HELLY-TYPE THEOREMS FOR HOLLOW AXIS-ALIGNED BOXES

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Abstract. A hollow axis-aligned box is the boundary of the cartesian product of $d$ compact intervals in $\mathbb{R}^d$. We show that for $d \geq 3$, if any $2d$ of a collection of hollow axis-aligned boxes have non-empty intersection, then the whole collection has non-empty intersection; and if any 5 of a collection of hollow axis-aligned rectangles in $\mathbb{R}^2$ have non-empty intersection, then the whole collection has non-empty intersection. The values $2d$ for $d \geq 3$ and 5 for $d = 2$ are the best possible in general. We also characterize the collections of hollow boxes which would be counterexamples if $2d$ were lowered to $2d - 1$, and 5 to 4, respectively.

1. General notation and definitions

We denote the cardinality of a set $S$ by $\#S$. Let $\Pi(S,k)$ denote the property that any subcollection of $S$ of at most $k$ sets has non-empty intersection (where $k$ is any positive integer), and $\Pi(S)$ the property that $S$ has non-empty intersection. For any set $S \subseteq \mathbb{R}^d$, we denote the convex hull, interior and boundary by $\text{co}S$, $\text{int}S$ and $\text{bd}S$, respectively. An axis-aligned box in $\mathbb{R}^d$ is the cartesian product of $d$ compact intervals, i.e. a set of the form

$$\prod_{i=1}^{d} [a_i, b_i] = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : a_i \leq x_i \leq b_i, i = 1, \ldots, d\} \quad (a_i < b_i).$$

An axis-aligned hollow box in $\mathbb{R}^d$ is the boundary of a box, i.e. a set of the form

$$\text{bd} \prod_{i=1}^{d} [a_i, b_i] \quad (a_i < b_i).$$

In the rest of the paper, the word axis-aligned is implicit whenever we refer to boxes or hollow boxes. In the next section we state our results (Theorems 1 and 2), together with examples showing that they are the best possible. In Section 3 we derive a combinatorial lemma needed in the proofs of these theorems in Section 4.
2. HELLY-TYPE THEOREMS

A Helly-type theorem may be loosely described as an analogue of

**Helly’s Theorem** ([6]). Let $S$ be a collection of convex sets in $\mathbb{R}^d$ that is finite or contains at least one compact set. Then

$$\Pi(S, d+1) \implies \Pi(S).$$

There is an abundance of literature on Helly-type theorems; see the surveys [1, 3, 5]. Most of these analogues consider collections of *convex* sets, exactly as in Helly’s Theorem. Here are two examples where non-convex sets are considered.

**Theorem** (Motzkin [8, 2]). Let $S$ be a collection of sets in $\mathbb{R}^d$, each of which is the set of common zeroes of a set of real polynomials in $d$ variables of degree at most $k$. Then

$$\Pi(S, (\binom{d+k}{k})) \implies \Pi(S).$$

**Theorem** (Maehara [7, 4]). Let $S$ be a collection of at least $d+3$ euclidean spheres in $\mathbb{R}^d$. Then

$$\Pi(S, d+1) \implies \Pi(S).$$

In both of these theorems the sets are algebraic. In this paper we find Helly-type theorems for certain non-algebraic sets, namely hollow boxes. It is well known (and immediately follows from the one-dimensional Helly Theorem) that for any collection $S$ of boxes in $\mathbb{R}^d$,

$$\Pi(S, 2) \implies \Pi(S).$$

If we want the boxes to intersect only in their boundaries, then the value 2 has to be greatly enlarged, as the following examples show.

**Example 1.** A class of collections $S$ of hollow boxes in $\mathbb{R}^d$ such that $\Pi(S, 2d)$ holds, but not $\Pi(S, 2d+1)$.

