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A STRUCTURE OF RING HOMOMORPHISMS ON COMMUTATIVE BANACH ALGEBRAS

SIN-EI TAKAHASI AND OSAMU HATORI

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Dedicated to Professor Jyunji Inoue on his sixtieth birthday

ABSTRACT. We give a structure theorem for a ring homomorphism of a commutative regular Banach algebra into another commutative Banach algebra. In particular, it is shown that:

(i) A ring homomorphism of a commutative C*-algebra onto another commutative C*-algebra with connected infinite Gelfand space is either linear or anti-linear.

(ii) A ring automorphism of $L^1(\boldsymbol{R}^N)$ is either linear or anti-linear.

(iii) $C^n([a,b])$, $L^1(\boldsymbol{R}^N)$ and the disc algebra $A(D)$ are neither ring homomorphic images of $\ell^1(S)$ nor $L^p(G)$ ($1 \le p < \infty$, G a compact abelian group).

Let A and B be two commutative Banach algebras with Gelfand spaces Φ_A and Φ_B, respectively. Let ρ be a ring homomorphism of A into B such that

$$(*) \qquad \{\rho(x)\hat{}\,(\psi) : x \in A\} = \boldsymbol{C}, \quad \text{the complex field,}$$

for each $\psi \in \Phi_B$ ("$\hat{}$" denotes the Gelfand transform). This, of course, holds if ρ is onto.

The purpose of this note is to show the following structure theorem of ρ applying the method which L. Molnar used in [5] to prove that a commutative semisimple Banach algebra which is the range of a ring homomorphism from a commutative C*-algebra must be C*-equivalent.

Theorem 1. *Suppose A is regular. Then there exist a continuous map $\hat{\rho}$ of Φ_B into Φ_A and a division $\{\Phi_B^0, \Phi_B^1, \Phi_B^2\}$ of Φ_B such that Φ_B^1 and Φ_B^2 are closed, and for each $a \in A$, $\rho(a)\hat{} = \hat{a} \circ \hat{\rho}$ on Φ_B^1, $\rho(a)\hat{} = \bar{\hat{a}} \circ \hat{\rho}$ on Φ_B^2 and $\rho(a)\hat{}\,(\psi) = \tau_\psi(\hat{a}(\hat{\rho}(\psi)))$ for every $\psi \in \Phi_B^0$ and for a certain discontinuous ring automorphism τ_ψ of the complex field $\boldsymbol{C}$.*

Moreover, if ρ is surjective, then $\hat{\rho}$ is injective, and if A satisfies the following condition (#), then $\hat{\rho}(\Phi_B^0)$ is a finite set:

(#) *For any $\lambda_n \in \boldsymbol{C}$ with $|\lambda_n| \le 1/2^n$ ($n = 1, 2, \dots$) and any sequence $\{\varphi_1, \varphi_2, \dots\}$ in Φ_A such that each φ_n is an isolated point in $\{\varphi_1, \varphi_2, \dots\}$, there exists an element $a \in A$ such that $\hat{a}(\varphi_n) = \lambda_n$ ($n = 1, 2, \dots$).*

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Remark 1. If A is a commutative regular Banach algebra which satisfies the condition (#) and if ρ is surjective, then Φ_B^0 is a finite set and hence both Φ_B^1 and Φ_B^2 are clopen.

Remark 2. A commutative C*-algebra, $\ell^1(S)$ (S a set), $L^1(\boldsymbol{R}^N)$ and $L^p(G)$ ($1 \leq p < \infty$, G a compact abelian group) are commutative regular Banach algebras which satisfy the condition (#). The details of these algebras can be seen just before Corollary 4.

Now in order to prove the theorem, we have to prepare some lemmas.

