PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 8, Pages 2343–2351
S 0002-9939(99)04812-1
Article electronically published on April 9, 1999

FOURIER MULTIPLIERS ON WEIGHTED L^p-SPACES

T. S. QUEK

(Communicated by J. Marshall Ash)

Dedicated to Professor Leonard Y. H. Yap on the ocassion of his sixtieth birthday

Abstract. In his 1986 paper in the Rev. Mat. Iberoamericana, A. Carbery proved that a singular integral operator is of weak type (p,p) on $L^p(\mathbb{R}^n)$ if its lacunary pieces satisfy a certain regularity condition. In this paper we prove that Carbery's result is sharp in a certain sense. We also obtain a weighted analogue of Carbery's result. Some applications of our results are also given.

1. Introduction

We introduce some notation to be used in this paper. Let $\mathcal{S}(\mathbb{R}^n)$ be the class of Schwartz functions on $\mathbb{R}^n$ and let ϕ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{|x| \le 1\}$ with $\int \phi = 1$. For $j \in \mathbb{Z}$ define the operator P_j on $\mathcal{S}(\mathbb{R}^n)$ by

$$P_j f = \phi_{2^j} * f,$$

where $\phi_{2^j}(x) = 2^{-jn}\phi(2^{-j}x), \quad x \in \mathbb{R}^n$. Now let ψ be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{1 \le |\xi| \le 4\}$ such that $\sum_{j\in\mathbb{Z}} \psi(2^j\xi) = 1$ for $\xi \ne 0$. For $j \in \mathbb{Z}$, define the operator Q_j on $\mathcal{S}(\mathbb{R}^n)$ by

$$(Q_j f)^\wedge(\xi) = \psi(2^j\xi)\hat{f}(\xi).$$

Let η be a non-negative $C^\infty(\mathbb{R}^n)$ function supported in $\{1 \le |x| \le 4\}$ such that $\sum_{j\in\mathbb{Z}} \eta(2^j x) = 1$ for $x \ne 0$. Let T and T_j be singular integral operators with kernels $K(x,y)$ and $K(x,y)\eta(2^{-j}(x-y))$, respectively.

Recently, Carbery proved the following theorem in [C, Theorem 1].

Theorem C. *Let T be a singular integral operator bounded on $L^2(\mathbb{R}^n)$ such that*

$$\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \|\sum_{\ell\ge 0} Q_{j+k}T_{j+\ell}(I-P_j)\|_{\mathcal{M}_p} < \infty,$$

where $\|\cdot\|_{\mathcal{M}_p}$ is the operator norm on $L^p(\mathbb{R}^n)$ and I is the identity operator on $L^2(\mathbb{R}^n)$. Then T is of weak type (p,p) on $L^p(\mathbb{R}^n)$.

We show in Theorem 4.1 that Theorem C is sharp. We also obtain in Theorem 2.1 a weighted analogue of Theorem C. Theorem 2.1 is then applied to obtain two multiplier results on power-weighted L^p-spaces; see Theorems 3.1 and 3.4.

Received by the editors August 20, 1996 and, in revised form, October 31, 1997.
1991 *Mathematics Subject Classification.* Primary 42A45.
Key words and phrases. Singular integral operators, Fourier multipliers, weighted L^p-spaces.

2. Weighted analogue of Theorem C

Let w be a non-negative locally integrable function on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$ define $w(E) = \int_E w(x)dx$. For $1 \le p < \infty$ let $L^p_w(\mathbb{R}^n)$ be the space of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{p,w} < \infty$, where

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p}.$$

We shall simply write $L^p_\alpha(\mathbb{R}^n)$ and $\|\cdot\|_{p,\alpha}$ if $w(x) = |x|^\alpha$.

The weight w is said to satisfy the Muckenhoupt A_p condition if there is a constant C such that

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1} \le C, \qquad 1 < p < \infty;$$

$$\frac{1}{|B|}\int_B w(x)dx \le C \operatorname*{ess\,inf}_{x\in B} w(x), \qquad p = 1,$$

for all balls $B \subset \mathbb{R}^n$, where $|B|$ denotes the Lebesgue measure of B.

The main result in this section is the following weighted version of Theorem C.

