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A FIXED POINT THEOREM AND ITS APPLICATION TO INTEGRAL EQUATIONS IN MODULAR FUNCTION SPACES

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Abstract. In this paper we present a fixed point theorem of Banach type in modular space. We give an application of this result to a nonlinear integral equation in Musielak-Orlicz space.

0. Introduction

It is well known that one of the standard proofs of Banach's fixed point theorem is based on Cantor's theorem in complete metric spaces [3, 4]. To this end, using some convenient constants in the contraction assumption, we present a generalization of Banach's fixed point theorem in some classes of modular spaces, where the modular is s-convex, having the Fatou property and satisfying the Δ_2-condition.

As an application we study the existence of a solution for an integral equation of Lipschitz type in a Musielak-Orlicz space.

We begin by recalling some basic concepts of modular spaces; for more information, we refer to the books by Musielak [8] and Kozlowski [7].

Definition 0-1. Let X be an arbitrary vector space over K $(= \mathbf{R}$ or $\mathbf{C})$.

a) A functionnal $\rho : X \to [0, +\infty]$ is called modular if:

i) $\rho(x) = 0 \iff x = 0$.

ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, $\forall x \in X$.

iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1, \forall x, y \in X$.

b) If iii) is replaced by:

iii′) $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ for $\alpha, \beta \geq 0, \alpha^s + \beta^s = 1$ with an $s \in]0, 1]$, then the modular ρ is called an s-convex modular; and if $s = 1$, ρ is called convex modular.

c) A modular ρ defines a corresponding modular space, i.e. the space X_ρ given by

$$X_\rho = \{x \in X | \rho(\lambda x) \to 0 \ \ as \ \lambda \to 0\}.$$

Remarks. 1) Note that in general there is no reason to expect the subadditivity of a modular ρ. Nevertheless, in view of iii) from Definition 0-1 the inequality $\rho(x + y) \leq \rho(2x) + \rho(2y)$ holds.

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2) If ρ is convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm defined by:

$$|x|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}.$$

3) As a classical example, we would like to mention the Musielak-Orlicz space denoted by L^φ [8] and the modular function space denoted by L^ρ [7].

Definition 0-2. Let X_ρ be a modular space.

a) A sequence $(x_n)_n$ in X_ρ is said to be:

i) ρ-convergent to x if $\rho(x_n - x) \to 0$ *as* $n \to +\infty$.

ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ *as* n and $m \to +\infty$.

b) $X\rho$ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_n \subset B$ with $x_n \to x$, then $x \in B$. $\bar{B}^\rho$ denotes the closure of B in the sense of ρ.

d) A subset $B \subset X_\rho$ is called ρ-bounded if:

$\delta_\rho(B) = \sup_{x,y\in B} \rho(x-y) < +\infty$, where $\delta_\rho(B)$ is called the ρ-diameter of B.

e) We say that ρ has the Fatou property if:

$$\rho(x-y) \leq \liminf \rho(x_n - y_n)$$

whenever

$$x_n \xrightarrow{\rho} x \text{ and } y_n \xrightarrow{\rho} y.$$

f) ρ is said to satisfy the Δ_2-condition if: $\rho(2x_n) \to 0$ *as* $n \to +\infty$ whenever $\rho(x_n) \to 0$ *as* $n \to +\infty$.

I. Fixed point theorem

I-1. Theorem I-1. *Let X_ρ be a ρ-complete modular space. Assume that ρ is an s-convex modular satisfying the Δ_2-condition and having the Fatou property. Let B be a ρ-closed subset of X_ρ and $T : B \to B$ a mapping such that:*

$$(*) \qquad \exists\ c, k \in \mathbf{R}^+ : c > max(1,k), \rho(c(Tx - Ty)) \leq k^s \rho(x-y)\ \forall\ x, y \in B.$$

Then T has a fixed point.

Remarks. 1) It is natural to introduce the constants c and k in the assumption of strict contraction in modular spaces. Note also that Theorem I-1 and its proof become more simple in the particular case where $s = 1$ (ρ is convex) and $c = 2 > k > 0$; see [1].