Lemma 1. *If I is a closed ideal of B such that $I = \bigcap_{\substack{I \subseteq \operatorname{Ker}(\psi) \\ \psi \in \Phi_B}} \operatorname{Ker}(\psi)$, then $\rho^{-1}(I)$ is a closed algebra ideal of A.*

Proof. We first show that $\rho^{-1}(I)$ is norm-closed. To do this, let a be any element in the norm-closure of $\rho^{-1}(I)$. We will show that

$$\rho(a)\hat{}(\psi)\rho(x)\hat{}(\psi) \neq 1 \tag{1}$$

for all $x \in A$ and $\psi \in \Phi_B$ with $I \subseteq \operatorname{Ker}(\psi)$. Actually, for any arbitrary element $x \in A$, choose an element of $y \in \rho^{-1}(I)$ such that $\|ax - y\| < 1$ since ax belongs to the norm-closure of $\rho^{-1}(I)$, and define

$$z = \sum_{n=1}^{\infty} (ax - y)^n.$$

We have $zax - zy = z - (ax - y)$ and hence $(\rho(z) + 1)\rho(a)\rho(x) - \rho(z) \in I$. This implies easily that $\rho(a)\hat{}(\psi)\rho(x)\hat{}(\psi) \neq 1$ for all $\psi \in \Phi_B$ with $I \subseteq \operatorname{Ker}(\psi)$. Let us show now that $a \in \rho^{-1}(I)$. Suppose, on the contrary, that $\rho(a) \notin I$. Since $I = \bigcap_{I \subseteq \operatorname{Ker}(\psi)} \operatorname{Ker}(\psi)$, there exists an element $\psi_0 \in \Phi_B$ such that $I \subseteq \operatorname{Ker}(\psi_0)$ and $\rho(a)\hat{}(\psi_0) \neq 0$. Then we can choose an element $x_0 \in A$ such that $\rho(x_0)\hat{}(\psi_0) = \frac{1}{\rho(a)\hat{}(\psi_0)}$ since ρ satisfies the condition $(*)$. But this contradicts (1) and then $\rho^{-1}(I)$ is norm-closed.

We next show that $\rho^{-1}(I)$ is an algebra ideal of A. To do this, let $x \in \rho^{-1}(I)$ and $\lambda \in \boldsymbol{C}$. Choose a sequence $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots\}$ of rational numbers such that $\lim_{n\to\infty}(\alpha_n + i\beta_n) = \lambda$. Then we have $\lim_{n\to\infty} \|\alpha_n x + i\beta_n x - \lambda x\| = 0$ and $\alpha_n x, \beta_n x \in \rho^{-1}(I)$ for each integer $n \geq 1$. For any $\psi \in \Phi_B$ with $I \subseteq \operatorname{Ker}(\psi)$, we have

$$\psi(\rho(i\beta_n x))^2 = -\psi(\rho(\beta_n x))^2 = 0,$$

and hence $\psi(\rho(i\beta_n x)) = 0$. Then $\rho(i\beta_n x) \in I$ since $I = \bigcap_{I \subseteq \operatorname{Ker}(\psi)} \operatorname{Ker}(\psi)$. Therefore we have $\alpha_n x + i\beta_n x \in \rho^{-1}(I)$ for each integer $n \geq 1$, and hence $\lambda x \in \rho^{-1}(I)$ since $\rho^{-1}(I)$ is norm-closed. Q.E.D.

Lemma 2. *There exists a mapping $\hat{\rho}$ of Φ_B into Φ_A such that*

$$\rho(a)\hat{}(\psi) = \tau_\psi(\hat{a}(\hat{\rho}(\psi))) \qquad (a \in A)$$

for every $\psi \in \Phi_B$ and for a certain ring automorphism τ_ψ of $\boldsymbol{C}$.

Proof. Let ψ be any element of Φ_B and define $\rho_\psi(a) = \rho(a)\hat{}(\psi)$ for each $a \in A$. Then ρ_ψ is a ring homomorphism of A onto $\boldsymbol{C}$ by the condition $(*)$. Hence we can easily see that $\operatorname{Ker}(\rho_\psi)$ is a closed algebra ideal of A by applying the preceding lemma. From this, $A/\operatorname{Ker}(\rho_\psi)$ is a unital commutative Banach algebra which is ring isomorphic to $\boldsymbol{C}$. Then there exists an algebra homomorphism of $A/\operatorname{Ker}(\rho_\psi)$

