THE MOD 2 COHOMOLOGY OF THE LINEAR GROUPS 
OVER THE RING OF INTEGERS

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Abstract. This paper completely determines the Hopf algebra structure of
the mod 2 cohomology of the linear groups $GL(Z)$, $SL(Z)$ and $St(Z)$ as a
module over the Steenrod algebra, and provides an explicit description of the
generators.

1. Introduction

Recently, J. Rognes and C. Weibel deduced from V. Voevodsky’s proof [V] of
the Milnor conjecture the complete calculation of the 2-torsion of the algebraic K-
theory of the ring of integers $Z$ (see Table 1 of [W] and Theorem 0.6 of [RW]). Of
course, this has immediate consequences on the mod 2 cohomology of the infinite
general linear group $GL(Z)$ and more generally on the understanding of the space
$BGL(Z)^+$. In [Bok], M. Bökstedt tried to construct a 2-adic model for the space $BGL(Z)^+$:
he considered any prime number $p \equiv 3$ or $5$ mod $8$ and introduced a space $J(p)$
which is defined by the pull-back diagram

$$
\begin{array}{ccc}
J(p) & \xrightarrow{h'} & BO \\
\downarrow \phi' & & \downarrow c \\
F\Psi p & \xrightarrow{b} & BU,
\end{array}
$$

where $F\Psi p$ is the fiber of $(\Psi p - 1) : BU \rightarrow BU$ (recall that $F\Psi p \simeq BGL(\mathbb{F}_p)^+$
by Theorem 7 of [Q2]), $b$ is the Brauer lifting and $c$ is the complexification. The
fibers of the horizontal maps are homotopy equivalent to the unitary group $U$.
He was actually more precisely interested in the covering space $JK(Z,p)$ of $J(p)$
corresponding to the cyclic subgroup of order 2 of $\pi_1 J(p) \cong Z \oplus Z/2$. Bökstedt’s
definition of the space $JK(Z,p)$ (see [Bok], Definition 1.7 and the proof of Lemma
2.1) is based on the Adams conjecture and on the calculation of the 2-primary part
of the homotopy groups of $(F\Psi p)^2$ which is the same, in dimensions $\equiv 3$ mod 4, for

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all primes \( p \equiv 3 \) or \( 5 \) mod 8 (this explains the choice of \( p \); see Section 3 of [Au] for more details). Notice that the space \( JK(\mathbb{Z}, p) \), in the case \( p = 3 \), appears also in Section 4 of [DF] and in [M]. After completion at the prime 2, Bökstedt constructed a map

\[
\varphi : (BGL(\mathbb{Z})^+)^\wedge_2 \longrightarrow JK(\mathbb{Z}, p)^\wedge_2
\]

which induces a split surjection on all homotopy groups (see [Bok], Diagram 1.9). Recall that the localization exact sequence in K-theory implies that

\[
(BGL(\mathbb{Z}[\frac{1}{2}])^+)^\wedge_2 \simeq (BGL(\mathbb{Z})^+)^\wedge_2 \times (S^1)^\wedge_2.
\]

Therefore, \( \varphi \) provides a map

\[
\tilde{\varphi} : (BGL(\mathbb{Z}[\frac{1}{2}])^+)^\wedge_2 \longrightarrow J(p)^\wedge_2
\]

which also induces a split surjection on all homotopy groups. Bökstedt’s idea was indeed excellent because now the 2-torsion of \( K_*(\mathbb{Z}) \) is known and turns out to be isomorphic to the 2-torsion of \( \pi_*(JK(\mathbb{Z}, p)) \) (according to Table 1 of [W] and Theorem 0.6 of [RW]); therefore, \( \varphi \) and \( \tilde{\varphi} \) are actually homotopy equivalences. Observe in particular that the homotopy type of \( (JK(\mathbb{Z}, p))^\wedge_2 \) does not depend on \( p \) (for \( p \equiv 3 \) or \( 5 \) mod 8). Consequently, we obtain for all primes \( p \equiv 3 \) or \( 5 \) mod 8 the pull-back diagram (see also Corollary 8 of [W])

\[
\begin{array}{ccc}
(BGL(\mathbb{Z})^+)^\wedge_2 \times (S^1)^\wedge_2 & \longrightarrow & BO^\wedge_2 \\
\downarrow f_p & & \downarrow \\
(F\Psi_p)^\wedge_2 & \longrightarrow & BU^\wedge_2,
\end{array}
\]

and the commutative diagram (where both rows are fibrations)

\[
(*)
\begin{array}{ccc}
SU^\wedge_2 & \eta \longrightarrow & (BGL(\mathbb{Z})^+)^\wedge_2 \\
\downarrow \zeta & & \downarrow f_p \\
U^\wedge_2 & \theta \longrightarrow & (F\Psi_p)^\wedge_2
\end{array}
\text{with } (BGL(\mathbb{Z})^+)^\wedge_2 \leftrightarrow (BGL(\mathbb{Z})^+)^\wedge_2 \times (S^1)^\wedge_2
\]

where \( f_p \) and \( h' \) respectively, and \( \zeta \) the 2-completion of the inclusion \( SU \hookrightarrow U \simeq SU \times S^1 \). According to Section 2 of [Bok], the map \( h \) is induced by the inclusion \( \mathbb{Z} \hookrightarrow \mathbb{R} \) and for all odd primes \( p \), the diagram

\[
\begin{array}{ccc}
(BGL(\mathbb{Z})^+)^\wedge_2 & \xrightarrow{\sim} & JK(\mathbb{Z}, p)^\wedge_2 \\
\downarrow \text{red}_p & & \downarrow f_p \\
(BGL(\mathbb{Z}/p)^+)^\wedge_2 & \xrightarrow{\sim} & (F\Psi_p)^\wedge_2,
\end{array}
\]

where \( \text{red}_p \) is the map induced by the reduction mod \( p : GL(\mathbb{Z}) \to GL(\mathbb{Z}/p) \), is homotopy commutative. Thus, we may assume that the map \( f_p \) in the diagram (\( * \)) is induced by the reduction mod \( p \).
S. Mitchell computed the mod 2 homology of the space $JK(Z,3)$ in Theorem 4.3 of [M]; because of the above homotopy equivalence $(BGL(Z)^+)^2 \simeq JK(Z,3)^2$, this provides the calculation of $H_*(BGL(Z)^+;Z/2)$ and by dualization the determination of the Hopf algebra structure of $H^*(BGL(Z)^+;Z/2)$ as a module over the Steenrod algebra $\mathcal{A}$ (see [M], Remark 4.5). However, Mitchell’s argument does not give explicit generators of $H^*(BGL(Z)^+;Z/2)$. The first goal of the present paper is to use the above commutative diagram (*) in order to get a direct proof of Michell’s result.