Theorem 2.1. *Let $w \in A_p$, $1 < p < 2$, and let T be a singular integral operator bounded on $L^2_w(\mathbb{R}^n)$ such that*

$$\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \|\sum_{\ell\ge 0} Q_{j+k}T_{j+\ell}(I - P_j)\|_{\mathcal{M}_{p,w}} < \infty, \tag{2.1}$$

where $\|\cdot\|_{\mathcal{M}_{p,w}}$ is the operator norm on $L^p_w(\mathbb{R}^n)$. Then T is of weak type (p,p) on $L^p_w(\mathbb{R}^n)$.

The proof of Theorem 2.1 depends on the following weighted version of Calderón-Zygmund decomposition whose proof is standard and is therefore omitted.

Theorem 2.2. *Let $w \in A_p$, $1 \le p < \infty$, and let $f \in \mathcal{S}(\mathbb{R}^n)$. For $\alpha > 0$, there exist a sequence of mutually disjoint balls $\{B_i\}$ and measurable functions g and b_i, $i \in \mathbb{N}$, such that*

(i) $f = g + \sum_{i=1}^{\infty} b_i$;
(ii) $b_i = f\chi_{B_i}$ *and* $\|b_i\|_1 \le 2^n\alpha|B_i|$;
(iii) $\sum_{i=1}^{\infty} \|b_i\|^p_{p,w} \le \|f\|^p_{p,w}, \quad 1 \le p < \infty$;
(iv) $\sum_{i=1}^{\infty} w(B_i) \le C_p\alpha^{-p}\|f\|^p_{p,w}, \quad 1 \le p < \infty$;
(v) $\|g\|_\infty \le \alpha$;
(vi) $\|g\|^2_{2,w} \le C_p\alpha^{2-p}\|f\|^p_{p,w}, \quad 1 \le p < 2$.

Proof of Theorem 2.1. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Fix $\alpha > 0$ and apply Theorem 2.2 to write $f = g + \sum_i b_i$ with $\|g\|_\infty \le \alpha$ and each b_i supported in a ball B_i of radius $2^{j(i)}$. Let

$$G = g + \sum_i b_i * \phi_{2^{j(i)}}$$

and

$$h = \sum_i (b_i - b_i * \phi_{2^{j(i)}}),$$

where ϕ is a non-negative radial $C^\infty(\mathbb{R}^n)$ function supported in $\{|x| \le 1\}$ with $\int \phi = 1$ and $\phi_{2^{j(i)}}(x) = 2^{-nj(i)}\phi(2^{-j(i)}x)$. Then we have

$$\begin{aligned}\{x : |Tf(x)| > \alpha\} &\subseteq \{x : |TG(x)| > \alpha/2\} \cup \{x : |Th(x)| > \alpha/2\}\\ &:= E_\alpha \cup F_\alpha.\end{aligned}$$

It follows from hypothesis (2.1) and Carbery's arguments in [C, Theorem 1] that $w(F_\alpha) \le C\alpha^{-p}\|f\|_{p,w}^p$. To estimate $w(E_\alpha)$, we note that $w(E_\alpha) \le C\alpha^{-2}\|G\|_{2,w}^2$ because T is bounded on $L_w^2(\mathbb{R}^n)$. Since $1 < p < 2$, Theorem 2.2(vi) implies that

$$\|g\|_{2,w}^2 \le C\alpha^{2-p}\|f\|_{p,w}^p. \tag{2.2}$$

Note that

$$\begin{aligned}\|b_i * \phi_{2^{j(i)}}\|_\infty &\le \|b_i\|_1\|\phi_{2^{j(i)}}\|_\infty\\ &\le C2^n\alpha|B_i|2^{-nj(i)}\\ &\le C\alpha\end{aligned}$$

for all $i \in \mathbb{N}$. Since $1 < p < 2$ we have

$$\begin{aligned}\sum_i \|b_i * \phi_{2^{j(i)}}\|_{2,w}^2 &\le C\alpha^{2-p}\sum_i \|b_i * \phi_{2^{j(i)}}\|_{p,w}^p\\ &\le C\alpha^{2-p}\sum_i \|b_i\|_{p,w}^p\\ &\le C\alpha^{2-p}\|f\|_{p,w}^p,\end{aligned} \tag{2.3}$$

where the second inequality follows from [ST, Theorem 6, p. 162] and the last inequality is by Theorem 2.2 (iii). It now follows from (2.2) and (2.3) that

$$\|G\|_{2,w}^2 \le C\alpha^{2-p}\|f\|_{p,w}^p.$$

Hence we also have

$$w(E_\alpha) \le C\alpha^{-p}\|f\|_{p,w}^p.$$

Consequently we have $w\{x : |Tf(x)| > \alpha\} \le C\alpha^{-p}\|f\|_{p,w}^p$; that is to say, T is of weak type (p,p) on $L_w^p(\mathbb{R}^n)$.