2) The contraction (*) in Theorem I-1 is also true for any constant c_0 such that $1 < c_0 \leq c$:

$$\begin{aligned}\rho(c_0(Tx - Ty)) &= \rho(\frac{c_0}{c}c(Tx - Ty)) \\ &\leq \frac{c_0^s}{c^s}\rho(c(Tx-Ty)) \leq k_0^s \rho(x-y)\end{aligned}$$

where $k_0 = \frac{c_0}{c}k < c_0$, since $\frac{k}{c} < 1$.

I-2. Proof of Theorem I-1. 1^{st} **step**: The proof of Theorem I-1 is based on the next result which is the modular formulation of Cantor's theorem:

Theorem I-2. *Let X_ρ be a ρ-complete modular space. Let $(F_n)_n$ be a decreasing sequence of nonempty, ρ-closed subset of X_ρ with $\delta_\rho(F_n) \to 0$ as $n \to +\infty$. Then $\bigcap_n F_n$ is reduced to one point.*

The proof of this result uses the same ideas as in complete metric spaces and it suffices to replace the distance by the modular ρ.

2^{nd} **step.** Let $(\epsilon_n)_n$ be a decreasing sequence of positive numbers such that $\epsilon_n \to 0$ *as* $n \to +\infty$, and consider the sets defined by:

$$M_{\epsilon_n} = \{x \in B | \rho(L(x - Tx)) \leq \epsilon_n\},$$

where $L = \max\{c, 2\alpha\}$ and α is the s-conjugate of c, i.e. $\frac{1}{c^s} + \frac{1}{\alpha^s} = 1$. Then the sequence (M_{ϵ_n}) has the following property:

1)$M_{\epsilon_n} \neq \emptyset, \forall n$.

We assume without any loss of generality that: $\exists x \in B$ such that $\rho(x - Tx) < +\infty$; then for $p \in \mathbf{N}^*$ we have:

$$\begin{aligned}\rho(c(T^{p+1}x - T^p x)) &\leq k^s \rho(T^p x - T^{p-1}x)\\ &= k^s \rho(\frac{1}{c} c(T^p x - T^{p-1}x))\\ &\leq (\frac{k^s}{c^s}) k^s \rho(T^{p-1}x - T^{p-2}x),\end{aligned}$$

and by induction we deduce:

$$\rho(c(T^{p+1}x - T^p x)) \leq (\frac{k}{c})^{s(p-1)} \rho(x - Tx).$$

Since $\frac{k}{c} < 1$, we have $\rho(c(T^{p+1}x - T^p x)) \to 0$ *as* $p \to +\infty$. Thus by (Δ_2)

$$\rho(L(T^{p+1}x - T^p x)) \to 0 \quad as\ p \to +\infty.$$

Hence: $\exists\ q \in \mathbf{N}^*$ such that $\rho(L(T^{q+1}x - T^q x)) \leq \epsilon_n$. Then: $y = T^q x \in M_{\epsilon_n}$.

2)M_{ϵ_n} is ρ-closed.

Let (x_p) be a sequence in M_{ϵ_n}. Assume that $(x_p)_p$ is ρ-convergent to $x \in X_\rho$, $(x_p)_p \subset B$ and B is ρ-closed; it follows that $x \in B$.

$\rho(x_p - x) \to 0$ *as* $p \to +\infty$, by (Δ_2), $\rho(L(x_p - x)) \to 0$ *as* $p \to +\infty$.

On the other hand $\rho(c(Tx_p - Tx)) \leq k^s \rho(x_p - x) \leq k^s \rho(L(x_p - x))$; then

$$\rho(c(Tx_p - Tx)) \to 0 \ as\ p \to +\infty.$$

Again by (Δ_2) $\rho(L(Tx_p - Tx)) \to 0$ *as* $p \to +\infty$. The Fatou property implies that:

$$\rho(L(Tx - x)) \leq \liminf \rho(L(Tx_p - x_p)) \leq \epsilon_n.$$

Therefore $x \in M_{\epsilon_n}$ and hence M_{ϵ_n} is ρ-closed.