Choose any box $B = \prod_{i=1}^d [x_i^0, x_i^1]$ (where $x_i^0 < x_i^1$), and $p = (p_1, \ldots, p_d) \in \text{int } B$. For $i = 1, \ldots, d$ and $j = 0, 1$, let $F_i^j$ denote the facet of $B$ contained in the hyperplane $\{x \in \mathbb{R}^d : x_i = x_i^j\}$. Let $S$ be any collection of hollow boxes such that

1. $\text{bd } B \in S$,
2. $p \in D$ for all $D \in S \setminus \{\text{bd } B\}$,
3. for each $D \in S \setminus \{\text{bd } B\}$ there is a facet of $B$ contained in $D$,
4. for each facet $F$ of $B$ there exists some $D \in S \setminus \{\text{bd } B\}$ such that $F \subseteq D$.

It is clear that there exist such collections $S$ (even infinite ones, provided that $d \neq 1$). Note that the facet in (3) is unique, by (2). See Figure 1 for an example in $\mathbb{R}^2$.

Choose any subcollection $T \subseteq S$ of $2d$ hollow boxes. If $\text{bd } B \not\in T$, then by (2), $\bigcap_{D \in T} D \neq \emptyset$. Otherwise, by (3), there is a facet of $B$ not contained in any $D \in T \setminus \{\text{bd } B\}$, say $F_1^0$. Then it easily follows from (2) and (3) that $(x_i^0, p_2, p_3, \ldots, p_d) \in \bigcap_{D \in T} D$. It follows that $\Pi(S, 2d)$ holds.

Secondly, use (4) to choose for each facet $F_i^j$ of $B$ a $D_i^j \in S$ containing $F_i^j$. Then $F_i^{1-j} \cap D_i^j = \emptyset$ by (2). It follows that $(\text{bd } B) \cap \bigcap_{i=1}^d (D_i^0 \cap D_i^1) = \emptyset$, and $\Pi(S, 2d+1)$ does not hold.
Example 2. A class of collections $\mathbf{S}$ of hollow boxes in $\mathbb{R}^d$ such that $\Pi(\mathbf{S}, 2^d - 1)$ holds, but not $\Pi(\mathbf{S}, 2^d)$.

Let $B = \prod_{i=1}^{d} [x_0^i, x_1^i]$ $(x_0^i < x_1^i)$, and let $\mathbf{S}$ be any collection of hollow boxes such that

(5) $B \subseteq \operatorname{co} D$ for all $D \in \mathbf{S}$,

(6) for each vertex $v$ of $B$ there exists a $D \in \mathbf{S}$ not containing $v$,

(7) each $D \in \mathbf{S}$ contains all the vertices of $B$ except at most one.

Thus it is clear there exist such collections, even infinite ones. See Figure 2 for an example in $\mathbb{R}^2$. Given a subcollection of $2^d - 1$ hollow boxes, then by (7) some vertex of $B$ is contained in all these boxes. Thus $\Pi(\mathbf{S}, 2^d - 1)$ holds.

Secondly, (6) gives a subcollection of $2^d$ boxes $D_v$ with $v \notin D_v$. But then, also using (5), it follows from Lemma 4.2 that for any vertex $w$ of $B$, $\bigcap_{v \neq w} D_v = \{w\}$. Thus, $\bigcap_v D_v = \emptyset$, and $\Pi(\mathbf{S}, 2^d)$ does not hold.

The following two theorems show that the collections in Example 1 in the case $d = 2$, and the collections in Example 2 in the case $d \geq 3$ are the worst cases.

Theorem 1. Let $\mathbf{S}$ be a collection of hollow boxes in $\mathbb{R}^2$. Then

$$\Pi(\mathbf{S}, 5) \implies \Pi(\mathbf{S}).$$
We first show that any minimal cover $C$ matches 0 or 1. A cover $C$ is a collection in $\mathbb{R}^d$ such that any string $\varepsilon \in \Pi(S)$, with equality if $C = \{0, 1\}^d$. Then $\Pi(S)$ is a cover of $\{0, 1\}^d$ such that no proper subset of $C$ is a cover of $\{0, 1\}^d$.

