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CODIMENSION 1 LINEAR ISOMETRIES ON FUNCTION ALGEBRAS

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ABSTRACT. Let A be a function algebra on a locally compact Hausdorff space. A linear isometry $T : A \longrightarrow A$ is said to be of codimension 1 if the range of T has codimension 1 in A. In this paper, we provide and study a classification of codimension 1 linear isometries on function algebras in general and on *Douglas algebras* in particular.

1. INTRODUCTION

Recently, several authors have begun the study of a special kind of linear isometry $T : C(X) \longrightarrow C(X)$, where X is a compact Hausdorff space, called (isometric) shift operators. The origin of these operators is the following: Let $\mathcal{H}$ be a separable Hilbert space and let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. It is said that $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a *shift operator* or a *unilateral shift* if $T(\phi_n) = \phi_{n+1}$ for $n = 1, 2, \ldots$. The connections between the shift operator and the rest of mathematics are numerous. Indeed this operator plays an essential role in many disciplines such as scattering theory or stationary stochastic processes (see [10]).

Several extensions of this type of operator to Banach spaces have been proposed. Among these, R.M. Crownover ([3]) was the first to give a definition of the concept of shift operator on Banach spaces without using basis: Let $\mathcal{K}$ be a Banach space. It is said that $T : \mathcal{K} \longrightarrow \mathcal{K}$ is a *(isometric) shift operator* if

(1) T is a linear isometry,
(2) The codimension of $T(\mathcal{K})$ in $\mathcal{K}$ is 1,
(3) $\bigcap_{n=1}^{\infty} T^n(\mathcal{K}) = \{0\}$.

If we omit condition (3), then we have a *codimension 1 linear isometry* or a *quasi-shift operator*.

Crownover justified his definition by showing that there exists a Banach space $\mathcal{K}'$ of complex-valued sequences such that $\mathcal{K}$ is isomorphic and isometric to $\mathcal{K}'$, and such that on $\mathcal{K}'$ the above operator T corresponds to the shift operator T' defined by the condition $T'(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$.

Most of the usefulness of shift operators comes from its equivalence to the operator multiplication by z on the Hilbert space of square-summable power series, i.e.,

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the Hardy space H^2. As a consequence, shift operators have an analytic structure which enables us to express each element of the Banach space $\mathcal{K}$ as an analytic function on a certain domain. This fact can be used, for example, to show that the bounded linear operators on $\mathcal{K}$ which commute with T (the commutant of T) are identified with a (not necessarily proper) subalgebra of H^∞. Indeed, the commutant of T when $\mathcal{K}$ is the Hardy space H^p $(1 \leq p \leq \infty)$ is the full algebra H^∞, whereas in the case of l^p-spaces $(1 \leq p \leq \infty)$ the commutant is a proper subalgebra of H^∞.

In [9], the authors studied isometric shift operators on the Banach space $C(X)$ (X compact). First, they classified these operators using the following result: Let $T : C(X) \longrightarrow C(X)$ be a codimension 1 linear isometry. Then there exists a closed subset X_0 of X such that either

(1) $X_0 = X \setminus \{p\}$

where p is an isolated point of X, or

(2) $X_0 = X$

and such that there exist a continuous map h of X_0 onto X and a function $a \in C(X_0)$, $|a| \equiv 1$, such that

$$(Tf)(x) = a(x) \cdot f(h(x))$$

for all $x \in X_0$.

The proof of this result is based on a well known theorem of Holsztyński ([7]). Those isometries that satisfy condition (1) are said to be of Type I. Those satisfying condition (2) are said to be of Type II. These two classes are not disjoint. In this paper, the authors focus on a question which had not been addressed yet: which Banach spaces admit (quasi-)shift operators, particularly, the conditions on the compact space X so that $C(X)$ admits such operators. They show, for instance, that if X is not separable, then it does not admit isometric shifts of Type II. In [6], R. Haydon proves the existence of isometric shifts of Type II when X is either connected or the Cantor set.

