PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 8, Pages 2325–2333
S 0002-9939(99)04793-0
Article electronically published on April 9, 1999

FINITE-DIMENSIONAL LEFT IDEALS IN SOME ALGEBRAS ASSOCIATED WITH A LOCALLY COMPACT GROUP

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(Communicated by Palle E. T. Jorgensen)

Abstract. Let G be a locally compact group, let $L^1(G)$ be its group algebra, let $M(G)$ be its usual measure algebra, let $L^1(G)^{**}$ be the second dual of $L^1(G)$ with an Arens product, and let $LUC(G)^*$ be the conjugate of the space $LUC(G)$ of bounded, left uniformly continuous, complex-valued functions on G with an Arens-type product. We find all the finite-dimensional left ideals of these algebras. We deduce that such ideals exist in $L^1(G)$ and $M(G)$ if and only if G is compact, and in $L^1(G)^{**}$ (except those generated by right annihilators of $L^1(G)^{**}$) and $LUC(G)^*$ if and only if G is amenable.

1. Introduction

Let G be a locally compact group, $L^1(G)$ be its group algebra, and $M(G)$ be its usual measure algebra. Other Banach algebras (usually larger than $L^1(G)$ and $M(G)$) can also be associated with G. For instance, the second dual $L^1(G)^{**}$ of $L^1(G)$ is a Banach algebra with an Arens product. One can also consider the space $LUC(G)$ of the bounded left uniformly continuous functions on G, or the space $WAP(G)$ of the weakly almost periodic functions on G. An Arens-type product can be introduced into the conjugate of each of these spaces of functions and make them into Banach algebras. Let $\mathcal{A}$ denote any of these algebras. Our main concern in this paper is with the finite-dimensional left ideals in $\mathcal{A}$. We start in Theorem 1 by giving examples of such ideals. These examples are obtained with the help of the U-invariant elements of $\mathcal{A}^n$ where U is a continuous and bounded representation of G on $\mathbb{C}^n$. This notion has already been introduced in earlier papers with $n = 1$. In [3], the so-called χ-invariant elements (where χ is a character of G) were introduced in order to determine the minimal left ideals of these algebras when G is abelian. In [4], these type of elements were referred to as λ-invariant elements with $\lambda \in \mathbb{T}$, and were used to solve some linear equations in $\ell^\infty(\mathbb{Z})^*$, then to determine the finite-dimensional left ideals in this case. In [5], the U-invariance was used to study the minimal ideals in these algebras. In Theorems 2 and 3, we determine all the finite-dimensional left ideals in $\mathcal{A}$ and show that they are in general of the form given in Theorem 1. This is a generalization of the result obtained in [4, 2.7(a)]. We deduce in the corollary which follows that such ideals exist in $LUC(G)^*$ and in $L^1(G)^{**}$ (apart from those generated by right annihilators of $L^1(G)^{**}$) if and only

Received by the editors October 28, 1997.
1991 *Mathematics Subject Classification.* Primary 43A10, 22D15.
Key words and phrases. Locally compact group, Arens product, representation, amenable, U-invariant, finite-dimensional left ideal.

if G is amenable. In $L^1(G)$ and $M(G)$, the ideals of finite-dimension exist if and only if G is compact.

2. Preliminaries

Let G be a locally compact group with a left Haar measure λ, and a Haar modular function Δ. For measurable functions f and ϕ on G and for $s \in G$, we write

$$f * \phi(s) = \int_G f(t)\phi(t^{-1}s)\, d\lambda(t)$$

whenever the integral exists. We shall be concerned with the following Banach algebras which are related to G. We begin by recalling, of course, the most intimate ones: the group algebra $L^1(G)$ and the measure algebra $M(G)$. The first one is the Banach algebra of all measurable complex-valued functions ϕ on G satisfying

$$\int_G |\phi(s)|\, d\lambda(s) < \infty.$$

The product of two elements ϕ and ψ in $L^1(G)$ is $\phi * \psi$. The second one is the Banach algebra of all bounded, regular, Borel measures on G. By the Riesz representation theorem (see [8, Chapter 3]), we shall regard a measure in $M(G)$ as an element of $C_0(G)^*$, where $C_0(G)$ is the space of continuous complex-valued functions on G vanishing at infinity. The product of two elements μ and ν of $M(G)$ is given then by

$$(\mu\,\nu)(f) = \int_G \int_G f(s\,t)\, d\mu(s) d\nu(t) = \int_G \int_G f(s\,t)\, d\nu(t) d\mu(s) \quad \text{for} \quad f \in C_0(G).$$