onto $\boldsymbol{C}$, say η. Let φ be the composition map of the canonical map from A onto $A/\operatorname{Ker}(\rho_\psi)$ and η. Then φ is an element of Φ_A such that $\operatorname{Ker}(\rho_\psi) \subseteq \operatorname{Ker}(\varphi)$. Since $\boldsymbol{C}$ is a simple ring and $A/\operatorname{Ker}(\rho_\psi)$ is ring isomorphic to $\boldsymbol{C}$, it follows that $\operatorname{Ker}(\rho_\psi) = \operatorname{Ker}(\varphi)$. Set $\hat{\rho}(\psi) = \varphi$. Then $\hat{\rho}$ is a mapping of Φ_B into Φ_A and we have

$$\boldsymbol{C} \cong A/\operatorname{Ker}(\hat{\rho}(\psi)) = A/\operatorname{Ker}(\rho_\psi) \cong \boldsymbol{C},$$
$$\hat{a}(\hat{\rho}(\psi)) \leftrightarrow a + \operatorname{Ker}(\hat{\rho}(\psi)) = \dot{a} + \operatorname{Ker}(\rho_\psi) \leftrightarrow \rho(a)\hat{}\,(\psi)$$

for each $a \in A$, where the former is an algebra isomorphism and the latter is a ring isomorphism. Therefore a desired map τ_ψ can be obtained as the above composition map from $\boldsymbol{C}$ onto itself. Q.E.D.

Lemma 3. *If A is regular, then $\hat{\rho}$ is continuous on Φ_B.*

Proof. Let ψ be any element of Φ_B, $\{\psi_\lambda\}$ any net in Φ_B which converges to ψ and U any open neighbourhood of $\hat{\rho}(\psi)$. Suppose that A is regular. Then we can find an element a of A such that $\hat{a}(\hat{\rho}(\psi)) = 1$ and $\hat{a} = 0$ on $\Phi_A \backslash U$. By the preceding lemma, we have

$$\lim_\lambda \tau_{\psi_\lambda}(\hat{a}(\hat{\rho}(\psi_\lambda))) = \lim_\lambda \rho(a)\hat{}\,(\psi_\lambda) = \rho(a)\hat{}\,(\psi) = \tau_\psi(\hat{a}(\hat{\rho}(\psi))) = \tau_\psi(1) = 1.$$

Then there exists a λ_0 such that

$$\tau_{\psi_\lambda}(\hat{a}(\hat{\rho}(\psi_\lambda))) \neq 0, \quad \text{i.e.,}\, \hat{a}(\hat{\rho}(\psi_\lambda)) \neq 0 \text{ and so } \hat{\rho}(\psi_\lambda) \in U$$

for every $\lambda \geq \lambda_0$. This means $\lim_\lambda \hat{\rho}(\psi_\lambda) = \hat{\rho}(\psi)$ and the proof is complete. Q.E.D.

Lemma 4. *If ρ is surjective, then $\hat{\rho}$ is injective and condition $(*)$ holds automatically.*

Proof. Suppose that there exist two elements ψ_1 and ψ_2 in Φ_B such that $\psi_1 \neq \psi_2$ and $\hat{\rho}(\psi_1) = \hat{\rho}(\psi_2)$ $(\equiv \varphi \in \Phi_A)$. By Lemma 2, we have

$$\rho(a)\hat{}\,(\psi_1) = \tau_{\psi_1}(\hat{a}(\hat{\rho}(\psi_1))) = \tau_{\psi_1}(\varphi(a)) = \tau_{\psi_1}(0) = 0$$

for every $a \in \operatorname{Ker}(\varphi)$, and hence $\rho(\operatorname{Ker}(\varphi)) \subseteq \operatorname{Ker}(\psi_1)$. Similarly we have $\rho(\operatorname{Ker}(\varphi)) \subseteq \operatorname{Ker}(\psi_2)$ and so $\rho(\operatorname{Ker}(\varphi)) \subseteq \operatorname{Ker}(\psi_1) \cap \operatorname{Ker}(\psi_2)$. But since $\psi_1 \neq \psi_2$, it follows that $\operatorname{Ker}(\psi_1) \cap \operatorname{Ker}(\psi_2) \subsetneqq \operatorname{Ker}(\psi_1)$. Then we obtain

$$\rho(\operatorname{Ker}(\varphi)) \subsetneqq \operatorname{Ker}(\psi_1). \tag{2}$$