On the other hand, in [4], F.O. Farid and K. Varadarajan deepened the above classification and obtained a method for constructing codimension 1 linear isometries on $C(X)$. Recently, similar questions have been studied by M. Rajagopalan and K. Sundaresan in [12].

In this paper, we will first prove that if B is a linear subspace of codimension 1 of a function algebra on a locally compact Hausdorff space X, then B separates strongly the points of X except, at most, two. This result and the main theorems of [1] allow us to provide a classification of codimension 1 linear isometries on such algebras. We will devote the remainder of this paper to the study of this classification on function algebras in general and on Douglas algebras in particular.

2. Preliminaries

Let $\mathbb{K}$ denote the field of real or complex numbers. X will denote, unless otherwise specified, a locally compact Hausdorff space. We denote by $C_0(X)$ the Banach algebra of all $\mathbb{K}$-valued continuous functions defined on X which vanish at infinity, equipped with its usual supremum norm. If X is compact, we will write $C(X)$ instead of $C_0(X)$.

Let A be a linear subspace of $C_0(X)$. We will denote by σA the set of all $x_0 \in X$ such that, for each neighborhood U of x_0, there is a function f in A such that $|f(x)| < \|f\|$ for all $x \in X \setminus U$.

If it exists, we will denote by ∂A the *Shilov boundary* for A, that is, the *unique* minimal closed boundary for A. Let us recall that a subset Δ of X is a *boundary* for A if each function in A assumes its maximum on Δ. It is easy to check (see [1, Lemma 2.1]) that $\partial A = \sigma A$.

On the other hand, it is said that $x_0 \in X$ is a *strong boundary point* for A if, for each neighborhood U of x_0, there is a function f in A such that $|f(x_0)| = \|f\|$ and $|f(x)| < \|f\|$ for all $x \in X \setminus U$.

Let us recall that a *function algebra* A on X is a uniformly closed subalgebra of $C_0(X)$ which separates the points of X and vanishes identically at no point of X. It is well known (see, e.g., [11, Theorem 3.3.1]) that, for such an algebra, ∂A always exists. Furthermore, the set of strong boundary points for the function algebra A is included in ∂A. Indeed, it is dense in ∂A (see [2, Theorem 2]).

We will say that a linear subspace A of $C_0(X)$ *separates strongly* two elements of X, x_1 and x_2, if there exists $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$. It is a routine matter to verify that a function algebra on X separates strongly all the points of X (see [1, Theorem 6.1]).

3. Some properties of linear subspaces of codimension 1 of function algebras

Proposition 3.1. *Let A be a function algebra on X. If B is a linear subspace of A of codimension 1 in A, then B separates strongly the points of X, except at most two.*

Proof. Since A is a function algebra on X, it separates strongly the points of X, that is, given two elements of X, x_1 and x_2, there exists $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$. Suppose that $f(x_1) \neq 0$ and $f(x_2) \neq 0$. Let us define the following function in A:

$$g := \bar{f}^2 - \bar{f},$$

where $\bar{f} := f/f(x_1)$. Clearly, $g(x_1) = 0$ and $g(x_2) \neq 0$. Therefore, we can assume that, given $x_1, x_2 \in X$, there exists a function $g \in A$ such that $g(x_1) = 0$ and $g(x_2) = 1$ and vice versa.

Claim 1. Let us suppose that there exist two elements of X, x_1 and x_2, which cannot be separated strongly with functions of B. Let us consider a third element of X, x_3. We claim that there exists a function in B which separates strongly x_3 from x_1 and x_2.