3) $\delta_\rho(M_{\epsilon_n}) \to 0$ *as* $n \to +\infty$.

Let $x, y \in M_{\epsilon_n}$; we have:

$$\rho(x - y) = \rho(\frac{\alpha(x - Tx)}{\alpha} + \frac{c(Tx - Ty)}{c} + \frac{\alpha(Ty - y)}{\alpha})$$

where

$$\begin{aligned}\alpha^s &= \frac{c^s}{c^s-1}\\ &\le \frac{1}{\alpha^s}\rho(\alpha(x-Tx)+\alpha(Ty-y))+\frac{1}{c^s}\rho(c(Tx-Ty))\\ &\le \frac{1}{2^s\alpha^s}\rho(2\alpha(x-Tx))+\frac{1}{2^s\alpha^s}\rho(2\alpha(Ty-y))+\frac{1}{c^s}\rho(c(Tx-Ty))\\ &\le \frac{1}{2^s\alpha^s}(\rho(L(x-Tx))+\rho(L(Ty-y)))+\frac{1}{c^s}\rho(c(Tx-Ty)).\end{aligned}$$

Then

$$\rho(x-y)\le \frac{1}{2^s\alpha^s}2\epsilon_n+\frac{k^s}{c^s}\rho(x-y).$$

Hence,

$$\rho(x-y)\le \frac{1}{2^s\alpha^s}2\epsilon_n\ \frac{c^s}{c^s-k^s}$$

and we deduce that

$$\delta_\rho(M_{\epsilon_n})=\sup_{x,y\in M_{\epsilon_n}}\rho(x-y)\le \epsilon_n\ 2^{1-s}\frac{c^s-1}{c^s-k^s},$$

and

$$\delta_\rho(M_{\epsilon_n})\to 0 \text{ as } n\to+\infty.$$

4) (M_{ϵ_n}) is decreasing. This is an immediate consequence of the fact that (ϵ_n) is decreasing.

It follows that the conditions of Cantor's theorem are satisfied and hence we have: $\bigcap_n M_{\epsilon_n}=\{x_0\}$.

But $x_0\in M_{\epsilon_n}\forall n \Rightarrow \rho(L(x_0-Tx_0))\le\epsilon_n\to 0$ as $n\to+\infty\Rightarrow Tx_0=x_0$.

Remark. We are unable to prove whether the conclusion of Theorem I-1 is true if we have $c=1$ and $0<k<1$. To this end, recall the following results by Khamsi-Kozlowski-Reich:

Theorem I-3 ([5])**.** *Let ρ be a modular function satisfying the Δ_2-condition and let B be a $|.|_\rho$-closed subset of L_ρ.*

$T:B\to B$ is a mapping such that: $\exists\ 0<k<1$ such that

$$\rho(Tf-Tg)\le k\rho(f-g)\quad \forall f,g\in B.$$

Then: T has a fixed point if one of the following assumptions is satisfied:
i) $\exists\ f_0\in B;\sup_n(2T^nf_0)<\infty$.
ii) B is ρ-bounded.

If *ii*) is satisfied, the fixed point is unique. Note also that the modular ρ in Theorem I-3 has the Fatou property and the Δ_2-condition as in Theorem I-1. Also by (Δ_2) we have: B is ρ-closed $\Leftrightarrow$ B is $|.|_\rho$-closed.

On the other hand the strict inequality (*) in Theorem I-1 implies the inequality of Theorem I-3

$$\rho(Tx-Ty)=\rho(\frac{1}{c}(c(Tx-Ty)))\le\frac{1}{c^s}\rho(c(Tx-Ty))\le\frac{k^s}{c^s}\rho(x-y).$$

Consequently with some reinforced assumptions in Theorem I-1, namely the s-convexity of ρ and the strict contraction (*), we prove the existence of a fixed point for T without restrictive conditions concerning the domain of T: i) or ii) in Theorem I-3 .

The following example shows that Theorem I-1 can be more appropriate for applications.