Lemma 1. Let $C$ be a minimal cover of $\{0, 1\}^d$. Then, for each $i = 1, \ldots, d$, $E_i := \{\varepsilon \in \varepsilon_1 \ldots \varepsilon_d \in C \} \text{ is equal to either } \{\ast\}, \{0, 1\} \text{ or } \{0, 1, \ast\}$. Let $s := \#(i : E_i = \{\ast\}).$ Then $\#C \leq 2^{d-s},$ with equality iff $C = \{\varepsilon : \varepsilon_i = \ast \text{ for all } i \in J\}$ for some $J \subseteq \{1, 2, \ldots, d\}$ with $\#J = s$.

Proof. We first show that any minimal cover $C$ satisfies $\#C \leq 2^d$, with equality if $C = \{0, 1\}^d$. For each pattern $\rho \in C$, the set $C \setminus \{\rho\}$ is not a cover of $\{0, 1\}^d$, and there exists a string $\varepsilon_\rho \in \{0, 1\}^d$ that matches $\rho$ but does not match any other pattern in $C$. Thus

$$\phi : C \to \{0, 1\}^d, \rho \mapsto \varepsilon_\rho$$

is an injection, and $\#C \leq 2^d$. If equality holds, $\phi$ is a bijection, and any string in $\{0, 1\}^d$ matches a unique pattern in $C$. Thus $C$ defines a partition of $\{0, 1\}^d$: a block of the partition consists of all strings matching a given pattern in $C$. Since there are $2^d$ blocks, each block must contain exactly 1 element. Thus no pattern in $C$ contains a $\ast$, and $C = \{0, 1\}$.

Secondly, we show that if 0 does not occur in the first position of any string in $C$, there are only $\ast$’s in the first position. Let $C^* = \{\varepsilon_2 \ldots \varepsilon_n : \ast \varepsilon_2 \ldots \varepsilon_n \in C\}$. It is easily seen that $C^*$ is a cover for $\{0, 1\}^{d-1}$. For any $\varepsilon \in \{0, 1\}^{d-1}$, $0\varepsilon$ matches some pattern in $C$ starting with $\ast$. But then, by putting back $\ast$ in the first position of every pattern in $C^*$, we already obtain a cover of $\{0, 1\}^d$. Thus, 1 does not occur in the first position in any string in $C$. Similarly, if 1 does not occur in the first position, then there are again only $\ast$’s in the first position.
Finally, to complete the proof, delete the positions for which \( E_i = \{ \} \), to obtain \( C' \subseteq \{0, 1, \}^{d-s} \). Then \( C' \) is clearly a minimal cover of \( \{0, 1\}^{d-s} \), and \( \#C = \#C' \). Now apply the first part of the proof.

We omit the proof of the following elementary inequality.

**Lemma 2.** Let \( d \geq s \geq 0 \) be integers. Then \( 2^{d-s} < 2^d - 2s \), except in the following cases:

1. If \((d, s) = (1, 1) \) or \((d, s) = (2, 2) \), the opposite inequality holds.
2. If \(s = 0\), or \((d, s) = (2, 1) \), there is equality.

**Proof.** It is easy to check everything for \( d = 1 \) and \( d = 2 \): The only minimal covers for \( d = 1 \) are \( \{ \} \) and \( \{0, 1\} \), and for \( d = 2 \), are equivalent (up to permutation of the positions, and interchange of 0 and 1) to one of

\[
\{\}, \{0, 1\}, \{0, 10, 11\}, \{0, 01, 11\}.
\]

For \( d \geq 3 \), if \( s \geq 1 \), then \( \#C \leq 2^{d-s} < 2^d - 2s \), by Lemmas 1 and 2. Otherwise, \(s = 0\), and by Lemma 1, \( \#C < 2^d \) unless \( C = \{0, 1\}^d \).