To see this, let us consider three functions in A, f_1, f_2 and f_3, such that

$$f_1(x_1) = 1 \text{ and } f_1(x_i) = 0 \text{ for } i = 2, 3.$$

$$f_2(x_2) = 1 \text{ and } f_2(x_i) = 0 \text{ for } i = 1, 3.$$

$$f_3(x_3) = 1 \text{ and } f_3(x_i) = 0 \text{ for } i = 1, 2.$$

It is clear that such functions exist since, for example, there exist $h_2, h_3 \in A$ with $h_2(x_1) = 1$, $h_3(x_1) = 1$, $h_2(x_2) = 0$ and $h_3(x_3) = 0$. Hence it suffices to define $f_1 := h_2 \cdot h_3$.

On the other hand, it is clear that f_1 and f_2 do not belong to B and, if f_3 belongs to B, then we are done. So that we will also assume that $f_3 \notin B$. Hence, since the codimension of B in A is 1 and, clearly, the functions f_1, f_2 and f_3 are linearly independent, we deduce that there exist two non-zero constants, α_1 and α_2, such that $\alpha_1 \cdot f_1 + f_3$ and $\alpha_2 \cdot f_2 + f_3$ belong to B. Hence, since

$$(\alpha_1 \cdot f_1 + f_3)(x_3) = 1 \text{ and } (\alpha_1 \cdot f_1 + f_3)(x_2) = 0,$$

and

$$(\alpha_2 \cdot f_2 + f_3)(x_3) = 1 \text{ and } (\alpha_2 \cdot f_2 + f_3)(x_1) = 0,$$

we deduce that x_3 can be separated strongly from x_1 and x_2 with functions of B.

Claim 2. Let x_1 and x_2 be as in Claim 1. Consider now two elements of X, x_3 and x_4, which are distinct from x_1 and x_2. We claim that x_3 and x_4 can be separated strongly with functions of B.

To see this, let us consider three functions in A, g_1, g_3 and g_4, such that

$$g_1(x_1) = 1 \text{ and } g_1(x_i) = 0 \text{ for } i = 2, 3, 4.$$
$$g_3(x_3) = 1 \text{ and } g_3(x_i) = 0 \text{ for } i = 1, 2, 4.$$
$$g_4(x_4) = 1 \text{ and } g_4(x_i) = 0 \text{ for } i = 1, 2, 3.$$

If either g_3 or g_4 is in B, then we are done. Hence we will assume that these two functions do not belong to B. As f_1 above, g_1 does not belong to B either. Hence, since the codimension of B in A is 1 and, clearly, the functions g_1, g_3 and g_4 are linearly independent, we deduce that there exist two non-zero constants β_1 and β_2 such that $\beta_1 \cdot g_1 + g_3$ and $\beta_2 \cdot g_1 + g_4$ belong to B. Hence, since

$$(\beta_1 \cdot g_1 + g_3)(x_3) = 1 \text{ and } (\beta_1 \cdot g_1 + g_3)(x_4) = 0,$$

and

$$(\beta_2 \cdot g_1 + g_4)(x_4) = 1 \text{ and } (\beta_2 \cdot g_1 + g_4)(x_3) = 0,$$

we infer that x_3 and x_4 can be separated strongly with functions of B. □

Remark 3.2. Let A be a function algebra on (a locally compact space) X. If $f \in A$, let $\hat{f}$ be the restriction of f to ∂A. It is apparent that the mapping $f \longrightarrow \hat{f}$ is an isometric isomorphism of A into $C_0(\partial A)$. Hence, in the sequel, we will assume that A is a function algebra on its Shilov boundary ∂A.

Proposition 3.3. *Let A be a function algebra. If B is a linear subspace of A of codimension 1 in A and Δ is any nonempty closed boundary for B, then either $\Delta = \partial A$ or there exists a strong boundary point p for A such that $\Delta = \partial A \setminus \{p\}$.*

Proof. We will first prove that there can only be a strong boundary point for A outside Δ. To this end, let us suppose that there exist two strong boundary points, x_1 and x_2, for A which do not belong to Δ. Let V_1 be an open neighborhood of x_1 such that $V_1 \cap (\Delta \cup \{x_2\}) = \emptyset$ and let V_2 be an open neighborhood of x_2 such that $V_2 \cap (\Delta \cup \{x_1\}) = \emptyset$. Since x_1 and x_2 are strong boundary points for A, there exist two functions, f_1 and f_2, in A such that $f_i(x_i) = 1 = \|f_i\|$ and $|f_i(x)| < 1$ for all x off V_i (i=1,2).