**Theorem.** There is an isomorphism of Hopf algebras and of modules over the Steenrod algebra

$$\alpha : H^*(BGL(Z)^+;Z/2) \cong H^*(BO;Z/2) \otimes H^*(SU;Z/2).$$

Recall that $H^*(BO;Z/2) \cong Z/2[w_1, w_2, \ldots]$ and $H^*(SU;Z/2) \cong \Lambda(v_3, v_5, \ldots)$, where $\deg(w_j) = j$ and $\deg(v_{2k-1}) = 2k - 1$.

In fact, the main objective of this paper is to describe explicitly the generators of $H^*(BGL(Z)^+;Z/2)$. The generators of the polynomial part are the Stiefel-Whitney classes, also denoted by $w$, coming from $H^*(BO;Z/2)$ via the homomorphism induced by $h$. On the other hand, we identify precisely (see Definitions 5 and 10 and Remark 4.5) the exterior generators $u_{2k-1}$ of degree $2k-1$ in $H^*(BGL(Z)^+;Z/2)$, corresponding to $1 \otimes v_{2k-1}$ under the above isomorphism $\alpha$, in terms of the image of the homomorphism

$$f_p^* : H^*(F\Psi^p;Z/2) \cong H^*(BGL(F_p)^+;Z/2) \to H^*(BGL(Z)^+;Z/2)$$

induced by the reduction mod $p$ for $p \equiv 5 \pmod{8}$ (they actually do not depend on the choice of $p$). We show that the classes $u_{2k-1}$ are primitive cohomology classes and compute the action of the Steenrod squares on them. Therefore, we get an isomorphism of Hopf algebras and of modules over the Steenrod algebra

$$H^*(BGL(Z)^+;Z/2) \cong Z/2[w_1, w_2, \ldots] \otimes \Lambda(u_3, u_5, \ldots)$$

and we deduce that the isomorphism $\alpha$ is unique (see Theorem 11). We also obtain an explicit formula relating the classes $u_{2k-1}$ to the image of the homomorphism $f_p^*$ for all primes $p \equiv 3 \pmod{8}$ (see Theorem 13).

This provides a complete description of the mod 2 cohomology of the infinite general linear group $GL(Z)$. In the remainder of the paper we compute the mod 2 cohomology of the infinite special linear group $SL(Z)$ and of the infinite Steinberg group $St(Z)$ (see Corollary 15, Theorem 17 and Remark 18).

2. The mod 2 cohomology of the linear groups $GL(Z)$ and $SL(Z)$

**Theorem 1.** There is an isomorphism of Hopf algebras and of modules over the Steenrod algebra

$$\alpha : H^*(BGL(Z)^+;Z/2) \cong H^*(BO;Z/2) \otimes H^*(SU;Z/2).$$

**Proof.** As mentioned in the introduction, this follows indirectly from [M], Theorem 4.3 and Remark 4.5. Here is a direct argument. Let $Q$ denote the subgroup of diagonal matrices in $GL(Z)$ and let $\lambda : BQ \to BGL(Z)^+$ be the map induced by the inclusion $Q \hookrightarrow GL(Z)$. It is known by Theorem 22.7 of [Bor] that the composition $h \lambda : BQ \to BO$ induces an injective homomorphism $\lambda^*h^* : H^*(BO;Z/2) \cong Z/2[w_1, w_2, \ldots] \to H^*(BQ;Z/2) \cong \lim_{\mapsto} \Z/2[z_1, z_2, \ldots, z_m]$ (with $\deg(z_i) = 1$) and that $\lambda^*h^*(w_j) = \sigma_j$, where $\sigma_j$ is the element of $H^j(BQ;Z/2)$ whose restriction
to $\mathbb{Z}/2[z_1, z_2, \ldots, z_m]$ is the $j$-th elementary symmetric function in the $m$ variables $z_1, \ldots, z_m$, for all $m \geq j$. This implies that the infinite loop map $h$ induces an injective homomorphism $h^* : H^*(BO; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. Therefore, Theorem 15.2 of [Bor] shows that the Serre spectral sequence of the fibration

$$SU_2 \cong (BGL(\mathbb{Z})^+)_2 \xrightarrow{h} BO_2$$

collapses (see also Corollary 4.3 of [DF]) and we get additively the desired isomorphism. Since $(BGL(\mathbb{Z})^+)_2$ is an $H$-space, the maps $\lambda$ and $\eta$ produce an $H$-map

$$\psi : BQ \times SU_2 \longrightarrow (BGL(\mathbb{Z})^+)_2$$

which induces an injective $A$-module Hopf algebra homomorphism

$$\psi^* : H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \longrightarrow H^*(BQ; \mathbb{Z}/2) \otimes H^*(SU; \mathbb{Z}/2).$$

Moreover, the fact that $\lambda^* : H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \to H^*(BO; \mathbb{Z}/2)$ also satisfies $\lambda^*(w_j) = \sigma_j$ (see Lemma 1.1 of [Ar1]) implies that the image of $\psi^*$ is isomorphic to $R \otimes H^*(SU; \mathbb{Z}/2)$, where $R$ is the subalgebra of $H^*(BO; \mathbb{Z}/2)$ generated by the elementary symmetric functions $\sigma_j$. On the other hand, the image of the injective $A$-module Hopf algebra homomorphism

$$\lambda^* h^* \otimes 1 : H^*(BO; \mathbb{Z}/2) \otimes H^*(SU; \mathbb{Z}/2) \longrightarrow H^*(BO; \mathbb{Z}/2) \otimes H^*(SU; \mathbb{Z}/2)$$

is also $R \otimes H^*(SU; \mathbb{Z}/2)$. This provides the statement of the theorem.