Furthermore, for each $\phi \in L^1(G)$, we define $\lambda_\phi \in M(G)$ by

$$\lambda_\phi(f) = \int_G f(s)\phi(s)\, d\lambda(s) \quad \text{for} \quad f \in C_0(G).$$

Then under the map $\phi \to \lambda_\phi$, we will regard $L^1(G)$ as a subalgebra of $M(G)$. In fact, $L^1(G)$ is a closed two-sided ideal of $M(G)$ (see for example [8, Theorem 19.18]). The other algebras which we shall consider are defined in the following way. Let $L^\infty(G)$ be the Banach space of all measurable complex-valued functions that are bounded almost everywhere with respect to λ, and for each $\phi \in L^\infty(G)$, let $\hat{\phi}$ be the function defined on G by $\hat{\phi}(s) = \Delta(s^{-1})\phi(s^{-1})$. The Banach space $L^1(G)^{**}$ becomes a Banach algebra with the first Arens product. This product is obtained by first letting

$$f_\nu(\phi) = \nu(\hat{\phi} * f) \quad \text{for all} \quad \nu \in L^1(G)^{**},\ f \in L^\infty(G) \text{ and } \phi \in L^1(G).$$

Then, for μ and ν in $L^1(G)^{**}$,

$$(\mu\,\nu)(f) = \mu(f_\nu) \quad \text{for all} \quad f \in L^\infty(G).$$

Let $C(G)$ denote the space of all bounded, complex-valued, continuous functions on G. The left translate of a function f on G by $s \in G$ is defined by $f_s(t) = f(s\,t)$ for all $t \in G$. Let $LUC(G)$ be the space of left uniformly continuous functions in $C(G)$, i.e.,

$$LUC(G) = \{f \in C(G) : s \mapsto f_s : G \to C(G) \text{ is norm continuous}\}.$$

Then $LUC(G)^*$ is also a Banach algebra under the product

$$(\mu\,\nu)(f) = \mu(f_\nu) \quad \text{for all} \quad f \in LUC(G), \quad \text{where}$$
$$f_\nu(s) = \nu(f_s) \quad \text{for all} \quad s \in G$$

(the function f_ν is easily seen to be in $LUC(G)$). Note that the product in $M(G)$ (and so in $L^1(G)$) is defined in the same way with $C_0(G)$ instead of $LUC(G)$.

More generally, one can start with a norm closed, conjugate closed subspace F of $C(G)$ containing the constant functions and having the property that the functions f_s and f_μ are in F for all $f \in F$, $s \in G$ and $\mu \in F^*$ (the functions f_s and f_μ are defined as earlier in $LUC(G)$). Following [2, Definition 2.2.10], such an F is said to be *admissible*; the space F^* also becomes a Banach algebra under the product

$$(\mu\,\nu)(f) = \mu(f_\nu) \quad \text{for all} \quad f \in F.$$

For more details, the reader is directed to [2, pages 72-78]. As we have already seen, $LUC(G)$ is admissible. Other examples are the space $WAP(G)$ of weakly almost periodic functions on G, and the space $AP(G)$ of almost periodic functions on G. These spaces are

$$WAP(G) = \{f \in C(G) : f_G \text{ is weakly relatively compact}\},$$
$$AP(G) = \{f \in C(G) : f_G \text{ is norm relatively compact}\}, \text{ where}$$
$$f_G = \{f_s : s \in G\}.$$

Let $\mathcal{A}$ denote each of the Banach algebras $L^1(G)$, $M(G)$, $L^1(G)^{**}$, and F^*, where F is an admissible subspace of $C(G)$ with $AP(G)F \subseteq F$. Apart from $L^1(G)$, the algebra $\mathcal{A}$ is the dual of some Banach space of functions of G, which we shall denote by $\mathcal{F}$. In the case of $L^1(G)$, we let $\mathcal{F} = C_0(G)$. When $\mathcal{A} = F^*$ or $M(G)$, the group G may be embedded continuously into $\mathcal{A}$ by the mapping $e : G \to \mathcal{A}$ which is defined by

$$e(s)(f) = f(s) \quad \text{for all} \quad s \in G \text{ and } f \in \mathcal{F}.$$