II. Application

II-1. General frame. Consider the following integral equation:

$$u(t) = \exp(-t)f + \int_0^t \exp(s-t)Tu(s)ds, \tag{I}$$

where:

i) $T : B \to B$ with B a ρ-closed, convex subset of a Musielak-Orlicz space L^φ satisfying the Δ_2-condition.

ii) T is ρ-Lipschitz:

$$\exists\ \gamma > 0, \rho(Tu - Tv) \leq \gamma\rho(u-v)\ u, v \in B.$$

iii) $f \in B$.

Theorem II-1. *Under these conditions, for all $A > 0$ the integral equation (I) has a solution $u \in C^\varphi = C([0,A], L^\varphi)$. C^φ is the modular space of continuous mappings from $[0,A]$ into L^φ*

By iterative techniques, Khamsi [6] has shown this result under supplementary conditions: B is ρ-bounded and T is ρ-Lipschitz with constant $\gamma = 1$.

To delete all restrictive assumptions on the Lipschitz constant γ we introduce the space C^φ equipped with a convenient modular.

Our method follows the standard technique of the resolution of differential equations or integral equations of Lipschitz type in Banach spaces. See Deimling [2, p. 39].

II-2. Functional frame: The modular space C^φ.

Definition 2-1. $u : I \to L^\varphi$, where $I = [0,A]$, is said to be continuous at $t_0 \in I$ if: for $t_n \in I,\ t_n \to t_0$ *as* $n \to +\infty \Rightarrow \rho(u(t_n) - u(t_0)) \to 0$ *as* $n \to +\infty$.

Since ρ satisfies the Δ_2-condition, it is equivalent to:

$$t_n \to t_0 \ as\ n \to +\infty \Rightarrow |u(t_n) - u(t_0)|_\rho \to 0 \ as\ n \to +\infty.$$

Let $C^\varphi = C(I, L^\varphi)$ be the space of all continuons mappings from $I = [0,A]$ into L^φ.

Proposition 2-1. *Suppose that the modular ρ satisfies (Δ_2), and $B \subset L^\varphi$ is a ρ-closed, convex subset of L^φ. For $a \geq 0$ let $\rho_a(u) = \sup_{t\in I} \exp(-at)\rho(u(t))$ for $u \in C^\varphi$. Then:*

1) (C^φ, ρ_a) is a modular space, and ρ_a is a convex modular satisfying the Fatou property and the Δ_2-condition.

2) C^φ is ρ_a-complete.

3) $C_0^\varphi = C(I,B)$ is a ρ_a-closed, convex subset of C^φ.

Proof. 1) i) C^φ is a real vector space.

Let $u, v \in C^\varphi, t_0 \in I$. Then for $t_n \in I$ such that $t_n \to t_0$ *as* $n \to +\infty$ we have:

$$\rho(\frac{(u+v)(t_n) - (u+v)(t_0)}{2}) \leq \frac{1}{2}\rho(u(t_n) - u(t_0)) + \frac{1}{2}\rho(v(t_n) - v(t_0)).$$

Hence, $\rho(\frac{(u+v)(t_n)-(u+v)(t_0)}{2}) \to 0 \ as \ n \to +\infty$. And by Δ_2:

$$\rho((u+v)(t_n)-(u+v)(t_0)) \to 0 \ as \ n \to +\infty,$$

which implies that $(u+v)$ is continuous at t_0.

Again by Δ_2 $\rho(\lambda(u(t_n)-u(t_0))) \to 0 \ as \ n \to +\infty$ for all $\lambda \in \mathbf{R}$; then λu is also continuous at t_0.

ii) ρ_a is well defined.

Since ρ satisfies the (Δ_2)-condition, the domain of ρ is $\{f \in L^\varphi, \rho(f) < +\infty\} = L^\varphi$, and since ρ is convex and $|.|_\rho$-continuous at 0 it follows that ρ is $|.|_\rho$-continuous on L^φ. See Zeidler [9, p. 383].

Consequently for all $u \in C^\varphi$, $\rho_a(u)$ has a meaning.

iii) ρ_a is a convex modular.

This is a simple consequence of the fact that ρ is a convex modular.

iv) ρ_a satisfies the Fatou property.