In order to get a more precise picture of $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$, let us identify its generators and understand the action of the Steenrod algebra on them. For $j \geq 1$ let us write $w_j \in H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$ for the image of the $j$-th universal Stiefel-Whitney class in $H^*(BO; \mathbb{Z}/2)$ under the homomorphism $h^* : H^*(BO; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. The action of the Steenrod algebra on the Stiefel-Whitney classes is known by Wu's formula (see for instance [MT], Part I, p. 141). It remains to identify the exterior generators of $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. This will be done by using the homomorphism $f_p^* : H^*(F\Psi^p; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$ induced by the map $f_p$.

Let us first recall some properties of $H^*(F\Psi^p; \mathbb{Z}/2) \cong H^*(BGL(\mathbb{F}_p)^+; \mathbb{Z}/2)$. According to Quillen's calculation and notation (see [Q2]), if $p$ is a prime $\equiv 5 \mod 8$, then

$$H^*(F\Psi^p; \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, \ldots] \otimes \Lambda(e_1, e_2, \ldots),$$

where $\deg c_j = 2j$ and $\deg e_k = 2k-1$; if $p$ is a prime $\equiv 3 \mod 8$, then $H^*(F\Psi^p; \mathbb{Z}/2)$ is also generated by the classes $c_j$ and $e_k$ $(j \geq 1, k \geq 1)$, but one has the relations

$$e_k^2 = c_{2k-1} + \sum_{j=1}^{k-1} c_j c_{2k-1-j}$$

for $k \geq 1$, and $H^*(F\Psi^p; \mathbb{Z}/2)$ is polynomial:

$$H^*(F\Psi^p; \mathbb{Z}/2) \cong \mathbb{Z}/2[e_1, e_2, \ldots, c_2, c_4, \ldots]$$

(see also Section IV.8 of [FP]). In both cases, $c_j$ is the image under $b^* : H^*(BU; \mathbb{Z}/2) \to H^*(F\Psi^p; \mathbb{Z}/2)$ of the reduction mod 2 of the $j$-th universal Chern class in $H^2(BU; \mathbb{Z})$ and a spectral sequence argument shows that

$$\theta^* : H^*(F\Psi^p; \mathbb{Z}/2) \to H^*(U; \mathbb{Z}/2) \cong \Lambda(v_1, v_2, \ldots)$$
satisfies $\theta^*(e_k) = v_{2k-1}$ for $k \geq 1$. For a prime $p \equiv 3$ or $5 \mod 8$, consider the homomorphism $f^*_p : H^*(F\Psi p; \mathbb{Z}/2) \rightarrow H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$ induced by $f_p$. For all $j \geq 1$, it is well known (see also Lemma 1.4 of [Ar1]) that

$$f^*_p(e_j) = w_j^2$$

and we established in [Ar2] for $k \geq 2$ the nonvanishing of the exterior class $f^*_p(e_k)$ if $p \equiv 5 \mod 8$, respectively of the exterior class

$$\gamma_k = f^*_p(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1}$$

of degree $2k - 1$ if $p \equiv 3 \mod 8$.

Let us mention the effect of the Steenrod squares on these cohomology classes.

**Lemma 2.** (a) In $H^*(SU; \mathbb{Z}/2)$, $Sq^{2i} v_{2k-1} = \binom{k-1}{i} v_{2k+2i-1}$ for $k \geq 2$, $1 \leq i < k$, and $Sq^{2i-1} v_{2k-1} = 0$ for $k \geq 2$, $1 \leq i < k$.

(b) In $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$, for any odd prime $p$, for $k \geq 1$ and $1 \leq i < k$,

$$Sq^{2i} f^*_p(e_k) = \left(\binom{k-1}{i} f^*_p(e_{k+i}) + \sum_{j=1}^i \binom{k-j-1}{i-j} (w_j^2 f^*_p(e_{k+i-j}) + w_{k+i-j}^2 f^*_p(e_j))\right).$$

(c) In $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$, for $k \geq 1$ and $1 \leq i < k$,

$$Sq^{2i-1} f^*_p(e_k) = \begin{cases} 0, & \text{if } p \equiv 1 \mod 4 \text{ or if } p \equiv 3 \mod 4 \text{ and } k - i \text{ is odd}, \\ \sum_{j=0}^{i-1} \binom{k-j-1}{i-j-1} w_j^2 w_{k+i-j-1}, & \text{if } p \equiv 3 \mod 4 \text{ and } k - i \text{ is even}. \end{cases}$$

(d) In $H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$, for any prime $p \equiv 3 \mod 8$ and for $k \geq 1$,

$$Sq^{2i} \gamma_k = \left(\binom{k-1}{i} \gamma_{k+i} + \sum_{j=1}^i \binom{k-j-1}{i-j} (w_j^2 \gamma_{k+i-j} + w_{k+i-j}^2 \gamma_j)\right)$$

for $1 \leq i < k$ and $Sq^{2i-1} \gamma_k = 0$ for $1 \leq i \leq k$.

**Proof.** Lemma 4 of [Ar2] gives the following information on the action of the Steenrod squares on the classes $e_k \in H^*(F\Psi p; \mathbb{Z}/2)$ for $k \geq 1$: for any odd prime $p$ and for $1 \leq i < k$,

$$Sq^{2i} e_k = \left(\binom{k-1}{i} e_{k+i} + \sum_{j=1}^i \binom{k-j-1}{i-j} (c_j e_{k+i-j} + e_{k+i-j} e_j)\right),$$

and for $1 \leq i \leq k$,

$$Sq^{2i-1} e_k = \begin{cases} 0, & \text{if } p \equiv 1 \mod 4 \text{ or if } p \equiv 3 \mod 4 \text{ and } k - i \text{ is odd}, \\ \sum_{j=0}^{i-1} \binom{k-j-1}{i-j-1} c_j e_{k+i-j-1}, & \text{if } p \equiv 3 \mod 4 \text{ and } k - i \text{ is even}. \end{cases}$$