We recall that an element μ of $\mathcal{A}$ is *left invariant* if

$$\mu(f_s) = \mu(f) \text{ for all } f \in \mathcal{F} \text{ and } s \in G.$$

When $L^1(G) * \mathcal{F} \subseteq \mathcal{F}$, we say that $\mu \in \mathcal{A}$ is *topologically left invariant* if

$$\mu(\phi * f) = \left(\int_G \phi(s)\, d\lambda(s)\right) \mu(f) \text{ for all } \phi \in L^1(G) \text{ and } f \in \mathcal{F}.$$

The notions of left invariance and topological left invariance are equivalent when $\mathcal{F} \subseteq LUC(G)$. But this is not so if $\mathcal{F} = L^\infty(G)$. We say that $\mathcal{F}$ is amenable if there is a non-zero left invariant (or equivalently a topologically left invariant) element in $\mathcal{A}$. When G is a compact topological group, $C_0(G) = C(G)$ is amenable since $\lambda \in C(G)^*$. The spaces $WAP(G)$ and $AP(G)$ are always amenable. But this is not so for $LUC(G)$ and $L^\infty(G)$; for example, when G is the free group on two generators (see [2, Example 3.4(e)]). So we say that the group G is *amenable* if $L^\infty(G)$, or equivalently $LUC(G)$, is amenable. See [2] or [7].

We shall also need representations of G on $\mathbb{C}^n$. Recall that a *representation* of G on a Hilbert space H is a homomorphism of G into the semigroup of bounded operators on H. We say that U is *continuous* when the function $s \mapsto U(s)\bar{x}$ is continuous on G for each $\bar{x} \in H$. We say that U is *irreducible* when $\{0\}$ and H are the only invariant (closed) subspaces under all $U(s)$, i.e., there is no (closed) subspace E other than $\{0\}$ and H satisfying $U(s)E \subseteq E$ for all $s \in G$. We also recall that there is a one-to-one correspondence between the representations of $L^1(G)$ on H and those of G on H. This is given by the formula

$$\langle U(\phi)\bar{x}, \bar{y}\rangle = \int_G \langle U(s)\bar{x}, \bar{y}\rangle\, \phi(s)\, d\lambda(s) \quad \phi \in L^1(G),\ \bar{x},\ \bar{y} \in H;$$

see [8, Section 22]. Since we shall be concerned solely with representations U on $H = \mathbb{C}^n$, we fix a basis $\{\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\}$ for $\mathbb{C}^n$, and correspond to U the matrix representation $U = (u_{ij})_{i,\, j=1}^n$, where u_{ij} are the coordinate functions defined on G by

$$u_{ij}(s) = \langle U(s)\bar{x}_i, \bar{y}_j\rangle.$$

Note that the letter U is used to denote the representations of G, $L^1(G)$, and their corresponding matrices, but we promise the reader that this will cause no confusion.

3. Left ideals of finite-dimension

The notion of χ-invariance, where χ is a continuous character of G was introduced in [3] to determine the minimal left ideals of $\mathcal{A}$ when G is abelian. In this section, we generalize this notion by defining the U-invariant vectors of $\mathcal{A}^n$, where U is a representation of G on $\mathbb{C}^n$. The U-invariance is essential to determine the finite-dimensional left ideals of these algebras. Some of the arguments in Theorems 1 and 2 have already been in [1]; they are, however, simplified here with the help of the U-invariance. Before we state our main definition, we introduce the following notations.

Notations. Let $\mu \in \mathcal{A}$, $f \in \mathcal{F}$, $\bar{\mu} = (\mu_i)_{i=1}^n$ be a column vector in $\mathcal{A}^n$, and A be an $n \times n$ matrix with entries a_{ij} $(i, j = 1, ..., n)$ in $\mathcal{F}$. Then we write $\mu(A) = (\mu(a_{ij}))_{i,j=1}^n$, $\bar{\mu}(f)$ is the column vector $(\mu_i(f))_{i=1}^n$ of $\mathbb{C}^n$, $(A)\bar{\mu}$ is the column vector $\left(\sum_{j=1}^n \mu_j(a_{ij})\right)_{i=1}^n$ of $\mathbb{C}^n$ (note that this is obtained with the matrix multiplication relative to the product given by the duality between $\mathcal{A}$ and $\mathcal{F}$), and $\mu\,\bar{\mu}$ is the column vector $(\mu\,\mu_i)_{i=1}^n$ of $\mathcal{A}^n$. Furthermore, when A and B are $n \times n$ matrices whose entries are measurable functions on G then, whenever the integrals exist, we write

$$A \cdot B = \int_G A(t)B(t)\, d\lambda(t),$$
$$A * B(s) = \int_G A(t)B(t^{-1}s)\, d\lambda(t),$$
$$A \tilde{*} B(s) = \int_G B(t^{-1}s)A(t)\, d\lambda(t).$$