Let $n \in \mathbb{N}$ such that $|f_i^n(x)| < \frac{1}{4}$ for every $x \notin V_i$ (i=1,2).

If either f_1^n or f_2^n belongs to B, then we easily deduce that Δ is not a boundary for B. Thus, let us assume that both $f_1^n \notin B$ and $f_2^n \notin B$. It is clear that f_1^n and f_2^n are linearly independent. Hence there exists, from the fact that B has codimension 1 in A, a nonzero constant α such that $\alpha \cdot f_1^n + f_2^n \in B$. As a consequence,

$$(\alpha \cdot f_1^n + f_2^n)(x_1) = \alpha + f_2^n(x_1),$$
$$(\alpha \cdot f_1^n + f_2^n)(x_2) = \alpha \cdot f_1^n(x_2) + 1,$$

$$(\alpha \cdot f_1^n + f_2^n)(x) = \alpha \cdot f_1^n(x) + f_2^n(x)$$

for all $x \in \Delta$.

If $|\alpha| \geq 1$, then, since $|f_i^n(x)| < \frac{1}{4}$ for every $x \in \Delta$ (i=1,2), we have that

$$\begin{aligned} |(\alpha \cdot f_1^n + f_2^n)(x_1)| &= |\alpha + f_2^n(x_1)| \\ &\geq |\alpha| - |f_2^n(x_1)| \\ &> |\alpha \cdot f_1^n(x)| + |f_2^n(x)| \\ &\geq |(\alpha \cdot f_1^n + f_2^n)(x)| \end{aligned}$$

for all $x \in \Delta$. On the other hand, if $|\alpha| < 1$, then

$$\begin{aligned} |(\alpha \cdot f_1^n + f_2^n)(x_2)| &= |\alpha \cdot f_1^n(x_2) + 1| \\ &\geq 1 - |\alpha \cdot f_1^n(x_2)| \\ &> |\alpha \cdot f_1^n(x)| + |f_2^n(x)| \\ &\geq |(\alpha \cdot f_1^n + f_2^n)(x)| \end{aligned}$$

for all $x \in \Delta$. That is, the function $(\alpha \cdot f_1^n + f_2^n) \in B$ attains its norm outside Δ, which contradicts the fact that Δ be a boundary for B. As a consequence, there can only be a strong boundary point for A outside Δ.

Finally, let us recall that the set of strong boundary points for A is dense in ∂A (see [2, Theorem 2]) and that Δ is a closed subset of ∂A. Consequently, since there can only be a strong boundary point for A outside Δ, if $\Delta \neq \partial A$, then $\Delta = \partial A \backslash \{p\}$, with p a strong boundary point for A. □

4. A classification of codimension 1 linear isometries on function algebras

The following theorem is a particular case of the main theorems (Theorems 3.1 and 4.1) of [1]:

Theorem $\mathcal{A}$. *Let A be a function algebra on ∂A and let $T : A \longrightarrow A$ be a linear isometry. Then there exist a closed boundary $(\partial A)_0 \subseteq \partial A$ for $T(A)$, a continuous map h of $(\partial A)_0$ onto ∂A and a continuous map $a : (\partial A)_0 \to \mathbb{K}$, such that $|a(x)| = 1$ for all $x \in (\partial A)_0$, and*

$$(Tf)(x) = a(x)f(h(x)) \text{ for all } x \in (\partial A)_0 \text{ and all } f \in A.$$

Furthermore, if $T(A)$ separates strongly the points of ∂A and vanishes identically at no point of ∂A, then $(\partial A)_0$ becomes the Shilov boundary for $T(A)$ ($(\partial A)_0 = \partial T(A) \setminus \{x_0\}$ if $T(A)$ vanishes identically at x_0) and h is a homeomorphism.