The formula (a) is well known but can be deduced from the previous equalities because the composition $\xi^* \theta^*: H^*(F\Psi p; \mathbb{Z}/2) \rightarrow H^*(SU; \mathbb{Z}/2)$ satisfies $\xi^* \theta^*(e_k) = v_{2k-1}$ for $k \geq 2$ and $\xi^* \theta^*(e_j) = 0$ for $j \geq 1$. The statements (b) and (c) follow directly since $Sq^{2i} f^*_p(e_k) = f^*_p(Sq^{2i} e_k)$ and $f^*_p(e_j) = w_j^2$ for $j \geq 1$. In order to get (d), let us consider again the homomorphism $\lambda^*: H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \rightarrow$
Since the classes $\gamma_k$ are exterior, they belong to the kernel of $\lambda^*$ and $\lambda^*(Sq^{2i}\gamma_k) = 0$. However, the injectivity of $\lambda^*$ on Stiefel-Whitney classes implies that the element of $\mathbb{Z}/2[w_1, w_2, \ldots]$ in the last formula vanishes. The assertion (c) shows that $Sq^{2i-1}\gamma_k$ is an element of $\mathbb{Z}/2[w_1, w_2, \ldots]$ and one deduces similarly that $Sq^{2i-1}\gamma_k = 0$.

Our argument will be based on the understanding of the homomorphism
\[ \mu^*: H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \to H^*(BGL(\mathbb{Z})^+ \times BGL(\mathbb{Z})^+; \mathbb{Z}/2) \]
\[ \cong H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \otimes H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \]
induced by the H-space structure $\mu$ of $BGL(\mathbb{Z})^+$.

**Lemma 3.** (a) For any $j \geq 1$,
\[ \mu^*(w_j) = \sum_{s=0}^{j} w_s \otimes w_{j-s} . \]

(b) For any prime $p \equiv 5 \mod 8$ and any integer $k \geq 2$,
\[ \mu^*(f_p^*(e_k)) = f_p^*(e_k) \otimes 1 + 1 \otimes f_p^*(e_k) + \sum_{\ell=1}^{k-2} (w_\ell^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_\ell^2) . \]

(c) For any prime $p \equiv 3 \mod 8$ and any integer $k \geq 2$,
\[ \mu^*(\gamma_k) = \gamma_k \otimes 1 + 1 \otimes \gamma_k + \sum_{\ell=1}^{k-2} (w_\ell^2 \otimes \gamma_{k-\ell} + \gamma_{k-\ell} \otimes w_\ell^2) . \]

**Proof.** Assertion (a) is known (see for instance [MT], Part I, p. 140). If $\nu$ denotes the H-space structure of $F\Psi^p$, Proposition 2 of [Q2] implies that
\[ \mu^*(f_p^*(e_k)) = f_p^*(\nu^*(e_k)) = f_p^*(\sum_{\ell=0}^{k} (c_\ell \otimes e_{k-\ell} + e_{k-\ell} \otimes c_\ell)) \]
\[ = \sum_{\ell=0}^{k} (w_\ell^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_\ell^2) . \]
for any odd prime \( p \). If \( p \equiv 5 \) mod 8, \( f_p^*(e_1) \) vanishes since \( e_1 \) is exterior and one gets immediately (b). If \( p \equiv 3 \) mod 8, the definition of \( \gamma_k \),

\[
\gamma_k = f_p^*(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1},
\]

shows that

\[
\mu^*(\gamma_k) = \sum_{\ell=0}^{k} \left( w_{\ell}^2 \otimes f_p^*(e_{k-\ell}) + f_p^*(e_{k-\ell}) \otimes w_{\ell}^2 \right) + (\text{element of } \mathbb{Z}/2[w_1, w_2, \ldots]).
\]

Since \( p \equiv 3 \) mod 8, it turns out that \( f_p^*(e_1) = w_1 \) and consequently that

\[
\mu^*(\gamma_k) = \gamma_k \otimes 1 + 1 \otimes \gamma_k + \sum_{\ell=1}^{k-2} \left( w_{\ell}^2 \otimes \gamma_{k-\ell} + \gamma_{k-\ell} \otimes w_{\ell}^2 \right)
+ (\text{element of } \mathbb{Z}/2[w_1, w_2, \ldots]).
\]

However, the element of \( \mathbb{Z}/2[w_1, w_2, \ldots] \) in that formula must be trivial since \( \mu^*(\gamma_k) \) is exterior. This implies the last assertion. \( \square \)

Now, let \( p \) be a prime \( \equiv 5 \) mod 8 and \( k \) an integer \( \geq 2 \). Consider an integer \( m \geq k \), \( C \) the cyclic group of order \( p - 1 \) and

\[
H^*(BC^m; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, \ldots, x_m] \otimes \Lambda(y_1, y_2, \ldots, y_m)
\]

defined by \( d \) and \( d(y_i) = 0 \). Then, look at the homomorphism \( \rho : H^*(F \Psi^p; \mathbb{Z}/2) \to H^*(BC^m; \mathbb{Z}/2) \), introduced in [Q2], p. 563–565, which is injective in dimensions \( \leq 2m \) (and in particular in dimensions \( \leq 2k \)) since its kernel is the ideal generated by the elements \( c_j \) and \( e_j \) for \( j > m \), and which fulfills \( \rho(c_j) = s_j \) and \( \rho(e_j) = d(s_j) \) for \( 1 \leq j \leq m \), where \( s_j \) denotes the \( j \)-th elementary symmetric function in \( x_1, x_2, \ldots, x_m \). For \( k \geq 1 \), define the exterior class

\[
\xi_k = \sum_{j=1}^{m} x_j^{k-1} y_j \in H^{2k-1}(BC^m; \mathbb{Z}/2).
\]

Since \( s_k = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \), one has

\[
d(s_k) = \sum_{i_1 < i_2 < \ldots < i_k} \sum_{\ell} x_{i_1} \cdots \widehat{x_{i_\ell}} \cdots x_{i_k} y_i.
\]