Definition. We say that a vector $\bar{\mu}$ of $\mathcal{A}^n$ is U-invariant if there exists a continuous and bounded representation U of G on $\mathbb{C}^n$ such that

$$\bar{\mu}(f_s) = U(s^{-1})\bar{\mu}(f) \quad \text{for each} \quad f \in \mathcal{F} \text{ and } s \in G.$$

When $\mathcal{A} = L^1(G)^{**}$, we say that $\bar{\mu}$ be topologically U-invariant if

$$\bar{\mu}(\phi * f) = U(\phi)\bar{\mu}(f) \quad \text{for each} \quad f \in L^\infty(G) \text{ and } \phi \in L^1(G).$$

Let $\widetilde{U}$ be defined on G by $\widetilde{U}(s) = U(s^{-1})$. A direct computation leads to the following lemma. We omit the proof.

Lemma 1. *Let A be an $n \times n$ matrix with entries in $\mathcal{F}$, and for $s \in G$, let A_s be the matrix whose entries are the left translates of those of A by s. Let Φ be an $n \times n$ matrix with entries in $L^1(G)$.*

1. *If $\bar{\mu} \in \mathcal{A}^n$ is U-invariant, then $(A_s)\bar{\mu} = \left(A\widetilde{U}(s)\right)\bar{\mu}$.*
2. *If $\mathcal{A} = L^1(G)^{**}$ and $\bar{\mu} \in \mathcal{A}^n$ is topologically U-invariant, then $(\Phi * A)\bar{\mu} = (\Phi \cdot A(.)U)\,\bar{\mu}$ and $(\Phi \tilde{*} A)\bar{\mu} = (A(\Phi \cdot U))\,\bar{\mu}$, where $\Phi \cdot A(.)U$ is the matrix-valued function defined on G by $\Phi \cdot A(s)U = \int_G \Phi(t)A(s)U(t)\,d\lambda(t)$.*

Lemma 2. *Let U be a continuous and bounded representation of G on $\mathbb{C}^n$, and let I be the identity representation. Let $\bar{\mu}$ and $\bar{\nu}$ be in $\mathcal{A}^n$ such that $\bar{\mu}(f) = (f\,U)\bar{\nu}$ for all $f \in \mathcal{F}$; or equivalently, $\bar{\nu}(f) = (f\,\widetilde{U})\bar{\mu}$ for all $f \in \mathcal{F}$. Then*

1. *$\bar{\nu}$ is I-invariant if and only if $\bar{\mu}$ is U-invariant,*
2. *when $\mathcal{A} = L^1(G)^{**}$, $\bar{\nu}$ is topologically I-invariant if and only if $\bar{\mu}$ is topologically U-invariant.*

Remark. Observe that $(f\,U)\bar{\nu}$ and $(f\,\widetilde{U})\bar{\mu}$ are well defined in the lemma. This is due to the fact that the coordinate functions of U and $\widetilde{U}$ are almost periodic (which is easy to verify, or see [1, Lemma 1]) and the extra assumption that $AP(G)F \subseteq F$.

Proof of Lemma 2. Let $\bar{\nu} \in \mathcal{A}^n$ be I-invariant, $s \in G$ and $f \in \mathcal{F}$. Then, by Lemma 1,

$$\bar{\mu}(f_s) = (f_s\,U)\bar{\nu} = (\widetilde{U}(s)f_s\,U_s)\bar{\nu} = \widetilde{U}(s)(f\,U)\bar{\nu} = \widetilde{U}(s)\bar{\mu}(f).$$

So $\bar{\mu}$ is U-invariant. For the converse, let $\bar{\mu}$ be U-invariant, $s \in G$ and $f \in \mathcal{F}$. Then

$$\bar{\nu}(f_s) = (f_s\,\widetilde{U})\bar{\mu} = \left(f_s\,\widetilde{U}_s U(s)\right)\bar{\mu} = \left(f\,\widetilde{U}U(s)\widetilde{U}(s)\right)\bar{\mu} = (f\,\widetilde{U})\bar{\mu} = \bar{\nu}(f).$$

So $\bar{\nu}$ is I-invariant.

