_0)}{N^{k-1}} = (\frac{z_N}{z_0})^N \frac{S_N(z_0) - \sigma^{(k)}(z_0)}{N^{k-1}} + o(1), \ \text{ as } \ N \to \infty.$$

Next, let $z_0 = e^{ix}, \ z_N = e^{i(x+\beta_N)}, \ \beta_N = O(\frac{1}{N})$. Then,

$$\frac{S_N(x + \beta_N) - \sigma^{(k)}(x)}{N^{k-1}} = e^{iN\beta_N} \frac{S_N(x) - \sigma^{(k)}(x)}{N^{k-1}} + o(1).$$

Assuming $k \geq 2$ (for $k = 1$ we have the Marcinkiewicz-Zygmund Theorem) and setting

$$t_N(x) = \frac{1}{N^{k-1}} S_N(x)$$

we find:

$$t_N(x+\beta_N) = e^{iN\beta_N} t_N(x) + o(1), \quad \text{as} \quad N \to \infty. \tag{5}$$

Now the rest of the proof is identical to the proof of the Marcinkiewicz- Zygmund Theorem (see [9], Vol. II, p. 178). Only for the sake of completeness (and because it is not so well known) we give here a sketch of proof.

Denote by $D(\zeta, r)$ the open disk centered at ζ with radius r, and by $A(r_1, r_2)$ the open ring centered at 0 with extremal radii r_1 and r_2. Remember that, for every $x \in E, S(x)$ is (C,k) summable and this implies (5) whenever $\beta_n = O(\frac{1}{n})$. To prove that $L(x)$ has circular structure for almost every x in E, it is enough to prove that, if $D(\zeta, r)$ is any disc with rational center $\zeta \neq 0$ and rational radius $r \leq |\zeta|$, then for almost every $x \in E$: if $L(x)$ does not cut $D(\zeta, r)$, then it does not cut $A(|\zeta| - r, |\zeta| + r)$ either. Now consider some increasing sequence of radii tending to $r, r_k \uparrow r$. Consider also the set $E_{k,N}$ of all $x \in E$ such that: $t_n(x)$ is not in $D(\zeta, r_k)$ for every $n \geq N$. It is enough to prove that for almost every $x \in E_{k,N}$ the set $L(x)$ does not cut $A(|\zeta| - r_k, |\zeta| + r_k)$.

Take any point of density x of $E_{k,N}$. If $L(x)$ cuts $A(|\zeta| - r_k, |\zeta| + r_k)$ then, for some sequence of n's, $t_n(x)$ will tend to some point of $A(|\zeta| - r_k, |\zeta| + r_k)$ making an angle, say γ, with ζ. Find a sequence β_n such that:

$$\text{(i) } x + \beta_n \in E_{k,N} \quad \text{and} \quad \text{(ii) } n\beta_n \to -\gamma.$$

Then (i) implies that $t_n(x+\beta_n)$ is not in $D(\zeta, r_k)$ for all $n \geq N$, while (ii), together with (5), implies that $t_n(x+\beta_n)$ is in $D(\zeta, r_k)$ for a sequence of n's.

Thus we arrive at a contradiction and we finish the proof of Theorem 1.

Note that a result of M. Riesz (see [1], Theorem 76) immediately implies that, if $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is (C,k) summable, then $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{(n+1)^{k-1}} z^n$ is (C,1) summable. Therefore, by the Marcinkiewicz-Zygmund Theorem the limit points of the partial sums $\displaystyle\sum_{n=0}^{N} \frac{a_n}{(n+1)^{k-1}} z^n$ have, for almost every $z \in E$, circular structure around the (C,1) sum of the last series (which depends on z).

3. An application

A Taylor series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence equal to 1 is called Universal if, on any compact subset K of the complex plane not intersecting the open unit disc and with connected complement, its partial sums approximate uniformly any given function continuous on K and holomorphic in the interior of K.

This definition is due to V. Nestoridis who proved the existence and the basic properties of such series in the framework of a project studying the behaviour of partial sums of Taylor series (see [7], [8]).

Natural questions arise about this class of series. Here we answer two of them by the following:

Theorem 3. *A Universal Taylor series cannot be (C,k) summable at any point of its circle of convergence. Also it cannot belong to any H^p space, $p > 0$.*

Note that [6] contains the result that any Universal Taylor series cannot belong to the class N of Nevanlinna, thus implying the last part of our Theorem 3. But since the method of proof is different and since it may have some independent interest we include it here.