Then, consider the difference

\[
d(s_k) - s_{k-1} \xi_1 = \sum_{i_1 < i_2 < \ldots < i_k} \sum_{\ell} x_{i_1} \cdots \widehat{x_{i_\ell}} \cdots x_{i_k} y_i
- \sum_{i_1 < i_2 < \ldots < i_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \sum_i y_i
= \sum_{i_1 < i_2 < \ldots < i_{k-1}} \sum_{\ell} x_{i_1} x_{i_2} \cdots x_{i_\ell} \cdots x_{i_{k-1}} y_i
= \sum_{i_1 < i_2 < \ldots < i_{k-1}} \sum_{\ell} x_{i_1} x_{i_2} \cdots \widehat{x_{i_\ell}} \cdots x_{i_{k-1}} x_{i_\ell} y_i.
\]
From this formula, one may compute the difference
$$d(s_k) - s_{k-1} \xi_1 - s_{k-2} \xi_2 = \sum_{i_1 < i_2 < \cdots < i_{k-2}} \sum_{\ell} x_{i_1} x_{i_2} \cdots \bar{x}_{i_\ell} \cdots x_{i_{k-2}} \bar{x}_{i_{\ell}} y_{i_\ell}$$
and obtain by induction
$$d(s_k) = \xi_k + \sum_{j=1}^{k-1} s_j \xi_{k-j}$$
for $k \geq 2$. Since $\rho(e_k) = d(s_k)$ and $\rho(c_j) = s_j$, we get
$$\rho(e_k) = \xi_k + \sum_{j=1}^{k-1} \rho(c_j) \xi_{k-j}.$$  
This implies inductively that the exterior class $\xi_k$ belongs to the image of $\rho$ and the injectivity of $\rho$ in dimensions $\leq 2k$ produces the following lemma.

**Lemma 4.** For $p \equiv 5 \mod 8$ and for any $k \geq 2$, the class $e_k \in H^{2k-1}(F^p; \mathbb{Z}/2)$ satisfies
$$e_k = \rho^{-1}(\xi_k) + \sum_{j=1}^{k-1} c_j \rho^{-1}(\xi_{k-j}).$$

**Definition 5.** Let $p$ be a prime $\equiv 5 \mod 8$. For all integers $k \geq 2$, let us define the exterior class $u_{2k-1}(p) = f_p^*(\rho^{-1}(\xi_k)) \in H^{2k-1}(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$, where $f_p^*$ denotes the homomorphism $H^*(F^p; \mathbb{Z}/2) \rightarrow H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$ induced by $f_p$. Observe that this definition does not depend on the choice of $m \geq k$. Notice also that $f_p^*(\rho^{-1}(\xi_1)) = f_p^*(e_1) = 0$.

**Proposition 6.** For any prime $p \equiv 5 \mod 8$ and for $k \geq 2$, one has:

(a) $u_{2k-1}(p) = f_p^*(e_k) + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}(p)$,

(b) the homomorphism $\eta^* : H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \rightarrow H^*(SU; \mathbb{Z}/2)$ fulfills $\eta^*(u_{2k-1}(p)) = v_{2k-1}$.

*Proof.* Lemma 4 implies that
$$f_p^*(e_k) = u_{2k-1}(p) + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}(p)$$
since $f_p^*(\rho^{-1}(\xi_1)) = 0$. Consequently, (b) follows directly from the commutativity of the following diagram induced by the diagram $(\ast)$ of the introduction

$$
\begin{array}{ccc}
H^*(SU; \mathbb{Z}/2) & \xrightarrow{\eta^*} & H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \\
\uparrow{\zeta^*} & & \uparrow{f_p^*} \\
H^*(U; \mathbb{Z}/2) & \xrightarrow{\theta^*} & H^*(F^p; \mathbb{Z}/2) \\
& & \downarrow{b^*} \\
& & H^*(BU; \mathbb{Z}/2),
\end{array}
$$

because $\eta^* f_p^*(e_k) = \zeta^* \theta^* (e_k) = v_{2k-1}$ and $\eta^* (w_j) = 0$ for all $k \geq 2, j \geq 1$.  

**Proposition 7.** For any prime $p \equiv 5 \mod 8$ and for any integer $k \geq 2$, the element $u_{2k-1}(p)$ is a primitive cohomology class in $H^{2k-1}(BGL(\mathbb{Z})^+; \mathbb{Z}/2)$. 


The last sum can be written as follows:

\[ u = \sum_{j=1}^{k-2} w_{j}^{2} u_{2k-2j-1}(p) . \]

Consequently, \( \mu \) is exactly one nontrivial primitive exterior class in each odd degree. We proceed by induction on \( k \). We just established in Proposition 6 that

\[ u_{2k-1}(p) = f_{p}^{*}(e_{k}) + \sum_{j=1}^{k-2} w_{j}^{2} u_{2k-2j-1}(p) . \]

For instance, \( u_{3}(p) = f_{p}^{*}(e_{2}) \) and it follows from Lemma 3 (b) that \( \mu^{*}(u_{3}(p)) = u_{3}(p) \otimes 1 + 1 \otimes u_{3}(p) \). We then may deduce from Lemma 3 (a) and (b) and the induction hypothesis that

\[ u^{*}(u_{2k-1}(p)) = f_{p}^{*}(e_{k}) \otimes 1 + 1 \otimes f_{p}^{*}(e_{k}) + \sum_{j=1}^{k-2} (w_{j}^{2} f_{p}^{*}(e_{k-j-1}) + f_{p}^{*}(e_{k-j-1}) \otimes w_{j}^{2} ) \]

and therefore that

\[ u^{*}(u_{2k-1}(p)) = u_{2k-1}(p) \otimes 1 + 1 \otimes u_{2k-1}(p) \]

\[ + \sum_{j=1}^{k-2} \left( w_{j}^{2} \otimes u_{2k-2j-1}(p) + u_{2k-2j-1}(p) \otimes w_{j}^{2} \right) \]

\[ + \sum_{j=1}^{k-3} \sum_{i=1}^{k-i} \left( w_{j}^{2} w_{i}^{2} u_{2k-2j-1}(p) + w_{i}^{2} u_{2k-2j-1}(p) \otimes w_{j}^{2} \right) \]

\[ + \sum_{j=1}^{k-2} \left( w_{j}^{2} \otimes u_{2k-2j-1}(p) + u_{2k-2j-1}(p) \otimes w_{j}^{2} \right) \]

\[ + \sum_{j=1}^{k-2} \sum_{s=1}^{j-1} \left( w_{s}^{2} u_{2k-2j-1}(p) \otimes w_{j-s}^{2} + w_{s}^{2} \otimes w_{j-s}^{2} u_{2k-2j-1}(p) \right) . \]