Statement (2) follows with the help of statement (2) of Lemma 1. Let $\bar{\nu}$ be topologically I-invariant, $\phi \in L^1(G)$ and $f \in L^\infty(G)$. We remark first that

$$\begin{aligned}(\phi * f)\,U(s) &= U(s)\int_G \phi(t)f(t^{-1}s)\,d\lambda(t)\\ &= \int_G U(t)U(t^{-1}s)\phi(t)f(t^{-1}s)\,d\lambda(t) = (\phi\,U) * (f\,U)(s).\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\mu}(\phi * f) &= ((\phi * f)\,U)\bar{\nu} = ((\phi\,U) * (f\,U))\bar{\nu} = ((\phi\,U)\cdot(f\,U)(.)I)\,\bar{\nu}\\ &= \left(\int_G U(s)\phi(s)\,d\lambda(s)\right)(f\,U)\bar{\nu} = U(\phi)\bar{\mu}(f),\end{aligned}$$

and so $\bar{\mu}$ is topologically U-invariant.

For the converse, let $\bar{\mu}$ be topologically U-invariant, $f \in L^\infty(G)$ and $\phi \in L^1(G)$. Then, for each $s \in G$,

$$\begin{aligned}(\phi * f)\,\widetilde{U}(s) &= \widetilde{U}(s)\int_G \phi(t)f(t^{-1}s)\,d\lambda(t)\\ &= \int_G \widetilde{U}(t^{-1}s)\widetilde{U}(t)\phi(t)f(t^{-1}s)\,d\lambda(t) = (\phi\,\widetilde{U})\tilde{*}(f\,\widetilde{U})(s),\end{aligned}$$

and so

$$\bar{\nu}(\phi * f) = ((\phi * f)\,\widetilde{U})\bar{\mu} = \left((\phi\,\widetilde{U})\tilde{*}(f\,\widetilde{U})\right)\bar{\mu} = \left((f\,\widetilde{U})((\phi\,\widetilde{U})\cdot U)\right)\bar{\mu}$$
$$= \left((f\,\widetilde{U})\int_G \phi(s)\widetilde{U}(s)U(s)\,d\lambda(s)\right)\bar{\mu}$$
$$= \left(\int_G \phi(s)\,d\lambda(s)\right)(f\,\widetilde{U})\bar{\mu} = \left(\int_G \phi(s)\,d\lambda(s)\right)\bar{\nu}(f).$$

So $\bar{\nu}$ is topologically I-invariant. □

Theorem 1. *Let G be a locally compact group. Let $\bar{\mu} \in \mathcal{A}^n$ and M be the linear span of the coordinates $\mu_1, \mu_2, ..., \mu_n$ of $\bar{\mu}$. Let U be a continuous and bounded representation of G on $\mathbb{C}^n$. Then M is a left ideal of $\mathcal{A}$ of dimension less or equal to n in each of the following cases:*

1. *F is amenable, $\mathcal{A} = F^*$ and $\bar{\mu}$ is U-invariant.*
2. *G is amenable, $\mathcal{A} = L^1(G)^{**}$ and $\bar{\mu}$ is topologically U-invariant.*
3. *G is compact, $\mathcal{A} = L^1(G)$ or $M(G)$, and $\bar{\mu}$ is U-invariant.*

Furthermore, M is minimal and of dimension n when U is irreducible.

Proof. We consider only the first two statements. The proof of statement (3) is similar. Let $\mathcal{A} = F^*$, and let $\bar{\mu}$ be U-invariant. That M is a left ideal follows directly from the lemma above. Let μ be arbitrary in F^* and $f \in F$. Then $(\mu\,\bar{\mu})(f) = \mu(f_{\bar{\mu}})$, where $f_{\bar{\mu}}(s) = \bar{\mu}(f_s) = \widetilde{U}(s)\bar{\mu}(f)$, and so

$$\mu\,\bar{\mu}(f) = \mu(\widetilde{U}\bar{\mu}(f)) = \mu(\widetilde{U})\bar{\mu}(f).$$

Thus, $\mu\,\bar{\mu} = \mu(\widetilde{U})\bar{\mu}$, which means obviously that M is a left ideal of F^*.