Also we have

$$\operatorname{Ker}(\rho) \subseteq \operatorname{Ker}(\varphi) \tag{3}$$

since $\varphi(a) = \hat{a}(\hat{\rho}(\psi_1)) = \tau_{\psi_1}^{-1}(\rho(a)\hat{}\,(\psi_1)) = \tau_{\psi_1}^{-1}(0) = 0$ for every $a \in \operatorname{Ker}(\rho)$. Therefore if ρ is surjective, then we have

$$\begin{aligned}
\boldsymbol{C} &\cong A/\operatorname{Ker}(\varphi) \quad \text{(algebra isomorphic)}\\
&\cong (A/\operatorname{Ker}(\rho))/(\operatorname{Ker}(\varphi)/\operatorname{Ker}(\rho)) \quad \text{(ring isomorphic) (by (3))}\\
&\cong B/\rho(\operatorname{Ker}(\varphi)) \quad \text{(ring isomorphic)}\\
&\supsetneqq \operatorname{Ker}(\psi_1)/\rho(\operatorname{Ker}(\varphi))\\
&\supsetneqq \{0\} \quad \text{(by (2))}.
\end{aligned}$$

But this is a contradiction since $\boldsymbol{C}$ is a simple ring and the proof is complete. Q.E.D.

We are now in a position to prove our main theorem.

Proof of Theorem 1. Let us consider the following three sets:

$$\Phi_B^0 = \{\psi \in \Phi_B : \tau_\psi \text{ is discontinuous}\},$$
$$\Phi_B^1 = \{\psi \in \Phi_B : \tau_\psi(\lambda) = \lambda \text{ for all } \lambda \in \boldsymbol{C}\},$$
$$\Phi_B^2 = \{\psi \in \Phi_B : \tau_\psi(\lambda) = \bar{\lambda} \text{ for all } \lambda \in \boldsymbol{C}\}.$$

In this case, it is easy to see that

$$\Phi_B = \Phi_B^0 \cup \Phi_B^1 \cup \Phi_B^2 \quad \text{(disjoint union)}.$$

Moreover, by definition and Lemma 2, we have that for each $a \in A$, $\rho(a)\hat{} = \hat{a} \circ \hat{\rho}$ on Φ_B^1, $\rho(a)\hat{} = \bar{\hat{a}} \circ \hat{\rho}$ on Φ_B^2 and $\rho(a)\hat{}(\psi) = \tau_\psi(\hat{a}(\hat{\rho}(\psi)))$ for every $\psi \in \Phi_B^0$ and for a certain discontinuous ring automorphism τ_ψ of $\boldsymbol{C}$.

Assume now that A is regular. Then $\hat{\rho}$ is continuous on Φ_B by Lemma 3. We shall show that Φ_B^1 is closed in Φ_B. To do this, let $\{\psi_\lambda\}$ be a net in Φ_B^1 which converges to an element $\psi \in \Phi_B$. Then

$$\hat{a}(\hat{\rho}(\psi)) = \lim_\lambda \hat{a}(\hat{\rho}(\psi_\lambda)) = \lim_\lambda \rho(a)\hat{}(\psi_\lambda) = \rho(a)\hat{}(\psi)$$

for all $a \in A$. Therefore we have

$$\tau_\psi(\rho(a)\hat{}(\psi)) = \tau_\psi(\hat{a}(\hat{\rho}(\psi))) = \rho(a)\hat{}(\psi)$$

for all $a \in A$. By the above fact and the condition $(*)$, we have $\tau_\psi(\lambda) = \lambda$ for every $\lambda \in \boldsymbol{C}$. In other words, $\psi \in \Phi_B^1$ and hence Φ_B^1 is closed in Φ_B. Similarly, it is shown that Φ_B^2 is also closed in Φ_B.

Let us assume that A additionally satisfies the condition (#). We shall show that $\hat{\rho}(\Phi_B^0)$ is a finite set. Suppose not. Then we can choose a sequence $\{\varphi_1, \varphi_2, \dots\}$ in $\hat{\rho}(\Phi_B^0)$ such that each φ_n is an isolated point of the subset $\{\varphi_1, \varphi_2, \dots\}$ (not necessarily an isolated point of $\hat{\rho}(\Phi_B^0)$). For each φ_n, choose an element ψ_n of Φ_B^0 with $\varphi_n = \hat{\rho}(\psi_n)$. Since each τ_{ψ_n} is a discontinuous automorphism of C, it follows from [2, Theorem 2, p. 360] that τ_{ψ_n} maps every disc onto an unbounded set and hence we can take a complex number λ_n such that $|\lambda_n| \leq \frac{1}{2^n}$ and $|\tau_{\psi_n}(\lambda_n)| \geq n$. By the condition (#), there exists an element $a \in A$ such that $\hat{a}(\varphi_n) = \lambda_n$ for each integer $n \geq 1$.