Let $(u_n)_n$ (resp $(v_n)_n$) be a sequence in C^φ, ρ_a-convergent to u (resp. to v) $\in C^\varphi$. Then $\forall t \in I,\ \rho(u_n(t)-u(t)) \to 0 \ as \ n \to +\infty$ and $\rho(v_n(t)-v(t)) \to 0 \ as \ n \to +\infty$.

Since ρ satisfies the Fatou property, we have:

$$\rho(u(t)-v(t)) \leq \liminf \rho(u_n(t)-v_n(t)) \ \forall \ t \in \ I,$$

$$\exp(-at)\rho(u(t)-v(t)) \leq \liminf \exp(-at)\rho(u_n(t)-v_n(t)) \leq \liminf \rho_a(u_n - v_n)$$

and then

$$\rho_a(u-v) \leq \liminf \rho_a(u_n - v_n).$$

v) ρ_a has the (Δ_2)-condition.

Since ρ satisfies the (Δ_2)-condition, one has:

$$\begin{aligned}\rho_a(u_n) \to 0 \ as \ n \to +\infty &\Leftrightarrow \forall t \in I,\ \exp(-at)\rho(u_n(t)) \to 0 \ as \ n \to +\infty \\ &\Leftrightarrow \forall t \in I,\ \exp(-at)\rho(2u_n(t)) \to 0 \ as \ n \to +\infty \\ &\Leftrightarrow \rho_a(2u_n) \to 0 \ as \ n \to +\infty.\end{aligned}$$

2) It is known that $(L^\varphi, |.|_\rho)$ is a Banach space. Then the space $(C^\varphi, |.|_{\rho_a})$ is also a Banach space. If $(u_n)_n$ is a ρ_a-Cauchy sequence in C^φ , by the Δ_2-condition it is $|.|_{\rho_a}$-Cauchy. Hence: $\exists \ u \in C^\varphi$ such that $u_n \to u$ in $(C^\varphi, |.|_{\rho_a})$.

Consequently $(u_n) \to u$ in (C^φ, ρ_a). It follows that C^φ is ρ_a-complete.

3) $C_0^\varphi = C(I,B)$ is convex and ρ_a-closed.

The convexity of C_0^ϕ and its ρ_a-closedness are clearly obtained by the convexity and ρ-closedness of B in L^ϕ.

II-3. Proof of Theorem II-1. Define the operator S over C_0^φ by:

$$\forall u \in C_0^\varphi, \quad Su(t) = \exp(-t)\ f + \int_0^t \exp(s-t)\ Tu(s)ds, \quad \forall t \in I.$$

1^{st} **step:** We show that $S : C_0^\varphi \to C_0^\varphi$.

i) Su is continuous from I into $(L^\varphi, |.|_\rho)$.

Let $t_n, t_0 \in I$ with $t_n \to t_0 \ as \ n \to +\infty$.

T is ρ-Lipschitz $\rho(Tu(t_n)-Tu(t_0)) \ \leq \ \gamma\rho(u(t_n)-u(t_0))$. Since u is ρ-continuous, Tu is ρ-continuous at t_0 and by (Δ_2), Tu is $|.|_\rho$-continuous at t_0. Hence Su is $|.|_\rho$-continuous at t_0.

ii) $Su(t) \in B, \quad \forall t \in I$.

It is well known that in the Banach space $(L^{\varphi}, |.|_{\rho})$

$$\int_0^t \exp(s-t)\ Tu(s)ds \in (\int_0^t \exp(s-t)ds)\ \overline{co}\{Tu(s), 0\ \leq\ s\ \leq\ t\}$$
$$\in\ (1-\exp(-t))\ \overline{co}B,$$

where $\overline{co}B$ is the closed convex hull of B in $(L^{\varphi}, |.|_{\rho})$. But B is convex and ρ-closed; thus, $\overline{co}B = \overline{B} \subset \overline{B^{\rho}} = B$. Hence, $Su(t) \in \exp(-t)\ B + (1-\exp(-t))\ B \subseteq B \quad \forall t \in I$.