Now, Proposition 3.1 and Theorem $\mathcal{A}$ allow us to classify codimension 1 linear isometries T on function algebras into three types:

Type I. The range of T separates strongly the points of ∂A, except two of them.

Type II. The range of T separates strongly the points of ∂A and there exists an element $x_0 \in \partial A$ such that $f(x_0) = 0$ for all $f \in T(A)$.

Type III. The range of T separates strongly the points of ∂A and, for each $x \in \partial A$, there exists $f \in T(A)$ such that $f(x) \neq 0$.

5. Codimension 1 linear isometries of Type I

Let us recall here the definition of the closed boundary, $(\partial A)_0$, for $T(A)$, which appears in Theorem $\mathcal{A}$:

$$(\partial A)_0 := \bigcup_{x\in\partial A} V_x$$

where

$$V_x := \{x' \in \partial A : |(Tf)(x')| = |f(x)| \text{ for all } f \in A\}.$$

For any $x \in \partial A$, the set V_x is nonempty (see [1, Lemma 2.3]).

Theorem 5.1. *Let A be a function algebra and let $T : A \longrightarrow A$ be a codimension 1 linear isometry of Type I. Then*

$$(\partial A)_0 = \partial A.$$

Proof. Let x_1 and x_2 be the points of ∂A which cannot be separated strongly with functions of $T(A)$. Since $T(A)$ has codimension 1 in A and $(\partial A)_0$ is a closed boundary for $T(A)$, we have, by Proposition 3.3, two possibilities: either

$$(\partial A)_0 = \partial A$$

or there exists a strong boundary point p for A such that

$$(\partial A)_0 = \partial A \setminus \{p\}.$$

Let us suppose that we have the second case. We first claim that p is distinct from x_1 and x_2. Otherwise, if, for example, $x_1 = p$, then x_2 belongs to $(\partial A)_0$. That is, there exists a point $x_2' \in \partial A$ such that $x_2 \in V_{x_2'}$. Hence, since x_1 and x_2 cannot be separated strongly with functions of the range of T, we deduce that $x_1 \in V_{x_2'} \subseteq (\partial A)_0$, which is absurd since $x_1 = p$ does not belong to $(\partial A)_0$.

As in the proof of Proposition 3.1, there exists a function $f_1 \in A$ such that

$$f_1(x_1) = 1$$

and

$$f_1(x_2) = f_1(p) = 0.$$

Let us define $M := \|f_1\|$. Since p is a strong boundary point for A, there exists a function $g \in A$ such that

$$g(p) = 1 = \|g\|$$

and

$$|g(x)| < 1$$

for all $x \in (\partial A)_0$. It is clear that f_1 does not belong to $T(A)$. Also, the function g^n does not belong to $T(A)$ for any $n \in \mathbb{N}$ since $(\partial A)_0$ is a boundary for $T(A)$ and $p \notin (\partial A)_0$. Hence, as $T(A)$ has codimension 1 in A and the functions f_1 and g^n are linearly independent, there exists a nonzero constant α_n such that

$$\alpha_n \cdot f_1 + g^n \in T(A)$$

for all $n \in \mathbb{N}$. Since x_1 and x_2 cannot be separated strongly with functions of $T(A)$, we infer that

$$|\alpha_n \cdot f_1(x_1) + g^n(x_1)| = |\alpha_n \cdot f_1(x_2) + g^n(x_2)| = |g^n(x_2)|.$$