Theorem 2.1 has the following simple corollary.

Corollary 2.3. *Let $\{\alpha(k)\}_{-\infty}^0$ be a sequence of positive real numbers satisfying $\sum_{k\le 0}|k|\alpha(k) < \infty$. Let $w \in A_p$, $1 \le p < 2$, and let $m \in L^\infty(\mathbb{R}^n)$ be such that m is a multiplier on $L_w^2(\mathbb{R}^n)$. Suppose that for all $j > i$, we have*

$$\|m_i * (\hat{\eta})_{2^{-j}}\|_{\mathcal{M}_{p,w}} \le \alpha(i-j),$$

where $m_i(\xi) = m(\xi)\psi(2^i\xi)$. Then m is a multiplier of weak type (p,p) on $L_w^p(\mathbb{R}^n)$.

Proof. Let T be the convolution operator defined on $\mathcal{S}(R^n)$ by $(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$. For $j \in \mathbb{Z}$ define the operator T_j on $\mathcal{S}(R^n)$ by

$$(T_jf)^\wedge(\xi) = (m * (\hat{\eta})_{2^j}(\xi))\hat{f}(\xi).$$

Using Theorem 2.1 and [ST, Theorem 6, p.162], we can prove the corollary in the same manner as Theorem 3 is proved in [C]. Details of the proof are therefore omitted.

3. Multipliers on power-weighted $L^p(\mathbb{R}^n)$

Corollary 2.3 indicates the amount of regularity needed for each m_i so that m is a multiplier of weak type (p,p) on $L^p_w(R^n)$. Our next theorem shows that for certain power weights, such a regularity condition is implied by m_i satisfying a certain Lipschitz condition. We say a distribution f is in the Lipschitz space $\Lambda^\beta_{r,s}(\mathbb{R}^n)$ for $\beta > 0,\ 1 \le r, s \le \infty,$ if $\|f\|_{\Lambda^\beta_{r,s}} < \infty$, where

$$\|f\|_{\Lambda^\beta_{r,s}} = \left\{ \sum_{j\in\mathbb{Z}} 2^{-sj\beta} \|f * (\hat{\eta})_{2^j}\|_r^s \right\}^{1/s},$$

with the usual modification if $s = \infty$.

Theorem 3.1. *Let $\beta > 0$ and let $m \in L^\infty(\mathbb{R}^n)$ be such that $m_i \in \Lambda^\beta_{\infty,2}(\mathbb{R}^n)$ and*

$$\sup_{i\in\mathbb{Z}} 2^{-i\beta} \|m_i\|_{\Lambda^\beta_{\infty,2}} < \infty, \tag{3.1}$$

where m_i is as in Corollary 2.3.

(i) *Let $1 < q < 2$ and let $n(2-q)/2q < \beta < n/2$. Then m is a multiplier on $L^r_\alpha(R^n)$ for all $q < r \le 2$ and $|\alpha| \le 2\beta(r-q)/(2-q)$.*
(ii) *If $n/2 \le \beta$, then m is a multiplier on $L^r_\alpha(R^n)$ for all $1 < r \le 2$ and $|\alpha| < n(r-1)$.*

We need the following two lemmas in the proof of Theorem 3.1.

Lemma 3.2. *Let $1 < q < 2$ and let $n(2-q)/2q < \beta < n/2$. Let $m \in L^\infty(\mathbb{R}^n)$ be such that $m_i \in \Lambda^\beta_{\infty,\infty}(\mathbb{R}^n)$ for all $i \in \mathbb{Z}$ and*

$$\sup_{i\in\mathbb{Z}} 2^{-i\beta} \|m_i\|_{\Lambda^\beta_{\infty,\infty}} < \infty, \tag{3.2}$$

where m_i is as in Corollary 2.3. Then m is a multiplier on $L^q(\mathbb{R}^n)$.