For statement (2), let $\mu \in L^1(G)^{**}$ and $f \in L^\infty(G)$. Then $(\mu\bar{\mu})(f) = \mu(f_{\bar{\mu}})$, where

$$f_{\bar{\mu}}(\phi) = \bar{\mu}(\hat{\phi} * f) = U(\hat{\phi})\bar{\mu}(f), \text{ and}$$
$$U(\hat{\phi}) = \int_G U(s)\hat{\phi}(s)\,d\lambda(s) = \int_G \widetilde{U}(s)\phi(s)\,d\lambda(s) = \widetilde{U}(\phi) \text{ for all } \phi \in L^1(G).$$

Thus, $(\mu\bar{\mu})(f) = \mu(f_{\bar{\mu}}) = \mu(\widetilde{U})\bar{\mu}$, and so M is a left ideal of $L^1(G)^{**}$. That M is of dimension less or equal to n is clear in each case.

Suppose now that U is irreducible, and let us prove that M is minimal. We start with the algebra F^*. Let $\mu \in M$ be arbitrary, and write $\mu = \sum_{i=1}^n x_i\mu_i = \bar{\mu}\underline{x} = \bar{x}\underline{\mu}$, where $\underline{x} = (x_1, x_2, ..., x_n)$ is a non-zero vector of $\mathbb{C}^n$. For each vector $\overline{y} \in \mathbb{C}^n$, we can find $s_1, s_2, ..., s_k$ in G and $\alpha_1, \alpha_2, ..., \alpha_k$ in $\mathbb{C}$ such that

$$\sum_{i=1}^n \alpha_i\widetilde{U}(s_i)\overline{x} = \overline{y}$$

since $\widetilde{U}$ is also irreducible. It follows that

$$\sum_{i=1}^n \alpha_i e(s_i)\,\mu = \sum_{i=1}^n \alpha_i e(s_i)\,(\bar{\mu}\,\underline{x}) = \sum_{i=1}^n \alpha_i (e(s_i)\,\bar{\mu})\underline{x}$$
$$= \sum_{i=1}^n \alpha_i\,(\widetilde{U}(s_i)\bar{\mu})\underline{x} = \sum_{i=1}^n \alpha_i\,(\widetilde{U}(s_i)\overline{x})\underline{\mu} = \overline{y}\,\underline{\mu}$$

(remember that $e(s) \in F^*$ and $e(s)(f) = f(s)$ for $s \in G$ and $f \in F$). This means first that $F^* \mu = M$, and so M is minimal. Secondly, if $\overline{y} \in \mathbb{C}^n$ is such that $\overline{y}\underline{\mu} \neq \bar{0}$, then this argument shows also that

$$\sum_{i=1}^{n} \alpha_i e(s_i)\bar{\mu}\underline{x} = \overline{y}\underline{\mu} \neq 0,$$

and implies that $\bar{\mu}\underline{x} \neq 0$. So the elements $\mu_1, \mu_2, ..., \mu_n$ are linearly independent and M is of dimension n.

In $L^1(G)^{**}$, the corresponding representation U of $L^1(G)$ is also irreducible, and so we can find, for each $\overline{y} \in \mathbb{C}^n$, $\phi \in L^1(G)$ such that $U(\phi)\overline{x} = \overline{y}$. As above, this shows that M is minimal and is of dimension n. □

Remark. Statement (1) means that G needs to be amenable if $F = LUC(G)$.

Theorem 2. *Let $\mathcal{A}$ be $L^1(G)$, $M(G)$ or F^*, and let M be a left ideal of $\mathcal{A}$ of dimension n. Then there exist m vectors $\bar{\mu}^i \in \mathcal{A}^{n_i}$ and m irreducible, unitary, bounded and continuous representations U^i $(i = 1, 2, ..., m)$ of G such that*

(1) *each $\bar{\mu}^i$ is U^i-invariant,*
(2) *for each $i = 1, 2, ..., m$, the coordinates of $\bar{\mu}^i$ span a minimal left ideal M_i of $\mathcal{A}$ of dimension n_i, and*
(3) *$M = M_1 \oplus M_2 \oplus ... \oplus M_m$.*

Proof. In the case of $\mathcal{A} = L^1(G)$, we regard M as a left ideal of $M(G)$. This is possible because M is closed and $L^1(G)$ is a closed ideal of $M(G)$. We start with a set of elements μ_1, μ_2, ..., μ_n which generate M, and let $\bar{\mu} = (\mu_i)_{i=1}^n$. Then, for each $\mu \in \mathcal{A}$ and for each $i = 1, 2, ..., n$, there exist $a_{i1}(\mu), a_{i2}(\mu), ..., a_{in}(\mu) \in \mathbb{C}$ such that

$$\mu\,\mu_i = a_{i1}(\mu)\mu_1 + a_{i2}(\mu)\mu_2 + \,...\, + a_{in}(\mu)\mu_n.$$

