PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 7, Pages 2123–2125
S 0002-9939(99)05219-3

A DESCRIPTION OF HILBERT C^*-MODULES IN WHICH ALL CLOSED SUBMODULES ARE ORTHOGONALLY CLOSED

JÜRGEN SCHWEIZER

(Communicated by David R. Larson)

Abstract. Let A, B be C^*-algebras and E a full Hilbert A-B-bimodule such that every closed right submodule $E_0 \subseteq E$ is orthogonally closed, i.e., $E_0 = (E_0^{\perp})^{\perp}$. Then there are families of Hilbert spaces $\{\mathcal{H}_i\}$, $\{\mathcal{V}_i\}$ such that A and B are isomorphic to c_0-direct sums $\sum \mathcal{K}(\mathcal{V}_i)$, resp. $\sum \mathcal{K}(\mathcal{H}_i)$, and E is isomorphic to the outer direct sum $\sum_0 \mathcal{K}(\mathcal{H}_i, \mathcal{V}_i)$.

In [5] B. Magajna proved that if B is a C^*-algebra which admits a full Hilbert B-module E such that every closed right submodule of E is orthogonally complemented, then B is necessarily isomorphic to a C^*-subalgebra of the algebra of compact operators on a Hilbert space.

It is the purpose of this note to show how to obtain Magajna's result in a very simple and conceptual manner by using a basic result of L. G. Brown [1]. Indeed, we shall show that not only B but also the algebra of "compact operators" $\mathcal{K}(E)$ on E and E itself is isomorphic to a C^*-subalgebra, resp. C^*-submodule, of the algebra of compact operators on a suitable Hilbert space. Moreover, to obtain this conclusion, we shall only require the weaker assumption that every closed submodule $E_0 \subseteq E$ is orthogonally closed in the sense that $E_0 = (E_0^{\perp})^{\perp}$.

Notation and main result

By a Hilbert B-module E we shall mean a right B-module E equipped with a B-valued inner product $(\cdot|\cdot)_B$, which is B-linear in its second variable, such that E is complete with respect to the norm $\|\xi\| = \sqrt{(\xi|\xi)_B}$, $\xi \in E$ ([6], [4]). The C^*-algebra of all bounded adjointable operators on E will be denoted by $\mathcal{L}(E)$. It contains the ideal of "compact operators" $\mathcal{K}(E)$ which is the closed linear span of all operators $\theta_{\xi,\eta}$, where $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)_B$ for all $\xi, \eta, \zeta \in E$.

Putting $A = \mathcal{K}(E)$ it is easy to see that ${}_A(\xi|\eta) := \theta_{\xi,\eta}$, $\xi, \eta \in E$, defines an A-valued inner product which is linear in the first variable. In fact, E becomes a Hilbert A-B-bimodule in the sense that

$$ {}_A(\xi|\eta)\zeta = \xi(\eta|\zeta)_B \qquad \text{for all} \quad \xi, \eta, \zeta \in E \tag{1}$$

Received by the editors October 23, 1997.
1991 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. Hilbert C^*-modules, complemented submodules.

The results of this paper are part of the author's doctoral dissertation at the University of Tübingen, which was completed before we received the preprint [5] by Magajna.

(cf. [2]). In the sequel, we shall thus concentrate on Hilbert A-B-bimodules which we assume to be *full*, i.e., the closed linear span of ${}_A(E|E)$ and $(E|E)_B$ coincides with A and with B, respectively.

We shall say that a closed right submodule $E_0 \subseteq E$ is *orthogonally complemented* if $E = E_0 + E_0^\perp$ where $E_0^\perp = \{\, \xi \in E \mid (\xi|E_0)_B = \{0\} \,\}$, and it is *orthogonally closed* if $E_0 = (E_0^\perp)^\perp$.

Clearly, every orthogonally complemented submodule is orthogonally closed. The converse, however, is not true. To see this, let B denote the C^*-algebra of continuous functions on some compact space X. We view $E = B$ as a Hilbert B-bimodule over itself, i.e., $(x|y)_B = x^*y$ $(x, y \in B)$. Here, closed submodules $E_0 \subseteq E$ coincide with closed ideals, which are in bijective correspondence with open subsets of X. Under this correspondence orthogonally complemented submodules map to clopen subsets, whereas a submodule is orthogonally closed if and only if the corresponding open subset coincides with the interior of its closure.

Finally, if for every $i \in \Lambda$, where Λ is some index set, E_i is a Hilbert A_i-B_i-bimodule, then $\sum_0 E_i$ will denote the collection of all families $\{\xi_i\}$, $\xi_i \in E_i$ $(i \in \Lambda)$, vanishing at infinity. Defining all operations componentwise, $\sum_0 E_i$ becomes a Hilbert $\sum A_i$-$\sum B_i$-bimodule in a natural way which will be referred to as the *outer direct sum* of $\{E_i\}_{i\in\Lambda}$.

As we have seen, for an individual submodule $E_0 \subseteq E$ to be orthogonally closed does not imply that E_0 is orthogonally complemented. The following theorem, however, shows that if *all* closed submodules of E are orthogonally closed, then they must be orthogonally complemented also. The simplest example where such a situation occurs is clearly if E is a Hilbert space. Only slightly more involved is the case $E = \mathcal{K}(\mathcal{H}, \mathcal{V})$, where $\mathcal{H}, \mathcal{V}$ are Hilbert spaces, with Hilbert $\mathcal{K}(\mathcal{V})$-$\mathcal{K}(\mathcal{H})$-bimodule structure defined by ${}_A(T|S) = TS^*$ and $(T|S)_B = T^*S$ $(T, S \in E)$.