Proof. Recall that $\sum_{k\in\mathbb{Z}} \psi(2^k\xi) = 1$ for $\xi \ne 0$ and write

$$\begin{aligned} m_i * (\hat{\eta})_{2^{-j}}(\xi) &= \sum_{k\in\mathbb{Z}} m_i * (\hat{\eta})_{2^{-j}}(\xi)\psi(2^k\xi) \\ &= \sum_{k<i-1} + \sum_{k=i-1}^{i+1} + \sum_{k>i+1} m_i * (\hat{\eta})_{2^{-j}}(\xi)\psi(2^k\xi) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Let τ_{ijk} be defined by $\tau_{ijk}(\xi) = (m_i * (\hat{\eta})_{2^{-j}}(\xi))\psi(2^k\xi)$. Then $\mathrm{II} = \sum_{k=i-1}^{i+1} \tau_{ijk}$.

Choose p such that $1 < p < q$ and $\beta > n(2-p)/2p$. We shall estimate $\|\tau_{ijk}\|_{\mathcal{M}_p}$ for $k = i$ by interpolation between $\|\tau_{iji}\|_{\mathcal{M}_2}$ and $\|\tau_{iji}\|_{\mathcal{M}_1}$. Clearly, we have

$$\begin{aligned} \|\tau_{iji}\|_{\mathcal{M}_2} &= \|\tau_{iji}\|_\infty \\ &\le \|m_i * (\hat{\eta})_{2^{-j}}\|_\infty \|\psi(2^i\cdot)\|_\infty \\ &\le C2^{(i-j)\beta}, \end{aligned} \tag{3.3}$$

where the last inequality follows from (3.2) and $\|\psi\|_\infty \le 1$. Note that $\|\tau_{iji}\|_{\mathcal{M}_1} = \|(\tau_{iji})^\vee\|_1$. Let $(\tau_{iji})^\vee = (\tau_{iji})^\vee\chi_A + (\tau_{iji})^\vee\chi_B$, where $A = \{|x| \le 2^{j+3}\}$, $B = \{|x| > 2^{j+3}\}$. Then

$$\begin{aligned}\|(\tau_{iji})^\vee\chi_A\|_1 &\le \|(\tau_{iji})^\vee\|_2\|\chi_A\|_2\\ &\le \|m_i * (\hat{\eta})_{2^{-j}}\|_\infty\|\psi(2^i\cdot)\|_2\|\chi_A\|_2\\ &\le C2^{(i-j)\beta}2^{-ni/2}2^{n(j+3)/2} \le C2^{(i-j)(\beta-n/2)},\end{aligned}$$

where the penultimate inequality follows from (3.2) and $\|\psi(2^i\cdot)\|_2 = 2^{-in/2}\|\psi\|_2$. To estimate $\|(\tau_{iji})^\vee\chi_B\|_1$ we write $B_\ell = \{2^\ell < |x| \le 2^{\ell+1}\}$ for $\ell \ge j+3$. Then

$$\|(\tau_{iji})^\vee\chi_B\|_1 = \sum_{\ell>j+2}\int_{B_\ell}|(\tau_{iji})^\vee(x)|dx.$$

Now for $x \in B_\ell$ and $t \in \mathbb{N}$, there exists a constant C_t such that

$$\begin{aligned}|(\tau_{iji})^\vee(x)| &\le 2^{-ni}\int_{\{1\le|2^{-j}y|\le 4\}}|(m_i)^\vee(y)\eta(2^{-j}y)||(\psi)^\vee(2^{-i}(x-y))|dy\\ &\le C_t2^{-2ni+nj+(i-\ell)t},\end{aligned}$$

where the last inequality follows from $\|(m_i)^\vee\|_\infty \le \|m_i\|_1 \le C2^{-ni}$. Now choose $t = 3n$ and we have

$$\|(\tau_{iji})^\vee\chi_B\|_1 \le C\sum_{\ell>j+2}2^{ni+nj-2n\ell} \le C2^{(i-j)n}.$$

Since $i < j$ and $\beta < n/2$, we have $\|(\tau_{iji})^\vee\|_1 \le C2^{(i-j)(\beta-n/2)}$. It follows that

$$\|\tau_{iji}\|_{\mathcal{M}_1} \le C2^{(i-j)(\beta-n/2)}. \tag{3.4}$$

Interpolating between (3.3) and (3.4) yields $\|\tau_{iji}\|_{\mathcal{M}_p} \le C2^{(i-j)(\beta-n(2-p)/2p)}$ for $1 < p < 2$.