Put $A(\mu) = (a_{ij}(\mu))_{i,j=1}^n$. Then, for μ and ν in $\mathcal{A}$,

$$A(\mu\,\nu)\bar{\mu} = (\mu\,\nu)\,\bar{\mu} = \mu\,\left(A(\nu)\,\bar{\mu}\right) = A(\nu)(\mu\,\bar{\mu}) = A(\nu)A(\mu)\,\bar{\mu}.$$

Hence, $A(\mu\,\nu) = A(\nu)A(\mu)$, i.e., A is an antirepresentation of $\mathcal{A}$. Moreover, for each $1 \leq j \leq n$, let $f \in \mathcal{F}$ be such that $\mu_j(f) = 1$ and $\mu_k(f) = 0$ for $k \neq j$. Then, for each $1 \leq i \leq n$, $\mu\,\mu_i(f) = a_{ij}(\mu)$, so

$$|a_{ij}(\mu)| \leq \|\mu\,\mu_i\|\|f\| \leq \|\mu\|\,\|\mu_i\|\|f\|,$$

which implies that A is bounded. Since the product in $\mathcal{A}$ is $\sigma(\mathcal{A},\mathcal{F})$-continuous on the left side, one can also see that the functions $\mu \mapsto a_{ij}(\mu)$ are $\sigma(\mathcal{A},\mathcal{F})$-continuous. When $\mathcal{A} = L^1(G)$, we let V be the antirepresentation of G associated to A; and when $\mathcal{F} = F$ is an admissible subspace of $C(G)$, we let

$$V(s) = (a_{ij}(e(s)))_{i,j=1}^n\,.$$

Then $\widetilde{V}$ is a bounded and continuous representation of G in each case. Furthermore, for $f \in \mathcal{F}$, we have

$$\bar{\mu}(f_s) = e(s)\,\bar{\mu}(f) = A(e(s))\bar{\mu}(f) = V(s)\bar{\mu}(f),$$

i.e., $\bar{\mu}$ is $\widetilde{V}$-invariant. Now it is easy to verify (or see [1, Lemma 3]) that $\widetilde{V}$ is in fact equivalent to a unitary representation U in the sense that $P\widetilde{V}(s) = U(s)P$ for all $s \in G$, where P is an invertible operator on $\mathbb{C}^n$. (This result is also true for the infinite-dimensional representations when G is amenable; see [9] or [7].) Put

$\bar{\gamma} = P\bar{\mu}$. It is clear that the coordinates of $\bar{\gamma}$ also generate the ideal M. We have also

$$\bar{\gamma}(f_s) = P\bar{\mu}(f_s) = PV(s)\bar{\mu}(f) = \widetilde{U}(s)P\bar{\mu}(f) = \widetilde{U}(s)\bar{\gamma}(f),$$

and so $\bar{\gamma}$ is U-invariant. Since U is unitary, it follows by [8, 21.40(a)] that U is a direct sum of continuous, irreducible representations U^1, U^2, ..., and U^m. This means that $\mathbb{C}^n$ is the direct sum of some invariant subspaces H_i, and each U^i is the restriction of U to H_i $(i = 1, 2, ..., m)$. For each $f \in \mathcal{F}$, we write $\bar{\gamma}(f) = \sum_i \bar{\mu}^i(f)$. Then

$$\sum_{i=1}^{m} \bar{\mu}^i(f_s) = \bar{\gamma}(f_s) = \widetilde{U}(s)\bar{\gamma}(f) = \sum_{i=1}^{m} \widetilde{U}^i(s)\bar{\mu}^i(f).$$

It follows that, for each $i = 1, 2, ..., m$, $\bar{\mu}^i(f_s) = \widetilde{U}^i(s)\bar{\mu}^i(f)$. This yields statement (1). Statement (2) follows from Theorem 1. Statement (3) is clear. □

Remark. In $L^1(G)^{**}$, the situation is slightly different. In fact, one can also produce finite-dimensional left ideals with the use of the right annihilators of $L^1(G)^{**}$. These are elements μ in $L^1(G)^{**}$ which satisfy $L^1(G)^{**}\,\mu = \{0\}$; see [6]. In such a situation, the representation $\widetilde{V}$ is trivial in the proof above.