Therefore we have

$$|\rho(a)\hat{}(\psi_n)| = |\tau_{\psi_n}(\hat{a}(\hat{\rho}(\psi_n)))| = |\tau_{\psi_n}(\hat{a}(\varphi_n))| = |\tau_{\psi_n}(\lambda_n)| \geq n$$

for each integer $n \geq 1$. On the other hand, we have

$$|\rho(a)\hat{}(\psi_n)| \leq \|\rho(a)\|$$

for each integer $n \geq 1$. This is a contradiction. Q.E.D.

Corollary 1. *Every ring homomorphism of a commutative* C*-*algebra onto another commutative* C*-*algebra with connected infinite Gelfand space is either linear or anti-linear.*

Proof. Since every commutative C*-algebra is a regular Banach algebra which satisfies the condition (#), the corollary follows immediately from Theorem 1. Q.E.D.

Corollary 2. *Every ring homomorphism of a commutative* C*-*algebra onto another commutative* C*-*algebra whose Gelfand space has no isolated points is continuous.*

Proof. Let A and B be two commutative C*-algebras and ρ a ring homomorphism of A onto B. Assume that the Gelfand space Φ_B of B has no isolated points. By Theorem 1, we can find two clopen subsets Φ_B^1 and Φ_B^2 of Φ_B such that $\Phi_B^1 \cap \Phi_B^2 = \varnothing$, $\Phi_B^1 \cup \Phi_B^2 = \Phi_B$ and $\rho(a)\hat{} = \hat{a} \circ \hat{\rho}$ on Φ_B^1, $\rho(a)\hat{} = \bar{\hat{a}} \circ \hat{\rho}$ on Φ_B^2 for each $a \in A$. Set

$$I_1 = \{x \in B : \hat{x} = 0 \text{ on } \Phi_B^1\} \quad \text{and} \quad I_2 = \{x \in B : \hat{x} = 0 \text{ on } \Phi_B^2\}.$$

Then I_1 and I_2 are closed ideals of B such that $\Phi_{B/I_1} \cong \Phi_B^1$ and $\Phi_{B/I_2} \cong \Phi_B^2$. Let ρ_1 (resp. ρ_2) be the composition map of the canonical map of B onto B/I_1 (resp. B/I_2) and ρ. Then it is easy to see that ρ_1 is an algebra homomorphism of A onto B/I_1 and ρ_2 is an anti-algebra homomorphism of A onto B/I_2. Hence both ρ_1 and ρ_2 are continuous by the Johnson theorem [1]. On the other hand, observe that

$$\|\rho(a)\| = \max(\|\rho_1(a)\|,\ \|\rho_2(a)\|)$$

for all $a \in A$. Therefore ρ must be continuous. Q.E.D.

Lemma 5. *The group algebra $L^1(\boldsymbol{R}^N)$ satisfies the condition (#).*

Proof. Choose a function $f \in L^1(\boldsymbol{R}^N)$ such that $\hat{f}(0) = 1$ and $\hat{f}(\xi) = 0$ for each $\xi \in \boldsymbol{R}^N$ with $\|\xi\| \geq 1$, where $\hat{f}$ denotes the Fourier transform of f. For any fixed $\alpha > 0$ and $b \in \boldsymbol{R}^N$, set

$$f_{a,b}(x) = \alpha^{-N} f(\alpha^{-1}x) e^{-i\alpha^{-1}\langle b,x\rangle}$$

for each $x \in \boldsymbol{R}^N$, where $\langle b, x\rangle$ denotes the inner product of b and x. Then a simple calculation implies that $\|f_{a,b}\|_1 = \|f\|_1$ and $\hat{f}_{a,b}(\xi) = \hat{f}(\alpha\xi + b)$ for every $\xi \in \boldsymbol{R}^N$.