**4. Proofs of Theorems 1 and 2**

We first prove a rather technical lemma, which gives some insight into the (not easily visualizable) intersections of hollow boxes.

**Lemma 4.** Let \( B = \prod_{i=1}^{d} [x^i_1, x^i_2, \ldots, x^i_{d_i}] \), with \( x^i_0 \leq x^i_1 \) for each \( i = 1, \ldots, d \). (Thus \( B \) is not necessarily full-dimensional.) For each string \( \epsilon \in \{0, 1\}^d \), let \( x_\epsilon := (x^{\epsilon_1}_1, x^{\epsilon_1}_2, \ldots, x^{\epsilon_1}_d) \), and let \( D_\epsilon \) be a hollow box such that \( x_{\epsilon'} \notin D_\epsilon \) and \( B \subseteq \text{co} D_\epsilon \).

Then,

1. \( B \cap \bigcap_\epsilon D_\epsilon = \emptyset \),
2. for any \( \gamma \in \{0, 1\}^d \), \( B \cap \bigcap_{\gamma \neq \epsilon} D_\epsilon \subseteq \{x_\gamma\} \),
3. for any \( \gamma, \delta \in \{0, 1\}^d \),

\[
B \cap \bigcap_{\epsilon \neq \gamma, \delta} D_\epsilon \subseteq \begin{cases} \text{co}\{x_\gamma, x_\delta\} & \text{if } x_\gamma \text{ and } x_\delta \text{ differ in exactly one coordinate}, \\
\{x_\gamma, x_\delta\} & \text{otherwise}. 
\end{cases}
\]

**Proof.** Clearly, part 1 follows from part 2: If \( B \) is a single point, each \( D_\epsilon \) is disjoint from \( B \). Otherwise, choose \( \gamma, \gamma' \) such that \( x_\gamma \neq x_{\gamma'} \). Then, by part 2, \( B \cap \bigcap_\epsilon D_\epsilon = \emptyset \).

Although part 2 also easily follows from part 3, we first prove part 2, as it clears the way for a proof of part 3. For each \( \epsilon \), write \( D_\epsilon = \text{bd} \prod_{i=1}^{d} [a^\epsilon_i, b^\epsilon_i] \). Let \( x = (x_1, x_2, \ldots, x_d) \in B \cap \bigcup_{\epsilon \neq \gamma} D_\epsilon \). Then \( x^0_i \leq x_1 \leq x^1_i \) for each \( i \). Define \( \epsilon_i \) by

\[
\epsilon_i := \begin{cases} \gamma_i & \text{if } x_i = x^{\gamma_i}_i, \\
1 - \gamma_i & \text{otherwise}. 
\end{cases}
\]

Since \( x_\epsilon \subseteq B \subseteq \text{co} D_\epsilon \), but \( x_\epsilon \notin D_\epsilon \), we have \( a^\epsilon_i \leq x^0_i \leq x^1_i \leq b^\epsilon_i \) and \( a^\epsilon_i < x^{\gamma_i}_i < b^\epsilon_i \) for all \( i \). If \( \epsilon_i = \gamma_i \), then \( x^{\epsilon_i}_i = x^{\gamma_i}_i = x_i \). If \( \epsilon_i = 1 - \gamma_i \), then \( x_i \neq x^{\gamma_i}_i \), and either
\[ \gamma_i = 1 \text{ and } x_i^\gamma_i = x_i^0 \leq x_i < x_i^1, \text{ or } \gamma_i = 0 \text{ and } x_i^\gamma_i = x_i^1 \geq x_i > x_i^0. \] In all cases, \( a_i^\gamma_i < x_i < b_i^\gamma_i, \) and it follows that \( x \not\in D_\varepsilon. \) Thus \( \varepsilon = \gamma, \) and \( x_i = x_i^{\gamma_i} \) for all \( i. \) It follows that \( x = x_\gamma. \)