Similar estimates of $\|\tau_{ijk}\|_{\mathcal{M}_p}$ for $k = i-1, i+1$ give

$$\|\text{ II}\|_{\mathcal{M}_p} \le C2^{(i-j)(\beta-n(2-p)/2p)}.$$

Routine calculations as in [C, p.395] show that for $|\gamma| \le n$ and $t > 2n$, there exists a constant C_t such that $\|\text{ I}\|_{\mathcal{M}_p} \le C_t\|m_i\|_\infty 2^{(j-i)(n+|\gamma|-t)}$ and $\|\text{ III}\|_{\mathcal{M}_p} \le C_t\|m_i\|_\infty 2^{(j-i)(n+|\gamma|-t)}$.

Consequently for $j > i$ and $t = 3n$, we have

$$\begin{aligned}\|m_i * (\hat{\eta})_{2^{-j}}\|_{\mathcal{M}_p} &\le \|\text{ I}\|_{\mathcal{M}_p} + \|\text{ II}\|_{\mathcal{M}_p} + \|\text{ III}\|_{\mathcal{M}_p}\\ &\le C(2^{(i-j)(\beta-n(2-p)/2p)} + 2^{(i-j)n}).\end{aligned}$$

It follows from Corollary 2.3 that m is a multiplier of weak type (p,p) on $L^p(\mathbb{R}^n)$. Since $1 < p < q < 2$ and m is a multiplier on $L^2(\mathbb{R}^n)$, we have m a multiplier on $L^q(\mathbb{R}^n)$.

Lemma 3.3. *Let $0 < \beta < n/2$. Let $m \in L^\infty(\mathbb{R}^n)$ be such that $m_i \in \Lambda^\beta_{n/\beta,2}(\mathbb{R}^n)$ and*

$$\sup_{i\in\mathbb{Z}}\|m_i\|_{\Lambda^\beta_{n/\beta,2}} < \infty,$$

where m_i is as in Corollary 2.3. Then m is a multiplier on $L^2_{2\beta}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\|(m_i\hat{f})^\vee\|_{2,2\beta} \sim \|m_i\hat{f}\|_{\Lambda^\beta_{2,2}}$, it follows from Herz [He, Lemma 1.5*] that

$$\begin{aligned}\|m_i\hat{f}\|_{\Lambda^\beta_{2,2}} &\le C\{\|m_i\|_\infty + \|m_i\|_{\Lambda^\beta_{n/\beta,2}}\}\|\hat{f}\|_{\Lambda^\beta_{2,2}}\\ &\le C\|f\|_{2,2\beta}.\end{aligned}$$

Thus we have $\sup_{i\in\mathbb{Z}} \|m_i\|_{\mathcal{M}_{2,2\beta}} < \infty$. Hence m is a multiplier on $L^2_{2\beta}(\mathbb{R}^n)$.

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. (i). Note that supp $m_i \subset \{2^{-i} \le |\xi| \le 2^{-i+2}\}$. Now Theorem 6.3.1 of [BL] and hypothesis (3.1) of Theorem 3.1 imply that m satisfies the hypotheses of both Lemmas 3.2 and 3.3. Thus m is both a multiplier on $L^q(\mathbb{R}^n)$, $1 < q < 2$, and a multiplier on $L^2_{2\beta}(\mathbb{R}^n)$. The result now follows from the Stein-Weiss interpolation of L^p spaces with change of measures; see [SW, Theorem 2.11].

(ii). The proof of Lemma 3.2 can be easily modified to show that m is a multiplier on $L^p(\mathbb{R}^n)$ for all $1 < p < 2$. Now $\Lambda^\beta_{n/\beta,2}(\mathbb{R}^n) \subset \Lambda^\alpha_{n/\alpha,2}(\mathbb{R}^n)$ for all $0 < \alpha < n/2$. It follows from Lemma 3.3 that m is a multiplier on $L^2_{2\alpha}(\mathbb{R}^n)$ for all $|\alpha| < n/2$. The result again follows from the Stein-Weiss interpolation with change of measures.