2^{nd} **step:** For $u, v \in C_0^{\varphi}$ and $\lambda > 0$ we have:

$$\lambda(Su(t) - Sv(t)) = \int_0^t \lambda \exp(s-t)\ (Tu(s) - Tv(s))ds.$$

Lemma 3-1 ([6]). *Let* $x \in C^{\varphi}$ *and* $0 < \lambda\ \leq\ \frac{\exp A}{\exp(A)-1}$. *Then:*

$$\rho(\int_0^t \lambda \exp(s-t)\ x(s)ds)\ \ \leq\ \ \lambda \frac{\exp(at) - \exp(-t)}{1+a} \rho_a(x).$$

Note that in [6] this result was been shown with $\lambda = 1$ and $a = 0$.

Proof. Let $T = \{t_0, t_1, t_2, \ldots, t_n\}$ be any subdivision of $[0, t]$.

$$\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t)\ x(t_i)$$

is $|.|_{\rho}$-convergent, and consequently, ρ-convergent to $\int_0^t \lambda \exp(s-t)\ x(s)ds$ in L^{φ} when $|T| = \sup\{|t_{i+1} - t_i|, i = 0, ..., n-1\} \to 0\ as\ n \to +\infty$.

By Fatou we have:

$$\rho(\int_0^t \lambda \exp(s-t)\ x(s)ds)\ \ \leq\ \ \liminf \rho(\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t) x(t_i)).$$

On the other hand:

$$\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t) \leq \int_0^t \lambda \exp(s-t)ds$$
$$= (1 - \exp(-t))\lambda\ \ \leq\ \ \lambda(1 - \exp(-A)).$$

Since $0 < \lambda < \frac{\exp(A)}{\exp(A)-1}$, it follows that $\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t) \leq 1$, and the convexity of ρ implies:

$$\rho(\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t) x(t_i))\ \ \leq\ \ \sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t) \rho(x(t_i))$$
$$= \sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp(t_i - t) \exp(at_i) \exp(-at_i) \rho(x(t_i))$$
$$\leq\ \sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) \exp((1+a)t_i - t) \rho_a(x)$$
$$\leq\ (\int_0^t \lambda \exp((1+a)s - t)\ \ ds)\ \rho_a(x).$$

Then:

$$\rho(\int_0^t \lambda \exp(s-t)\ x(s)ds) \ \leq \ \lambda \frac{\exp(at) - \exp(-t)}{1+a} \rho_a(x).$$

3^{rd} **step:**

$$\rho(\lambda(Su(t) - Sv(t))) \ \leq \ \lambda \frac{\exp(at) - \exp(-t)}{1+a} \rho_a(Tu - Tv).$$

But

$$\rho_a(Tu - Tv) = \sup_{t \in I} \ \exp(-at)\rho(Tu(t) - Tv(t)) \ \leq \ \gamma \rho_a(u - v).$$

Then:

$$\begin{aligned} \exp(-at)\ \rho(\lambda(Su(t) - Sv(t)) &\leq \ \lambda \frac{\gamma}{1+a}(1 - \exp((-t)(1+a)))\ \rho_a(u-v) \\ &\leq \ \lambda \frac{\gamma}{1+a}(1 - \exp((-A)(1+a)))\rho_a(u-v), \ \forall t \in I. \end{aligned}$$

Hence

$$\rho_a(\lambda(Su - Sv)) \ \leq \ \lambda \frac{\gamma}{1+a}(1 - \exp((-A)(1+a)))\ \rho_a(u-v).$$

We consider $\lambda, 1 < \lambda \ \leq \ \frac{\exp(A)}{\exp(A) - 1}$. Then S has a fixed point if:

$$\lambda \frac{\gamma}{1+a}(1 - \exp((-A)(1+a))) < \lambda \Leftrightarrow \frac{\gamma}{1+a}(1 - \exp((-A)(1+a))) < 1.$$

The last inequality is satisfied if we take for example $a \geq \gamma$.

In the end, by Theorem I-1, S has a fixed point which is a solution of the integral equation (I).

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