Now to show that $L^1(\boldsymbol{R}^N)$ satisfies the condition (#), let $\lambda_n \in \boldsymbol{C}$ with $|\lambda_n| \leq 1/2^n$ $(n = 1, 2, \dots)$ and $\{\xi_1, \xi_2, \dots\} \subseteq \boldsymbol{R}^N$ such that each ξ_n is an isolated point in $\{\xi_1, \xi_2, \dots\}$. Set

$$\alpha_n = \sup_{n \neq k} \|\xi_n - \xi_k\|^{-1} \qquad (n = 1, 2, \dots).$$

Then all α_n are positive numbers since each ξ_n is an isolated point in $\{\xi_1, \xi_2, \dots\}$. Consider the following function $g(x)$ on $\boldsymbol{R}^N$ defined by

$$g(x) = \sum_{n=1}^{\infty} \lambda_n f_{\alpha_n, -\alpha_n\xi_n}(x) \qquad (x \in \boldsymbol{R}^N).$$

Since $\|\lambda_n f_{\alpha_n,-\alpha_n\xi_n}\|_1 = |\lambda_n|\,\|f\|_1 \leq \frac{1}{2^n}\|f\|_1$ $(n = 1, 2, \dots)$, it follows that g belongs to $L^1(\boldsymbol{R}^N)$ and $\hat{g}(\xi_n) = \lambda_n$ $(n = 1, 2, \dots)$ by the simple calculation. In other words, $L^1(\boldsymbol{R}^N)$ satisfies the condition (#). Q.E.D.

Corollary 3. *A ring automorphism of $L^1(\boldsymbol{R}^N)$ is either linear or anti-linear.*

Proof. This is an immediate consequence of Theorem 1 and Lemma 5. Q.E.D.

For each nonnegative integer n, let $C^n([a, b])$ denote the family of all n-times continuously differentiable complex-valued functions defined on the closed interval $[a, b]$ on $\boldsymbol{R}$. Then $C^n([a, b])$ becomes a semisimple commutative Banach algebra in the usual way and its Gelfand space is homeomorphic to $[a, b]$ (cf. Larsen [4, p. 92]). Let G be a compact abelian group and $1 \leq p < \infty$. Then the L^p-space $L^p(G)$ on G becomes a semisimple commutative Banach algebra under convolution and its Gelfand space is homeomorphic to the dual group of G (cf. Larsen [3, p. 250]). Let S be a set and $\ell^1(S)$ the family of all complex-valued functions f on S such that

$\|f\|_1 = \sum_{s \in S} |f(s)| < \infty$. Then $\ell^1(S)$ becomes a semisimple commutative Banach algebra under the pointwise operations and the norm $\|f\|_1$ and its Gelfand space is homeomorphic to S endowed with the discrete topology (cf. [6, p. 611]).

For these algebras, we have the following result which is similar to Molnar's result [5, Corollary] which asserts that the group algebras $L^1(\boldsymbol{R})$, $L^1(\boldsymbol{T})$ and the disc algebra $A(\boldsymbol{D})$ are not ring homomorphic images of commutative C*-algebras.

Corollary 4. *$L^1(\boldsymbol{R}^N)$, $A(\boldsymbol{D})$ and $C^n([a,b])$ are neither ring homomorphic images of $\ell^1(S)$ nor $L^p(G)$ ($1 \le p < \infty$, G a compact abelian group).*

Proof. Let A be one of the Banach algebras $\ell^1(S)$ and $L^p(G)$, and let B be one of the Banach algebras $C^n([a,b])$, $L^1(\boldsymbol{R}^N)$ and $A(\boldsymbol{D})$. Then A is a commutative regular Banach algebra which satisfies the condition (#). Also the Gelfand space of B is a connected infinite set. Assume that there exists a ring homomorphism of A onto B, say ρ. Then by Theorem 1, ρ is either linear or anti-linear and hence continuous. This implies that $\Phi_{A/\operatorname{Ker}(\rho)}$ is homeomorphic to Φ_B. But since Φ_A is a discrete space, it follows that $\Phi_{A/\operatorname{Ker}(\rho)}$ is also a discrete space. This is a contradiction. Q.E.D.

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Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510, Japan

Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950-2102, Japan

E-mail address: hatori@math.sc.niigata-u.ac.jp