Theorem 1. *Let E be a full Hilbert A-B-bimodule with the property that every closed right submodule of E is orthogonally closed. Then there are families $\{\mathcal{H}_i\}$, $\{\mathcal{V}_i\}$ of Hilbert spaces such that $A \cong \sum \mathcal{K}(\mathcal{V}_i)$, $B \cong \sum \mathcal{K}(\mathcal{H}_i)$ and*

$$E \cong \sum\nolimits_0 \mathcal{K}(\mathcal{H}_i, \mathcal{V}_i).$$

PROOF OF THEOREM 1

The subsequent discussion is based on the following result of L. G. Brown.

Theorem 2 ([1, 2.5])**.** *Let E be a full Hilbert A-B-bimodule. The mapping*

$$E_0 \mapsto \overline{{}_A(E_0|E)}$$

employs a bijective correspondence between closed right submodules of E and closed right ideals of A. The inverse mapping is given by $R_0 \mapsto \overline{R_0 E}$.

Considering A as a Hilbert C^*-module over itself, we may reformulate the usual condition for A to be a dual C^*-algebra [3] as follows: $(R^\perp)^\perp = R$ for every closed right ideal $R \subseteq A$. But it is easy to see, using (1) and the inverse map provided in Theorem 2, that the bijective correspondence of Theorem 2 commutes with the operation ${}^\perp$. Hence A is a dual C^*-algebra if and only if every closed submodule of E is orthogonally closed. In this case there exists a family of Hilbert spaces $\{\mathcal{V}_i\}$ such that $A \cong \sum \mathcal{K}(\mathcal{V}_i)$ ([3, 4.7.20]).

Theorem 2 implies immediately that if $J \subseteq A$ is an ideal, then $\overline{JE}$ is a subbimodule and vice versa.

Now, let J_i denote the minimal ideal in A corresponding to the ith component of $\sum \mathcal{K}(\mathcal{V}_i)$. Clearly, $J_i J_j = \{0\}$ whenever $i \neq j$. We obtain thus a family $\{\overline{J_i E}\}$ of pairwise orthogonal closed subbimodules, whence $E \cong \sum_0 \overline{J_i E}$ since $\overline{AE} = E$. In order to conclude Theorem 1 it remains thus to show the following lemma.

Lemma 3. *Let E be a full Hilbert A-B-bimodule with $A = \mathcal{K}(\mathcal{V})$. Then there is a Hilbert space $\mathcal{H}$ such that $B \cong \mathcal{K}(\mathcal{H})$ and $E \cong \mathcal{K}(\mathcal{H}, \mathcal{V})$.*

Proof. It is possible, and in fact convenient, to assume that E is a Hilbert B-$\mathcal{K}(\mathcal{V})$-module by simply passing to the conjugate Hilbert C^*-bimodule E^* (cf. [2]).

We write π for the identity representation of $\mathcal{K}(\mathcal{V})$ on $\mathcal{V}$ and put $\mathcal{H} = E \otimes_\pi \mathcal{V}$. Left multiplication in the first component of $\mathcal{H} = E \otimes_\pi \mathcal{V}$ then defines a faithful representation of B onto $\mathcal{K}(\mathcal{H})$ ([4, Proposition 4.7]).

To see the final assertion we define a mapping

$$\Theta \colon E \to \mathcal{L}(\mathcal{V}, \mathcal{H}), \qquad \xi \mapsto (\zeta \mapsto \xi \otimes \zeta)$$

and observe that $\Theta(\xi)^* \Theta(\eta) = (\xi|\eta)_A$. If $x \in A$ is a minimal projection, then $\Theta(\xi x)\zeta = \xi x \otimes \zeta = \xi \otimes x\zeta$, so that $\Theta(Ex)$ consists of finite rank operators. But as $Ax\xi$ is dense in $\mathcal{V}$ and $E \otimes_\pi Ax\zeta = EA \otimes_\pi x\zeta \subseteq E \otimes_\pi x\zeta$ we see that $\Theta(Ex)$ consists of all operators with support $x\mathcal{V}$. Since x was arbitrary we conclude that $\Theta(E) = \mathcal{K}(\mathcal{V}, \mathcal{H})$. □

REFERENCES

1. L. G. Brown, *Close hereditary C^*-subalgebras and the structure of quasi-multipliers*, MSRI preprint #11211-85, 1985.
2. L. G. Brown, J. A. Mingo, and N.-T. Shen, *Quasi-multipliers and embeddings of Hilbert C^*-bimodules*, Canad. J. Math. **46** (1994), 1150–1174. MR **95k:**46091
3. J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1981. MR **56:**16388
4. E. C. Lance, *Hilbert C^*-modules – A toolkit for operator algebraists*, London Math. Soc. Lecture Notes no. 210, Cambridge University Press, Cambridge, 1995. MR **96k:**46100
5. B. Magajna, *Hilbert C^*-modules in which all closed submodules are complemented*, Proc. Amer. Math. Soc. **125** (1997), 849–852. MR **97e:**46079
6. N. E. Wegge-Olsen, *K-theory and C^*-algebras – a friendly approach*, Oxford University Press, Oxford, 1993. MR **95c:**46116

MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY

E-mail address: `juergen.schweizer@uni-tuebingen.de`