The last sum can be written as follows:

\[ \sum_{j=1}^{k-2} \sum_{s=1}^{j-1} \left( w_{s}^{2} u_{2k-2j-1}(p) \otimes w_{j-s}^{2} + w_{s}^{2} \otimes w_{j-s}^{2} u_{2k-2j-1}(p) \right) \]

\[ = \sum_{j=1}^{k-3} \sum_{s=1}^{j-1} \left( w_{s}^{2} \otimes u_{2k-2j-1}(p) + u_{2k-2j-1}(p) \otimes w_{s}^{2} \right) \]

\[ = \sum_{s=1}^{k-3} \sum_{t=1}^{s-1} \left( w_{s}^{2} \otimes u_{2k-2s-2t-1}(p) + u_{2k-2s-2t-1}(p) \otimes w_{s}^{2} \right) . \]

Consequently, \( \mu^{*}(u_{2k-1}(p)) = u_{2k-1}(p) \otimes 1 + 1 \otimes u_{2k-1}(p) \) and \( u_{2k-1}(p) \) is primitive.

Remark 8. Since we know that the Hopf algebra structure of \( H^{*}(BGL(\mathbb{Z}^{+}; \mathbb{Z}/2) \) by Theorem 1 (or [M], Theorem 4.3 and Remark 4.5), it is obvious that there is exactly one nontrivial primitive exterior class in each odd degree \( \geq 3 \) of \( H^{*}(BGL(\mathbb{Z}^{+}; \mathbb{Z}/2) \). However, let us show it again by a computational argument.
Lemma 9. Consider the homomorphism \( \eta^*: H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \rightarrow H^*(SU; \mathbb{Z}/2) \). For \( k \geq 2 \), let \( u''_{2k-1} \) and \( u'_j \) be primitive exterior classes of degree \( 2k-1 \) in \( H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \) such that \( \eta^*(u''_{2k-1}) = \eta^*(u''_{2k-1}) = v_{2k-1} \). Then \( u''_{2k-1} = u'_j \).

Proof. Observe first that \( u'_j = u''_{2j-1} \) since there is only one exterior class of degree 3 in \( H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \). Then, let us define \( \tilde{u}_{2k-1} = u''_{2k-1} - u''_{2k-1} \) for all \( k \geq 2 \) and prove by induction on \( k \) that \( \tilde{u}_{2k-1} = 0 \). Since \( \tilde{u}_{2k-1} \) is exterior and belongs to the kernel of \( \eta^* \), the induction hypothesis shows that one can write

\[
\tilde{u}_{2k-1} = \sum_{s=3}^{2k-2} u'(s)w(s),
\]

where \( u'(s) \) is an element of degree \( s \) in \( \Lambda(u'_3, u'_5, \ldots, u'_s) \) and \( w(s) \) is an element of degree \( 2k-s-1 \) in \( \mathbb{Z}/2[w_1, w_2, \ldots] \). However, the primitivity of the classes \( u'_j \) and Lemma 3 (a) provide an explicit computation of \( \mu^*(\tilde{u}_{2k-1}) \) which contradicts the primitivity of \( \tilde{u}_{2k-1} \) unless one has \( \tilde{u}_{2k-1} = 0 \).

Thus, we are finally able to define the exterior generators of \( H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \) (see also Remark 14 below).

Definition 10. Because of Proposition 7 and Remark 8, we may conclude that the classes \( u_{2k-1}(p) \) do not depend on \( p \). Therefore, for \( k \geq 2 \), we can define \( u_{2k-1} = u_{2k-1}(p) \) in \( H^{2k-1}(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \) for any prime \( p \equiv 5 \, \text{mod} \, 8 \). Since the image of \( u_{2k-1} \) under \( \eta^* \) is \( v_{2k-1} \), the classes \( u_{2k-1} \) are nontrivial algebraically independent exterior classes in \( H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \). See also Remark 14 for another definition of the classes \( u_{2k-1} \).

The following consequence follows immediately from Proposition 6 (b) and Remark 8.

Theorem 11. The isomorphism \( \alpha : H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \cong H^*(SU; \mathbb{Z}/2) \otimes H^*(BO; \mathbb{Z}/2) \) given by Theorem 1 is unique and satisfies \( \alpha(u_{2k-1}) = 1 \otimes v_{2k-1} \) for \( k \geq 2 \). Therefore, there is an isomorphism of \( \mathcal{A} \)-module Hopf algebras

\[
H^*(BGL(\mathbb{Z})^+; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \ldots] \otimes \Lambda(u_3, u_5, \ldots).
\]

It follows from Theorem 11 and Lemma 2 (a) that the action of the Steenrod algebra on the classes \( u_{2k-1} \) is described by the following Lemma 12. However, we mention here another proof, based on the definition of \( \xi_k \), which provides an explicit computational argument for the existence of the isomorphism of \( \mathcal{A} \)-module Hopf algebras \( \alpha \).

Lemma 12. For all \( k \geq 2 \), \( Sq^{2i}u_{2k-1} = \binom{k-1}{i}u_{2k+2i-1} \) for \( 1 \leq i < k \) and \( Sq^{2i-1}u_{2k-1} = 0 \) for \( 1 \leq i \leq k \).