As an application of Theorem 3.1 we consider the following multiplier discussed in (3.6) of Baernstein and Sawyer [BS, p.22].

Let a and b be positive real numbers and let s be an integer larger than b/a. Let m be a strongly singular multiplier such that

$$\begin{cases}|D^\beta m(\xi)| \le C|\xi|^{-b}|\xi|^{(a-1)|\beta|}, & 0 \le |\beta| \le s,\\ m(\xi) = 0, & |\xi| \le 1,\end{cases} \tag{3.5}$$

where $\beta = (\beta_1, ..., \beta_n)$, $\quad D^\beta = D_1^{\beta_1}...D_n^{\beta_n}$ and $D_j = \frac{\partial}{\partial\xi_j}$.

The prototypical example is the function $m_{a,b}$ defined by $m_{a,b}(\xi) = \Theta(\xi)|\xi|^{-b}e^{i|\xi|^a}$, where $\Theta \in C^\infty(\mathbb{R}^n)$, $\Theta = 0$ on $|\xi| < 1$, $\Theta = 1$ on $|\xi| \ge 2$.

The next theorem shows that m is a multiplier on certain power-weighted $L^p(\mathbb{R}^n)$.

Theorem 3.4. *Let $m \in L^\infty(\mathbb{R}^n)$ satisfy* (3.5) *above. Then we have*

(i) *if $1 < q < 2$ and $n(2-q)/2q < b/a < n/2$, then m is a multiplier on $L^r_\alpha(\mathbb{R}^n)$ for all $q < r \le 2$ and $|\alpha| \le 2b(r-q)/a(2-q)$;*

(ii) *if $n/2 \le b/a$, then m is a multiplier on $L^r_\alpha(\mathbb{R}^n)$ for all $1 < r \le 2$ and $|\alpha| < n(r-1)$.*

Proof. Let s be an integer larger than b/a. Since $m_i(\xi) = m(\xi)\psi(2^i\xi)$ and $m = 0$ for $|\xi| \le 1$, we have $m_i = 0$ for $i \ge 2$. For $i < 2$ our hypothesis (3.5) implies that

$$\|D^\beta m_i\|_\infty \le C2^{i\{b-(a-1)|\beta|\}}, \quad 0 \le |\beta| \le s,$$

where C is independent of i. Assume that b/a is not an integer and let $b/a = \nu + \sigma$, where ν is a non-negative integer and $0 < \sigma < 1$. For $1 \le j \le n$ we have

$$\begin{aligned}\left\|\Delta_h \frac{\partial^\nu m_i}{\partial\xi_j^\nu}\right\|_\infty &\le C \ \min\{|h| \max_{|\beta|=\nu+1}\|D^\beta m_i\|_\infty, \ \|\frac{\partial^\nu m_i}{\partial\xi_j^\nu}\|_\infty\}\\ &\le C \ \min\{|h|2^{i\{b-(a-1)(\nu+1)\}}, \ 2^{i\{b-(a-1)\nu\}}\},\end{aligned}$$

where $\Delta_h f(x) = f(x+h) - f(x)$. Let $\omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}) = \sup_{|h|<t} \|\Delta_h \frac{\partial^\nu m_i}{\partial \xi_j^\nu}\|_\infty$. Then we have

$$\begin{aligned}
&\sum_{j=1}^n \left(\int_0^\infty (t^{-\sigma}\omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}))^2 \, dt/t \right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^n \Bigg\{ \left(\int_0^{2^{i(a-1)}} (t^{-\sigma}\omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}))^2 \, dt/t \right)^{\frac{1}{2}} \\
&\qquad + \left(\int_{2^{i(a-1)}}^\infty (t^{-\sigma}\omega_\infty(t, \frac{\partial^\nu m_i}{\partial \xi_j^\nu}))^2 \, dt/t \right)^{\frac{1}{2}} \Bigg\} \\
&\le C 2^{ib/a}.
\end{aligned}$$

It follows from [BL, Theorem 6.3.1] that $m_i \in \Lambda^{b/a}_{\infty,2}$ and $\|m_i\|_{\Lambda^{b/a}_{\infty,2}} \le C 2^{ib/a}$.