Now let \( x \in B \cap \bigcap_{\varepsilon \neq \gamma, \delta} D_\varepsilon, \) and suppose \( x \neq x_\gamma, x_\delta. \) Let \( j \) be any position such that \( x_j \neq x_j^{\gamma_j}. \) Define \( \varepsilon \) by

\[
\varepsilon_i := \begin{cases} 
1 - \gamma_i & \text{if } i = j, \\
\delta_i & \text{if } x_i = x_i^{\delta_i}, i \neq j, \\
1 - \delta_i & \text{if } x_i \neq x_i^{\delta_i}, i \neq j.
\end{cases}
\]

As in the proof of part 2, for each \( i \) we obtain \( a_i^\varepsilon < x_i < b_i^\varepsilon, \) and therefore, \( x \not\in D_\varepsilon. \) Thus, \( \varepsilon = \gamma \) or \( \varepsilon = \delta. \) But, since \( \varepsilon_j \neq \gamma_j, \) we must have \( \varepsilon = \delta. \) Thus, \( \gamma_j = 1 - \delta_j, \) and for all \( i \neq j, x_i = x_i^{\delta_i}. \) Since \( x \neq x_\delta, \) we then must have \( x_j \neq x_j^{\delta_j}. \) By repeating the above argument with \( x_\delta \) instead of \( x_\gamma, \) we also obtain that for all \( i \neq j, x_i = x_i^{\gamma_i}. \) It follows that \( x \in \text{co}\{x_\gamma, x_\delta\}, \) and \( x_\gamma \) and \( x_\delta \) differ in only one coordinate.  

**Proof of Theorem 2.** Note that the first part of the theorem follows from the second part, since \( \Pi(S, 2^d) \) does not hold in Example 2. By compactness, we only have to prove the theorem for finite \( S. \) We assume that \( \Pi(S, 2^d - 1). \) Let \( B = \bigcap_{D \in S} \text{co } D = \prod_{i=1}^d [a_i^0, b_i^0]. \) (Since any two \( D \)'s intersect, \( x_i^0 \leq x_i^1 \) for all \( i.) \) We denote the vertices of \( B \) by \( x_\varepsilon, \varepsilon \in \{0, 1\}^d, \) as in Lemma 4. We now show that if \( x_\varepsilon \not\in \bigcap_{D \in S} D \) for all \( \varepsilon, \) then \( S \) is as in Example 2.

For each \( \varepsilon, \) choose \( D_\varepsilon = \text{bd} \prod_{i=1}^d [a_i^\varepsilon, b_i^\varepsilon] \in S \text{ such that } x_\varepsilon \not\in D_\varepsilon, \) and let

\[ X_\varepsilon := \{x_\delta : \delta \in \{0, 1\}^d, x_\delta \not\in D_\varepsilon\}. \]

Then \( X_\varepsilon = \{x_\delta : \delta \text{ matches } \rho_\varepsilon\}, \) where \( \rho_\varepsilon = \rho_1 \ldots \rho_d \) is the pattern defined by

\[
\rho_i := \begin{cases} 
0 & \text{if } a_i^\varepsilon < x_i^0 \text{ and } x_i^1 = b_i^\varepsilon, \\
1 & \text{if } a_i^\varepsilon = x_i^0 \text{ and } x_i^1 < b_i^\varepsilon, \\
* & \text{if } a_i^\varepsilon < x_i^0 \text{ and } x_i^1 < b_i^\varepsilon.
\end{cases}
\]