Proof of Theorem 3. Let $S(z)=\sum_{n=0}^{\infty}a_nz^n$ be a Universal Taylor series. Assume that it is (C,k) summable at a certain point $z_0, |z_0|=1$. Let $K=\{\zeta : |\zeta|\geq 1, |\zeta-z_0|\leq\delta\}$ for small $\delta>0$. Consider the constant function $\sigma^{(k)}(z_0)+1$ on K and (since $S(z)$ is Universal) a subsequence of S_N's converging uniformly on K towards this constant. Let

$$z_N=\frac{z_0}{1-\frac{x}{N}},\quad \text{where } x>0, N>x.$$

Then, the formula of Theorem 2 implies, for this subsequence of N's, that

$$e^{-x}=\lim_N\sum_{\mu=0}^{k}x^{\mu}A_{\mu,N},\quad x>0,$$

where

$$A_{\mu,N}=\frac{1}{N^k}\sum_{m=\mu}^{k}\binom{k-\mu}{m-\mu}(-1)^mA_{N-m}^{(k)}(\sigma_{N-m}^{(k)}(z_0)-\sigma^{(k)}(z_0)).$$

Therefore, a sequence of polynomials in x of degree not exceeding the fixed k converges on the positive real axis towards e^{-x}. This is clearly impossible!

The proof that $S(z)$ does not belong to any H^p space, $p>0$, will be an immediate consequence of the following proposition:

Proposition 1. *If the Taylor series $S(z)=\sum_{n=0}^{\infty}a_nz^n$, convergent in $|z|<1$, defines a function in some H^p space, $p>0$, then the series is (C,k) summable at almost every point of the unit circle for an appropriate k (depending only on p).*

Proof of the proposition. It is enough to assume that $S(z)$ never vanishes in $|z|<1$. Indeed, considering the standard decomposition $S(z)=B(z)G(z)$, where $B(z)$ is a Blaschke product and $G(z)$ never vanishes in $|z|<1$, we can write

$$S(z)=\frac{B(z)+1}{2}G(z)+\frac{B(z)-1}{2}G(z)=S_1(z)+S_2(z).$$

Thus both $S_j(z)$ never vanish in $|z|<1$, and it is enough to work with each $S_j(z)$.

First of all we observe that if $1\leq p$, then $S(z)$ is a Fourier series and thus it is (C,ϵ) summable almost everywhere on $|z|=1$, for every $\epsilon>0$ (see [9]). Also, if we accept the Carleson–Hunt Theorem, we have that if $1<p$, then $S(z)$ is (C,0) summable a.e.

Next, let $\frac{1}{2}<p<1$. Write $S=S^rS^t$, where r,t are chosen so that $r+t=1$ and $r<p,\ t<p$. Then, $S^r\in H^{p/r},\ S^t\in H^{p/t}$ and they are both (C,0) summable a.e.

Now we use a standard theorem (see [1], Theorem 164) saying that if two series are (C,k) and (C,ℓ) summable, then their Cauchy product is (C,k+ℓ+1) summable. Therefore $S(z)$ is (C,1) summable a.e.

If $p=\frac{1}{2}$, then $r=t=\frac{1}{2}$ gives that $S(z)$ is (C,$1+\epsilon$) summable a.e. for every $\epsilon>0$.

Proceed inductively: If $\frac{1}{k+1}<p<\frac{1}{k}$, we write $S=S^rS^t$, where $r+t=1,\ r<kp,\ t<p$. Then $S^r\in H^{p/r},\ S^t\in H^{p/t}$. Hence S^t is (C,0) summable a.e. and (we assume that) S^r is (C,$k-1$) summable a.e. Therefore S is (C,k) summable a.e. If

$p = \frac{1}{k+1}$, the choice $r = kp,\ t = p$ gives that S is (C,$k+\epsilon$) summable a.e. for every $\epsilon > 0$.

This finishes the proof of the proposition and of Theorem 3.

4. Remarks and comments

We initially offer two remarks on Theorem 1.

1. For simplicity in Theorem 1 we restricted ourselves to the case of integral k. We believe that this restriction is unnecessary and one can prove Theorem 1 for k real, $k > 0$, using suitably the formula of Theorem 2; the reader will find such kind of arguments in [5].

In [5] Marcinkiewicz and Zygmund actually proved that:

"If the series $S(z)$ is summable (C,k+1) (where $k \in \mathbb{R},\ k > -1$) at every point z of a set $E \subseteq T$, to sum $\sigma^{(k+1)}(z)$, then at almost every point z of E the set $L^{(k)}(z)$ of limit points of the sequence $\sigma_N^{(k)}(z)$ is of circular structure, with center $\sigma^{(k+1)}(z)$".