If b/a is an integer, we write $b/a = \nu + 1$, where $\nu \ge 0$. Then for $1 \le j \le n$ we have

$$\left\| \Delta_h^2 \frac{\partial^\nu m_i}{\partial \xi_j^\nu} \right\|_\infty \le C \ \min\{|h|^2 2^{i\{b-(a-1)(\nu+2)\}}, \ 2^{i\{b-(a-1)\nu\}}\},$$

where $\Delta_h^2 f = \Delta(\Delta_h f)$.

Routine calculation as in the case where b/a is a non-integer then shows that $m_i \in \Lambda^{b/a}_{\infty,2}$ and $\|m_i\|_{\Lambda^{b/a}_{\infty,2}} \le C 2^{ib/a}$. The theorem now follows from Theorem 3.1.

4. Sharpness of Theorem C

In this section we prove that Theorem C is sharp in the following sense.

Theorem 4.1. *Let $1 < p < 2$. There exists a convolution operator T bounded on $L^2(\mathbb{R}^n)$ so that*

(i) $\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \| \sum_{\ell\ge 0} Q_{j+k} T_{j+\ell}(I - P_j)\|_{\mathcal{M}_p} < \infty$;
(ii) *T is bounded on $L^p(\mathbb{R}^n)$;*
(iii) *T is not of weak type (r,r) on $L^r(\mathbb{R}^n)$ for $1 < r < p$.*

Proof. Let $1 < r < p < 2$. Choose q such that $r < q < p$ and choose α such that $0 < \alpha < 1$, $q < 2/(2-\alpha) < p$. Following Onneweer [O, p.56] we construct a sequence $(P_k)_{k\ge 0}$ of Rudin-Shapiro-like polynomials on the n-dimensional torus $\mathbb{T}^n$ such that $\|P_k\|_\infty \le 2^{n(k+1)/2}$ and $|\hat{P}_k(j)| = 1$ for $j = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n$, where $0 \le j_i < 2^k$ for $i = 1, \cdots, n$. Note that $\hat{P}_k$ is supported on $\mathbb{Z}^n$. Now for $k \ge 0$ define γ_k on $\mathbb{Z}^n$ by $\gamma_k = 2^{n(\alpha-1)k/2}\hat{P}_k$. Let $\tilde{k}$ denote the center of the cube $[2^{k+2}, 2^{k+3}]^n$ and define Φ_k on $\mathbb{Z}^n$ by $\Phi_k(j) = \gamma_k(j - \tilde{k})$. Write $\Phi = \sum_{k\ge 0} \Phi_k$ and it can be shown as in Figà-Talamanca and Gaudry [FG, Theorem B] that Φ is a multiplier on $L^p(\mathbb{T}^n)$ but not a multiplier on $L^q(\mathbb{T}^n)$.

Let m be the function defined on $\mathbb{R}^n$ by $m(\xi) = \sum_{j\in\mathbb{Z}^n} S(\xi - j)\Phi(j)$, where $S(\xi) := (\max\{1 - |\xi|^2, 0\})^n$ for $\xi \in \mathbb{R}^n$. Now define the convolution operator T on $\mathcal{S}(R^n)$ by $(Tf)^\wedge = m\hat{f}$. Then T is bounded on $L^2(\mathbb{R}^n)$. Since $n(1-\alpha)/2 >$

$n(2-p)/2p$, it follows from the proofs of Theorem 3 of [C] and Lemma 3.2 that T will satisfy (i) if we have

$$\sup_{k\in\mathbb{Z}} 2^{-nk(1-\alpha)/2}\|m_k\|_{\Lambda^{n(1-\alpha)/2}_{\infty,\infty}} < \infty, \tag{4.1}$$