This equality implies, if we previously choose a suitable integer n, that $|\alpha_n| < \frac{1}{2M^2}$. Since furthermore, and with no loss of generality, we can choose the above mentioned integer n such that

$$|g^n(x)| < \frac{1}{4M^2}$$

for all $x \in (\partial A)_0$, it turns out that

$$(\alpha_n \cdot f_1 + g^n)(p) = 1$$

and

$$|\alpha_n \cdot f_1(x) + g^n(x)| < \frac{M}{2M^2} + \frac{1}{4M^2} < 1$$

for all $x \in (\partial A)_0$, which contradicts the fact that $(\partial A)_0$ is a boundary for $T(A)$. We are done. □

Example 5.2. Let $\mathbb{N}^*$ be the one point compactification of the natural numbers. Thus, the function algebra $C(\mathbb{N}^*)$ can be identified with the space of all convergent sequences of naturals. Let $T : C(\mathbb{N}^*) \longrightarrow C(\mathbb{N}^*)$ be the linear isometry

$$(x_1, x_2, ...) \longrightarrow (-x_1, x_1, x_2, ...).$$

It is easy to verify that the range of T has codimension 1 in $C(\mathbb{N}^*)$. Furthermore, $\mathbb{N}^* = \partial C(\mathbb{N}^*)$ and the integers 1 and 2 cannot be separated strongly by $T(C(\mathbb{N}^*))$. Hence, T is a codimension 1 linear isometry of Type I. Finally, it is also clear that $(\mathbb{N}^*)_0 = \mathbb{N}^*$.

Remark 5.3. In the example above, it can be easily checked that the subsets $\{1, 3, 4, ...\}$ and $\{2, 3, 4, ...\}$ of $\mathbb{N}^*$ are minimal closed boundaries of $T(C(\mathbb{N}^*))$. Hence, the Shilov boundary for $T(C(\mathbb{N}^*))$ does not exist. This fact shows that we cannot strengthen Theorem 5.1 to the effect $\partial A = \partial T(A)$.

6. Codimension 1 linear isometries of Type II

Theorem 6.1. *Let A be a function algebra and let $T : A \longrightarrow A$ be a codimension 1 linear isometry of Type II. Then*

$$\partial T(A) = \partial A \setminus \{x_0\},$$

where x_0 is isolated in ∂A and $f(x_0) = 0$ for all $f \in T(A)$.

Proof. By Proposition 3.3, and since $\partial T(A)$, which exists by [2, Theorem 1], is a closed boundary for $T(A)$, it suffices to check that $x_0 \notin \partial T(A)$, i.e., $x_0 = p$. Let us suppose that $x_0 \in \partial T(A)$. Then, by Theorem $\mathcal{A}$, we know that $(\partial A)_0 = \partial T(A) \setminus \{x_0\}$, but this is impossible since $(\partial A)_0$ is a closed boundary for $T(A)$ and $\partial T(A)$ is the minimal closed boundary for $T(A)$. □

Example 6.2. Let $T : C(\mathbb{N}^*) \longrightarrow C(\mathbb{N}^*)$ be the linear isometry

$$(x_1, x_2, ...) \longrightarrow (0, x_1, x_2, ...).$$

It is easy to verify that the range of T has codimension 1 in $C(\mathbb{N}^*)$ and that T is a codimension 1 linear isometry of Type II with $x_0 = 1$. It is also clear that $\partial T(C(\mathbb{N}^*)) = \mathbb{N}^* \setminus \{1\}$.

7. Codimension 1 linear isometries of Type III

For any codimension 1 linear isometry of Type III, we know, by Theorem $\mathcal{A}$, that $(\partial A)_0 = \partial T(A)$. However, as we will see in the examples below, we can have either of the possibilities of Proposition 3.3, that is,

$$\partial T(A) = \partial A \quad \text{or} \quad \partial T(A) = \partial A \setminus \{p\}.$$

In any case, Theorem $\mathcal{A}$ implies that $\partial T(A)$ is homeomorphic to ∂A.