Theorem 3. *Let M be a left ideal of $L^1(G)^{**}$ of dimension n, and suppose that M contains l linearly independent right annihilators $\gamma_1, \gamma_2, ..., \gamma_l$ of $L^1(G)^{**}$. Then there exist m vectors $\bar{\mu}^i \in \left(L^1(G)^{**}\right)^{n_i}$ and m irreducible, unitary, bounded and continuous representations U^i $(i = 1, 2, ..., m)$ of G such that*

1. *each $\bar{\mu}^i$ is topologically U^i-invariant,*
2. *for each $i = 1, 2, ..., m$, the coordinates of $\bar{\mu}^i$ span a minimal left ideal M_i of $L^1(G)^{**}$ of dimension n_i, and*
3. *$M = \mathbb{C}\,\gamma_1 \oplus \mathbb{C}\,\gamma_2 \oplus ... \oplus \mathbb{C}\,\gamma_l \oplus M_1 \oplus M_2 \oplus ... \oplus M_m$.*

Proof. We take $n - l$ linearly independent elements $\mu_1, \mu_2, ..., \mu_{n-l}$ in M which are not right annihilators of $L^1(G)^{**}$, let $\bar{\mu} = (\mu_i)_{i=1}^{n-l}$. Then, form the matrices $A(\mu)$ such that $\mu\bar{\mu} = A(\mu)\bar{\mu}$ for $\mu \in L^1(G)^{**}$, restrict A to $L^1(G)$, and let $\widetilde{V}$ be the corresponding representation of G. Then, for $\phi \in L^1(G)$ and $f \in L^\infty(G)$, we have

$$\bar{\mu}(\phi * f) = \bar{\mu}(\hat{\hat{\phi}} * f) = \hat{\phi}\,\bar{\mu}(f) = A(\hat{\phi})\bar{\mu}(f) = V(\hat{\phi})\bar{\mu}(f) = \widetilde{V}(\phi)\bar{\mu}(f),$$

and so $\bar{\mu}$ is topologically $\widetilde{V}$-invariant. The proof is completed as that of Theorem 2. □

Corollary. *Let G be a locally compact group. Then*

1. *finite-dimensional (left) ideals exist in $M(G)$ and $L^1(G)$ if and only if G is compact,*
2. *finite-dimensional left ideals exist in $LUC(G)^*$ if and only if G is amenable,*
3. *finite-dimensional left ideals which are not generated by right annihilators of $L^1(G)^{**}$ exist in $L^1(G)^{**}$ if and only if G is amenable.*

Proof. This follows from Lemma 2 and Theorems 2 and 3. □

Remark. The finite-dimensional right ideals in $WAP(G)^*$ are determined in the same way because the two Arens product conicide in this case; see [2, Section 4.2]. When G is compact, one proceeds also in the same way to find these ideals in $L^1(G)$ and $M(G)$. These facts were already observed for the minimal right ideals in [1, Section 4]. However, in [4, Remark 2.7(b)], we have proved that the non-trivial right ideals are all of infinite dimension in $LUC(\mathbb{Z})^* = \ell^\infty(\mathbb{Z})^*$, where $\mathbb{Z}$ is the additive group of the integers. In [1, Section 4], we have given a class of locally compact abelian groups, which includes $\mathbb{Z}$, for which the non-trivial right ideals are all of infinite dimension in $LUC(G)^*$. Now we can prove that, for a locally compact abelian group G, the finite-dimensional right ideals exist in $LUC(G)^*$ if and only if G is compact. We hope to publish this result in another paper.

The author wishes to thank the referee for the very careful reading the paper was given.

References

[1] J. W. Baker and M. Filali, *Minimal ideals in group algebras and their biduals*, Math. Proc. Cambridge Philos. Soc. **120** (1996), 475–488. CMP 96:12

[2] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, Wiley, New York, 1989. MR **91b:**43001

[3] M. Filali, *The ideal structure of some Banach algebras*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 567–576. MR **93e:**22005

[4] M. Filali, *Linear equations in* $B(\mathbb{Z})^*$, Proc. Roy. Soc. Edinburgh Sect. A **123A** (1993), 1001–1009. MR **95j:**39035

[5] M. Filali, *Minimal left ideals in the group algebra and its second conjugate*, Mathematica Japonica (to appear).

[6] E. E. Granirer, *The radical of* $L^\infty(G)^*$, Proc. Amer. Math. Soc. **41** (1973), 321–324. MR **48:**4646

[7] F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand, New York, 1969. MR **40:**4776

[8] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Berlin,, 1963, 1970. MR **28:**158

[9] B. Sz.-Nagy, *Uniformly bounded linear transformations in Hilbert space*, Acta Math. (Szeged) **11** (1947), 152–157.

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