This result suggests that Theorem 1 may be extended as follows:

Let the power series $S(z)$ converge for $|z| < 1$. Also let it be (C,k) summable (where $k \in \mathbb{R},\ k > 0$) for every z in a certain subset E of the unit circle T, with (C,k) sum $\sigma^{(k)}(z)$. Then, for almost every z of E, the set $L^{(m)}(z)$ of limit points of the sequence

$$\frac{\sigma_N^{(m)}(z) - \sigma^{(k)}(z)}{N^{k-m-1}}, \quad N = 1, 2, 3, \dots, \ \ 0 \le m < k,$$

has circular structure with center 0.

Observe that, although, in the case $k \neq m+1$, the actual value of $\sigma^{(k)}(z)$ plays no role in the structure of $L^{(m)}(z)$, we include it in order to cover the case $k = m+1$ (which corresponds to the theorem of Marcinkiewicz and Zygmund).

Next, we note that we can replace the factor $(1 - \frac{z_0}{z_N})^\mu$, appearing in the right member of the formula of Theorem 2, by the factor $(\log \frac{z_N}{z_0})^\mu$. Thus, another extension of the formula of Rogosinski is the following:

$$S_N(z_N) - \sigma^{(k)}(z_0)$$
$$= (\frac{z_N}{z_0})^N \sum_{\mu=0}^{k} (\log \frac{z_N}{z_0})^\mu \sum_{m=\mu}^{k} \binom{k-\mu}{m-\mu} (-1)^m A_{N-m}^{(k)} (\sigma_{N-m}^{(k)}(z_0) - \sigma^{(k)}(z_0)) + o(1),$$

as $N \to \infty$.

Now we shall make some comments for the class of Universal Taylor Series. As we mentioned before, several questions arise naturally about this class of series (see [3], [4], [6] and [7]). Although some of them have been answered, there are others which remain open. For example, is a Universal Taylor series always non-continuable across T? To establish such properties of this class of series is a natural direction of research and may be difficult, as is mentioned in [3].

On the other hand we observe that, according to the Theorem 8.37, Ch. V in [9], if $S(z) = \sum_{n=0}^{\infty} a_n z^n$ is a Universal Taylor series, then almost all the functions

$$S_t(z) = \sum_{n=0}^{\infty} a_n z^n \phi_n(t), \quad \text{and} \quad S_t^*(z) = \sum_{n=0}^{\infty} a_n z^n \phi_n^*(t),$$

where $\phi_n(t),\ 0 < t < 1$, are the sequence of Rademacher's functions and $\phi_n^*(t) = \frac{1}{2}(1 + \phi_n(t))$, are not continuable across T. Moreover, for almost all t, the series

$S_t(z)$ and $S_t^*(z)$ are not Universal, since otherwise they shall have subsequences convergent on some arc of T, which amounts to an application of a linear method of summation to each of them - a contradiction according to the results of paragraph 8, Chapter V in [9] (let us notice that $\sum_{n=0}^{\infty} |a_n|^2 = \infty$). Writing now $S(z) = 2S_t^*(z) - S_t(z)$ we see that every Universal Taylor series can be expressed as the sum of two non-Universal and not continuable Taylor series.

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References

[1] Hardy, G. H., Divergent Series, Oxford University Press, 1967. MR **93g:**01100 (1992 reprint)

[2] Kahane, J. -P., Sur la structure circulaire des ensembles de points limites des sommes partielles d'une serie de Taylor, *Acta Sci. Math. (Szeged)* **45**, 1-4 (1983) 247–251. MR **85f:**30003

[3] Kahane, J. -P., Baire Theory in Fourier and Taylor series, *Conference in the honor of Donald Newman*, Philadelphia, March 1996.

[4] Kahane, J. -P., General Properties of Taylor series 1896-1996, L'Escurial, Spain, June 1996. CMP 98:09

[5] Marcinkiewicz, J. and A. Zygmund, On the behavior of trigonometric series and power series, *Trans. Amer. Math. Soc.* **50** (1941) 407–453. MR **3:**105d

[6] Melas, A., V. Nestoridis and I. Papadoperakis, Growth of coefficients of universal Taylor series and comparison of two classes of functions, *Journal d'Analyse Mathematique* **73** (1997) 187–202. CMP 98:10

[7] Nestoridis, V., Universal Taylor Series, *Ann. Inst. Fourier, Grenoble* **46** 5 (1996) 1293–1306. MR **97k:**30001

[8] Nestoridis, V. and S. K. Pichorides, The circular structure of the set of the limit points of partial sums of Taylor series, *Seminaire d'Analyse Harmonique, Univ. de Paris-Sud, Mathematiques, Orsay, France* (1989-90) 71–77.

[9] Zygmund, A., *Trigonometric Series*, second edition reprinted Vol. I, II, Cambridge University Press, 1979. MR **58:**29731

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