Example 7.1. Let $T : C(\mathbb{N}^*) \longrightarrow C(\mathbb{N}^*)$ be the linear isometry

$$(x_1, x_2, ...) \longrightarrow \left(\frac{-(x_1 + x_2)}{2}, x_1, x_2, \dots \right).$$

It is easy to verify that the range of T has codimension 1 in $C(\mathbb{N}^*)$ and that T is a codimension 1 linear isometry of Type III. Furthermore, $\partial T(C(\mathbb{N}^*)) = \mathbb{N}^* \setminus \{1\}$ since there is not a function $f \in C(\mathbb{N}^*)$ such that

$$(Tf)(1) = 1 = \|Tf\|$$

and

$$|(Tf)(x)| < 1$$

for all x in $\mathbb{N}^* \setminus \{1\}$.

Example 7.2. It is well known that H^∞ (the space of bounded analytic functions on the open unit disk) is a function algebra. Furthermore, H_0^∞ (the maximal ideal consisting of all $f \in H^\infty$ that vanish at the origin) has codimension 1 in H^∞. Hence, the range of the linear isometry $T : H^\infty \longrightarrow H^\infty$ defined to be $T(f) := z \cdot f$ has codimension 1 in H^∞. Since $|z| \equiv 1$ on ∂H^∞ ([5, p. 194]), we have that $\partial H_0^\infty = \partial H^\infty$, and that the functions of H_0^∞ separate strongly the points of ∂H^∞. On the other hand, since for all $x \in \partial H^\infty$, there exists $f \in T(H^\infty) = H_0^\infty$ such that $f(x) \neq 0$, we infer that T is a codimension 1 linear isometry of Type III with $\partial H^\infty = \partial T(H^\infty)$.

8. Codimension 1 linear isometries on Douglas algebras

Let L^∞ be the algebra of essentially bounded measurable functions on the unit circle $\mathbb{T}$. A *Douglas algebra* is said to be any closed subalgebra of L^∞ containing H^∞. It is well known that the Shilov boundary for any Douglas algebra A is the maximal ideal space of L^∞, which implies that ∂A is extremally disconnected and has no isolated points.

Theorem 8.1. *Douglas algebras only admit, if any, codimension 1 linear isometries of type III.*

Proof. Suppose first that a Douglas algebra A admits a codimension 1 linear isometry of type I and let x_1 and x_2 be the points which cannot be separated. Then, by [1, Theorem 5.1] and by Theorem 5.1, we infer that ∂A is homeomorphic to the quotient space $\frac{\partial A}{\sim}$, where $\sim$ is the following equivalence relation: $x \sim y$ means either $x = y$ or $\{x, y\} = \{x_1, x_2\}$. This is a contradiction since it can be proved (see, e.g., [9, Corollary 2.5]) that $\frac{\partial A}{\sim}$ is not extremally disconnected.

On the other hand, suppose that a Douglas algebra A admits a codimension 1 linear isometry of type II and let x_0 be the point in ∂A such that $f(x_0) = 0$ for all $f \in T(A)$. By Theorem 6.1 we know that x_0 is an isolated point in ∂A, which is absurd. $\square$

Remark 8.2. It is apparent that Theorem 8.1 can be applied to any function algebra A included in L^∞ provided ∂A be the maximal ideal space of L^∞ (even though it does not contain the constant functions). For example, it can be applied to H_0^∞.

On the other hand, Example 7.2 shows that H^∞ does admit codimension 1 linear isometries. A similar example is valid for H_0^∞. However, it is not known (at least to the authors) whether the same is true for proper Douglas algebras. Indeed, the linear isometry of Example 7.2 turns out to be surjective if we replace H^∞ by the Douglas algebra $H^\infty + C(\mathbb{T})$, since z is invertible in $H^\infty + C(\mathbb{T})$ (see [5, p. 395]). We conjecture that proper Douglas algebras admit no codimension 1 linear isometries. This conjecture is supported by the fact that there are no such isometries on the full algebra L^∞, which can be easily deduced from Theorem 8.1 (see also [9, Corollary 2.5]).

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