Proof. It is sufficient to prove the assertion for the classes \( u_{2k-1}(p) \) where \( p \) is any prime \( \equiv 5 \, \text{mod} \, 8 \). This follows from the injectivity of the map \( \rho \) which was explained just after the proof of Lemma 3 (if \( m \) is large enough) and from the computations \( Sq^{2i-1}\xi_k = 0 \) and

\[
Sq^{2i}\xi_k = \sum_{j=1}^{m} Sq^{2i}x_j\xi_k y_j = \sum_{j=1}^{m} \binom{k-1}{i}x_j^{k+i-1}y_j = \binom{k-1}{i} \xi_k^{k+i}.
\]
It is even possible to describe the classes $u_{2k-1}$ in terms of the image of $f_p^*$ for all primes $p \equiv 3$ or $5 \mod 8$.

**Theorem 13.** For $k \geq 2$, the classes $u_{2k-1} \in H^{2k-1}(BGL(Z)^+; \mathbb{Z}/2)$ satisfy

$$u_{2k-1} = \begin{cases} 
  f_p^*(e_k) + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}, & \text{if } p \equiv 5 \mod 8, \\
  f_p^*(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1} + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}, & \text{if } p \equiv 3 \mod 8,
\end{cases}$$

where $f_p^*$ denotes the homomorphism $H^*(F\Psi; \mathbb{Z}/2) \cong H^*(BGL(\mathbb{F}_p)^+; \mathbb{Z}/2) \to H^*(BGL(Z)^+; \mathbb{Z}/2)$ induced by the reduction mod $p : GL(Z) \to GL(\mathbb{F}_p)$.

**Proof.** If $p \equiv 5 \mod 8$, the statement is given by Proposition 6 (a). Observe in particular that $u_{2k-1}$ can be written as follows: $u_{2k-1} = F_k(f_p^*(e_2), f_p^*(e_3), \ldots, f_p^*(e_k))$, where $F_k$ is a polynomial with coefficients in $\mathbb{Z}/2[w_1, w_2, \ldots]$. If $p \equiv 3 \mod 8$, consider again

$$\gamma_k = f_p^*(e_k) + w_{2k-1} + \sum_{j=1}^{k-1} w_j w_{2k-j-1}$$

and define $\tilde{u}_{2k-1} = F_k(\gamma_2, \gamma_3, \ldots, \gamma_k)$. It is obvious that $\tilde{u}_{2k-1}$ is an exterior class and easy to check as in the proof of Proposition 6 that $\eta^*(\tilde{u}_{2k-1}) = u_{2k-1}$. Moreover, observe that the homomorphism

$$\mu^* : H^*(BGL(Z)^+; \mathbb{Z}/2) \to H^*(BGL(Z)^+ \times BGL(Z)^+; \mathbb{Z}/2)$$

acts on $\gamma_k$ (for $p \equiv 3 \mod 8$) and on $f_p^*(e_k)$ (for $p \equiv 5 \mod 8$) exactly in the same way, according to Lemma 3 (b) and (c). Thus, the argument of the proof of Proposition 7 implies that $\tilde{u}_{2k-1}$ is also primitive if $p \equiv 3 \mod 8$. It finally follows from Remark 8 that

$$u_{2k-1} = \tilde{u}_{2k-1} = F_k(\gamma_2, \gamma_3, \ldots, \gamma_k) = \gamma_k + \sum_{j=1}^{k-2} w_j^2 u_{2k-2j-1}.$$  

**Remark 14.** The formula provided by Theorem 13 can be used as an alternative recursive definition of the classes $u_{2k-1}$ in $H^*(BGL(Z)^+; \mathbb{Z}/2)$.

It is known that $BGL(Z)^+ \cong BSL(Z)^+ \times BZ/2$ and one deduces immediately the calculation of the mod 2 cohomology of the space $BSL(Z)^+$ (recall that $H^*(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2)$ is obtained from $H^*(BGL(\mathbb{F}_p)^+; \mathbb{Z}/2)$ by dividing out $e_1$ and $c_0$):

**Corollary 15.** There is an isomorphism of $A$-module Hopf algebras

$$H^*(BSL(Z)^+; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, \ldots] \otimes \Lambda(w_3, w_5, \ldots),$$

where $w_k$ and $u_{2k-1}$ are also written for the image of $w_k$ and $u_{2k-1}$ under the homomorphism induced by the inclusion $SL(Z) \hookrightarrow GL(Z)$. The formulas for $u_{2k-1}$ given by Theorem 13 do still hold but observe that the first Stiefel-Whitney class of $SL(Z)$ is trivial.
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Remark 16. The results of this section determine also the mod 2 cohomology of the groups $GL(\mathbb{Z})$ and $SL(\mathbb{Z})$ because $H^*(BG^+; \mathbb{Z}/2) \cong H^*(G; \mathbb{Z}/2)$ for $G = GL(\mathbb{Z})$ or $SL(\mathbb{Z})$.

3. THE MOD 2 COHOMOLOGY OF THE STEINBERG GROUP $St(\mathbb{Z})$

The goal of this last section is to compute $H^*(St(\mathbb{Z}); \mathbb{Z}/2)$ by looking at the universal central extension

$$\mathbb{Z}/2 \cong K_2(\mathbb{Z}) \to St(\mathbb{Z}) \xrightarrow{\pi} SL(\mathbb{Z})$$

and at the associated Serre spectral sequence

$$E_2^{p,q} \cong H^p(SL(\mathbb{Z}); \mathbb{Z}/2) \otimes H^q(\mathbb{Z}/2; \mathbb{Z}/2) \to H^*(St(\mathbb{Z}); \mathbb{Z}/2).$$

Let us use the notation $Q_0 = Sq^1$ and $Q_r = Sq^{2r} Q_{r-1} + Q_{r-1} Sq^{2r}$ and observe that $Q_r(w_2) = Sq^{2r} Sq^{2r-1} \cdots Sq_1 w_2$ because $Sq^{2r} w_2 = 0$ for $r \geq 2$ and $Sq_1 Sq^2 w_2 = 0$.

Theorem 17. (a) There is an isomorphism of $A$-module Hopf algebras

$$H^*(St(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\bar{w}_2, \bar{w}_3, \ldots]/(\bar{w}_2, Q_r(\bar{w}_2), r \geq 0) \otimes \Lambda(\bar{u}_3, \bar{u}_5, \ldots),$$

where $\bar{w}_k$ and $\bar{u}_{2k-1}$ denote the image of $w_k$ and $u_{2k-1}$ under $\pi^* : H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \to H^*(St(\mathbb{Z}); \mathbb{Z}/2)$.