Thus \( C := \{\rho_\varepsilon : \varepsilon \in \{0, 1\}^d\} \) is a cover of \( \{0, 1\}^d. \) If \( \rho_\varepsilon = \rho_\varepsilon', \) then \( x_\varepsilon \neq x_\varepsilon' \not\in D_\varepsilon, \) so we may choose the \( D_\varepsilon \)'s such that if \( \rho_\varepsilon = \rho_\varepsilon', \) then \( D_\varepsilon = D_\varepsilon'. \) We now write \( D_\rho \) for \( D_\varepsilon \) whenever \( \rho = \rho_\varepsilon \in C. \) Let \( C' \) be a minimal cover contained in \( C. \) For each \( \varepsilon \in \{0, 1\}^d \) there now exists a \( \rho \in C' \) matching \( \varepsilon \) such that \( x_\varepsilon \not\in D_\rho. \) Applying Lemma 4.1 to \( \{D_\rho : \rho \in C'\}, \) we find \( B \cap \bigcap_{\rho} D_\rho = \emptyset. \) Let \( J \subseteq \{1, \ldots, d\} \) be the set of positions in which there are only *’s in \( C'. \) For each \( j \in J, \) choose \( D_j^0 = \text{bd} \prod_{i \neq j} [r_j^0, s_j^0] \) and \( D_j^1 = \text{bd} \prod_{i \neq j} [r_j^1, u_j^1] \) from \( S \) such that \( r_j^0 = x_j^0 \) and \( u_j^1 = x_j^1 \) (which is possible since \( S \) is finite). Since (by Lemma 1) for each \( i \not\in J \) there exist \( \rho, \rho' \in C' \) such that \( \rho_i = 0 \) and \( \rho'_i = 1, \) we obtain

\[ \bigcap_{j \in J} \left( \text{co } D_j^0 \cap \text{co } D_j^1 \right) \cap \bigcap_{\rho \in C'} \text{co } D_\rho = B. \]

Thus, letting \( T := \{D_\rho : \rho \in C' \} \cup \{D_j^0, D_j^1 : j \in J\}, \) we obtain \( \bigcap_{D \in T} D = \emptyset. \) Thus, \( \# T \geq 2^d. \) Also, \( \# T \leq \# C' + 2 \# J. \) Thus, by Lemma 3, \( C' = \{0, 1\}^d. \) It follows that \( x_\varepsilon \not\in D_\varepsilon \) iff \( \varepsilon = \varepsilon. \) Thus, all \( x_i \)'s are distinct, and \( B \) is full-dimensional. Also, \( J = \emptyset \) and \( B = \bigcap_{\rho \in C'} \text{co } D_\rho. \) In fact, if we take any \( \varepsilon \) and \( \varepsilon' \) which differ in each position, then \( B = \text{co } D_\varepsilon \cap \text{co } D_{\varepsilon'}. \)
We already have that \( S \) satisfies (5) and (6) in Example 2. Consider any \( D \in S \) with \( D \neq D_x \) for all \( x \). Suppose there exist distinct \( \gamma, \delta \) such that \( x_\gamma, x_\delta \notin D \). By Lemma 4.3, \( D \cap B \cap \bigcap_{\varepsilon \neq \gamma, \delta} D_\varepsilon = \emptyset \). But there exist \( \varepsilon, \varepsilon' \notin \{\gamma, \delta\} \) differing in each position. Thus \( \bigcap_{\varepsilon \neq \gamma, \delta} D_\varepsilon \subseteq B \), and \( D \cap \bigcap_{\varepsilon \neq \gamma, \delta} D_\varepsilon = \emptyset \), contradicting \( \Pi(S, 2^d - 1) \). Thus \( D \) contains all \( x_\gamma \)'s, except at most one and (7) is satisfied.

**Proof of Theorem 1.** Proceeding as in the proof of Theorem 2, we assume that \( \Pi(S, 4) \) holds and that no vertex of \( B \) is in \( \bigcap_{D \in S} D \), and obtain \( C' = \{**\} \) and \# \( T = 5 \).

We now show that \( S \) is as in Example 1. Since \( C' = \{**\} \), there is only one \( D_p \), say \( D = D_{**} \), which is disjoint from \( B \). Also, \( T = \{D_0^1, D_1^1, D_2^1, D_3^1, D\} \), with the \( D_j^i \)'s as in the proof of Theorem 2. Thus \( \bigcap_{i,j} \text{co} D_j^i = B \).