where $m_k(\xi) = m(\xi)\psi(2^k\xi)$. Note that $\operatorname{supp} m_k \subset \{2^{-k} \le |\xi| \le 2^{-k+2}\}$ and so $m_k = 0$ for $k \ge 0$. Thus we only need to prove (4.1) for $k < 0$. Let $\xi \in \mathbb{R}^n$ and let $A_\xi = \{j \in \mathbb{Z}^n : |\xi - j| \le 1\}$. Then A_ξ is a finite set with at most 2^n elements. Let $\beta = n(1-\alpha)/2$ and we have $|\Phi(j)| \le 2^{(k+2)\beta}$ for $j \in A_\xi$, $\xi \in \operatorname{supp} m_k$. Furthermore, $k < 0$ implies that there exists a constant C so that $\|D^\mu S(\cdot - j)\psi(2^k\cdot)\|_\infty \le C$ for all multi-indices μ with $0 \le |\mu| \le n$ and all $j \in A_\xi$, $\xi \in \operatorname{supp} m_k$. Consequently we have

$$|D^\mu m_k(\xi)| \le \sum_{j\in A_\xi} |D^\mu S(\xi - j)\Phi(j)\psi(2^k\xi)| \le C2^{k\beta}.$$

If β is not an integer, we write $\beta = \nu + \sigma$ with $0 < \sigma < 1$. Then ν is an integer less than $n/2$. For $i = 1, 2, \cdots, n$ we have

$$\left\|\Delta_h \frac{\partial^\nu m_k}{\partial \xi_i^\nu}\right\|_\infty \le C|h|^\sigma 2^{k\beta}.$$

Thus we have $\|m_k\|_{\Lambda^\beta_{\infty,\infty}} \le C2^{k\beta}$ if β is not an integer. If β is an integer, then routine calculation as in the proof of Theorem 3.4 shows that $\|m_k\|_{\Lambda^\beta_{\infty,\infty}} \le C2^{k\beta}$. Consequently, the sequence $\{m_k\}_{-\infty}^{\infty}$ satisfies (4.1) and we have T satisfying (i).

Since $\beta = n(1-\alpha)/2 > n(2-p)/2p$, (4.1) and Lemma 3.2 imply that m is a multiplier on $L^p(\mathbb{R}^n)$. Hence T satisfies (ii). Lastly, if T were of weak type (r, r) on $L^r(\mathbb{R}^n)$ for $1 < r < p$, then T would be bounded on $L^q(\mathbb{R}^n)$ since $r < q < p$. Hence m would be a multiplier on $L^q(\mathbb{R}^n)$. By deLeeuw's theorem [L, Proposition 3.3] the restriction Φ of m to $\mathbb{Z}^n$ would then be a multiplier on $L^q(\mathbb{T}^n)$, but this contradicts our earlier observation that Φ is not a multiplier on $L^q(\mathbb{T}^n)$. Thus T satisfies (iii).

Acknowledgement

The author is grateful to the referee for his/her valuable comments and suggestions on several parts of the paper.

References

[BL] J. Bergh and J. Löfström, Interpolation spaces; an introduction, Grundl. der math. Wiss. 223, Springer-Verlag, Berlin and New York, 1976. MR **58:**2349

[BS] A. Baernstein II and E. T. Sawyer, Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$, Mem. Amer. Math. Soc., No. 318, Amer. Math. Soc., Providence, RI, 1985. MR **86g:**42036

[C] A. Carbery, Variants of the Calderón-Zygmund theory for L^p-spaces, Rev. Mat. Iberoamericana 2(1986), 381-396. MR **89f:**42011

[FG] A Figà-Talamanca and G. I. Gaudry, Multipliers of L^p which vanish at infinity, J. Funct. Anal. 7(1971), 475-486. MR **43:**2429

[He] C. S. Herz, Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms, J. Math. Mech. 18(1968), 283-323. MR **55:**11028

[L] K. DeLeeuw, On L^p multipliers, Ann. of Math. 81(1965), 364-379. MR **30:**5127

[O] C. W. Onneweer, On absolutely convergent Fourier series, Ark. Mat. 12(1974), 51-58. MR **50:**5338

[SW] E. M. Stein and G. Weiss, Interpolation of operators with change of measures, Trans. Amer. Math. Soc. 87(958), 59-72. MR **19:**1184d

[ST] J. O. Strömberg and A. Torchinsky, Weighted Hardy spaces, Lecture Notes in Mathematics 1381, Springer-Verlag, Berlin and New York, MR **90j:**42053

Department of Mathematics, National University of Singapore, Singapore 119260, Republic of Singapore

E-mail address: matqts@leonis.nus.edu.sg