(b) For $k \geq 2$,

$$\bar{u}_{2k-1} = \begin{cases} f_p^*(e_k) + \sum_{j=4}^{k-2} \bar{w}_j^2 \bar{u}_{2k-2j-1}, & \text{if } p \equiv 5 \text{ mod } 8, \\ f_p^*(e_k) + \bar{w}_{2k-1} + \sum_{j=4}^{k-1} \bar{w}_j \bar{w}_{2k-j-1} + \sum_{j=4}^{k-2} \bar{w}_j^2 \bar{u}_{2k-2j-1}, & \text{if } p \equiv 3 \text{ mod } 8, \end{cases}$$

where $f_p^*$ is written here for the homomorphism $H^*(SL(\mathbb{F}_p); \mathbb{Z}/2) \to H^*(St(\mathbb{Z}); \mathbb{Z}/2)$ induced by the reduction mod $p : St(\mathbb{Z}) \to St(\mathbb{F}_p) \cong SL(\mathbb{F}_p)$.

Proof. Because $H^*(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[z]$ with deg $z = 1$, one can compute the differentials in the above spectral sequence:

$$d_2(z) = w_2, \quad d_3(z^2) = Sq^1 d_2(z) = Sq^1 w_2 = w_3, \quad d_5(z^4) = Sq^2 d_3(z) = Sq^2 w_3$$

and inductively, $d_2r+1(z^{2r}) = d_2r+1(Q_r(z)) = Q_r d_2(z) = Q_r(w_2) = w_{2r+1}$ + (decomposable element of $\mathbb{Z}/2[w_2, w_3, \ldots]$) by Wu’s formula ([MT], Part I, p. 141). Therefore, the sequence $(w_2, Q_0(w_2), Q_1(w_2), \ldots)$ is regular and we obtain $E^\infty_{i,t} = 0$ if $t > 0$ and $E^0_{i,0} \cong H^*(SL(\mathbb{Z}); \mathbb{Z}/2)/(w_2, Q_r(w_2), r \geq 0)$. This gives the mod 2 cohomology of $St(\mathbb{Z})$ as described by statement (a) and assertion (b) follows directly from Theorem 13 and Corollary 15 since $\bar{w}_2 = \bar{w}_3 = 0$. \hfill $\square$

Remark 18. The above argument exhibits a surjective homomorphism from $H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$ to $H^*(BS\text{Spin}(\mathbb{Z})^+; \mathbb{Z}/2)$. However, it is actually possible to find a nice map from $BS\text{Spin}(\mathbb{Z})^+$ to the space $BS\text{Spin}$ inducing an injective homomorphism on mod 2 cohomology. More precisely, consider the map $\varepsilon : BSL(\mathbb{Z})^+ \to BSL(\mathbb{R})^+$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ and the map $\kappa : BSL(\mathbb{R})^+ \to BSL(\mathbb{R})^\text{top} \cong BSO$ induced by the obvious map $SL(\mathbb{R}) \to SL(\mathbb{R})^\text{top}$, where the first group $SL(\mathbb{R})$ is
endowed with the discrete topology and $\text{SL}(\mathbb{R})^{\text{top}}$ with the usual topology. Then, look at the commutative diagram

$$
\begin{array}{ccc}
\text{BSO} & \xrightarrow{\tau} & \text{BSpin} \\
\text{BSL}(\mathbb{R})^+ & \xrightarrow{\beta'} & K(\pi_2\text{BSO}, 2) \\
\text{BSO} & \xrightarrow{\kappa} & \text{BSpin} \\
\text{BSL}(\mathbb{R})^+ & \xrightarrow{\beta'} & K(\pi_2\text{BSO}, 2) \\
\end{array}
$$

where the rows are fibrations in which the maps $\beta, \beta', \beta''$ are the second Postnikov sections of the corresponding spaces ($\text{BSpin}$ is the fiber of $\beta''$), the maps $\bar{\varepsilon}$ and $\bar{\kappa}$ are the second Postnikov sections of $\varepsilon$ and $\kappa$, and the vertical maps on the left are the restrictions of $\varepsilon$ and $\kappa$ to the fibers. The composition $\bar{\kappa} \bar{\varepsilon}$ is a homotopy equivalence because $\bar{\kappa} \bar{\varepsilon} : K_2(\mathbb{Z}) \to \pi_2\text{BSO}$ is an isomorphism (see Corollary 4.6 of [Br] or p. 25-26 of [Be]). Let us denote the composition $\varepsilon \beta$ by $\chi$ and its restriction to $\text{BSL}(\mathbb{Z})^+$ by $\bar{\chi} : \text{BSL}(\mathbb{Z})^+ \to \text{BSpin}$ (note that the 2-completion of $\chi$ is the universal cover of the map $h$ defined in the introduction and that the fiber of the 2-completion of $\bar{\chi}$ is $SU_2$ because of the diagram (*)). We get the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}/2 & \xrightarrow{\chi} & \text{BSL}(\mathbb{Z})^+ \\
\mathbb{Z}/2 & \xrightarrow{\bar{\chi}} & \text{BSpin} \\
\end{array}
$$

The ring structure of the mod 2 cohomology of $\text{BSpin}$ is known by Proposition 6.5 of [Q1]:

$$
H^*(\text{BSpin}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\bar{w}_2, \bar{w}_3, \ldots]/(\bar{w}_2, Q_r(\bar{w}_2), r \geq 0),
$$

where the $\bar{w}_k$'s are written here for the image of the universal Stiefel-Whitney classes under the homomorphism $\tau^* : H^*(\text{BSO}; \mathbb{Z}/2) \to H^*(\text{BSpin}; \mathbb{Z}/2)$. Since $\chi^* : H^*(\text{BSO}; \mathbb{Z}/2) \to H^*(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}/2)$ is injective, the map $\bar{\chi}$ induces an injective $\mathcal{A}$-module Hopf algebra homomorphism

$$
\bar{\chi}^* : H^*(\text{BSpin}; \mathbb{Z}/2) \to H^*(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}/2).
$$

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