Suppose that for each \( \varepsilon \in \{0, 1\}^2 \) there exists a \( D_j^i \) not containing \( x_\varepsilon \). Then by Lemma 4.1, \( \bigcap_{i,j} D_j^i = \emptyset \), contradicting \( \Pi(S, 4) \). Thus, some \( x_\varepsilon \in \bigcap_{i,j} D_j^i \), say \( x_{00} \).

Suppose that \( B \) is two-dimensional, i.e. \( x_0^1 < x_1^1 \) and \( x_2^0 < x_3^0 \). Then, since \( x_{00} \in D_1^1 \) \( D_1^1 \) contains at least two sides of \( B \), and it follows that \( B = \text{co} D_1^1 \cap \text{co} D_2^1 \) or \( B = \text{co} D_1^1 \cap \text{co} D_0^1 \cap \text{co} D_1^1 \). Thus

\[
D_1^1 \cap D_0^1 \cap D_2^1 \cap D = \emptyset \quad \text{or} \quad D_1^1 \cap D_0^1 \cap D_1^1 \cap D = \emptyset,
\]

both cases contradicting \( \Pi(S, 4) \).

Suppose \( B \) is one-dimensional, say \( x_0^1 < x_1^1 \) and \( x_2^0 = x_3^0 \). Then \( D_0^1 \cap D_1^1 \) is a horizontal segment containing \( B \). If \( D_1^1 \) intersects \( D_0^1 \cap D_2^1 \) only in \( x_{00} \) and \( x_{10} \), then \( D_1^1 \cap D_0^1 \cap D_2^1 \cap D = \emptyset \), a contradiction. Thus, \( B \subseteq D_1^1 \). We may assume that \( D_2^1 \) and \( D_1^1 \) are on opposite sides of \( B \) (otherwise consider \( D_2^1 \) and \( D_1^1 \)). Then

\[
D_1^1 \cap D_1^1 \cap D_2^1 \subseteq B \quad \text{and} \quad D_0^1 \cap D_1^1 \cap D_2^1 \cap D = \emptyset,
\]
a contradiction.

Thus \( B \) is zero-dimensional, say \( B = \{p\} \), where \( p = x_{00} = (x_1, x_2) \) and \( x_1 = x_0^1, x_2 = x_1^1, x_0 = x_3^0 \). Then \( D_0^1 \cap D_1^1 \) is a vertical line segment through \( p \) which must intersect \( D_0^1 \cap D \) in a point \( b \neq p \), and \( D_2^1 \cap D \) in a point \( a \neq p \). Similarly, \( D_0^1 \cap D_0^1 \) is a horizontal segment through \( p \) which must intersect \( D_0^1 \cap D \) in a point \( d \neq p \), and \( D_1^1 \cap D \) in a point \( c \neq p \). See Figure 3. Now \( S \) already satisfies (1) and (4) of Example 1, if we take \( B \) there as \( \text{co} D \).

Consider any \( E \in S \setminus T \). By considering the intersection of three sets at a time from \( T \), we see that \( E \) must intersect each of the sets \( \{a, b\}, \{c, d\}, \{p, a\}, \{p, b\}, \{p, c\}, \{p, d\} \). If \( p \notin E \), then \( a, b, c, d \in E \), and \( E = D \), a contradiction.

Thus \( p \in E \), and (2) is satisfied. Also, \( a \in E \) or \( b \in E \). We may assume without loss that \( a \in E \), and similarly, \( c \in E \). But then, since \( E \cap D \cap D_0^1 \cap D_1^1 \neq \emptyset \), we
must have either \( b \in E \) or \( d \in E \), and (3) is satisfied. It follows that \( S \) is as in Example 1.

\[ \square \]

**Acknowledgement**

This paper is based on part of the author’s Ph.D. thesis written under supervision of Professor W. L. Fouché at the University of Pretoria. I thank the referee for pointing out a few small errors in a previous version of this paper.

**References**


