ONE-TO-ONE BOREL SELECTION THEOREMS

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Abstract. For $X = [0, 1]$ we obtain new theorems stating that a Borel set in $X^2$ with large sets of large vertical and large horizontal sections admits a one-to-one Borel selection with large domain and large range. Largeness is meant mainly in measure or category sense. Our proofs combine a result of Graf and Mauldin with a modified result of Sarbadhikari.

1. Introduction

Measurable selections have been extensively studied in recent years. For an expository survey, see [W]. Interesting problems concerning one-to-one Borel selections were investigated in [M1], [M2], [GM], [MS], [S] and [DSR]. Some results contained in these papers can be formulated by the use of $\sigma$-ideals.

Let $X = [0, 1]$. Assume that $A \subseteq X^2$. Recall that $f$ forms a (one-to-one) selection of a set $A$ if $f$ is a (one-to-one) function such that $f \subseteq A$. We denote by $\text{dom } A$ and $\text{ran } A$ the projections of $A$ on the first and the second axis, respectively. For $x, y \in X$ we write $A_x = \{ t \in X : \langle x, t \rangle \in A \}$, $A^y = \{ t \in X : \langle t, y \rangle \in A \}$. These are vertical and horizontal sections of $A$. If $f$ is a selection of $A \subseteq X^2$, one usually requires that $\text{dom } f = \text{dom } A$. For a one-to-one selection $f$ of $A \subseteq X^2$, we will require that the differences $\text{dom } A \setminus \text{dom } f$ and $\text{ran } A \setminus \text{ran } f$ are small in the respective sense. This standpoint was used in [M1], [GM], [S]. The reason is that in several cases a Borel set has all vertical and horizontal sections large and it does not admit a one-to-one Borel selection defined everywhere. (See [M1, Example, p.828], [GM, Remark, p.422] and [DSR, Th.3].) The largeness of sections is described in the language of $\sigma$-ideals, mainly those of meager sets or of measure zero sets. Note, that the case when $\sigma$-ideals establishing the largeness of sections can vary also seems interesting. This was used in [GM] and [M1] for probability transition kernels. We will consider Borel $\sigma$-ideals of subsets of $X$, i.e., $\sigma$-ideals $I$ such that each set $A \in I$ is contained in a Borel set $B \in I$.

Let $I_1, I_2, I_3$ be Borel $\sigma$-ideals of subsets of $X$. A Borel set $A \subseteq X^2$ will be called $\langle I_1, I_2 \rangle$-wide if there are Borel sets $E \in I_1, F \in I_2$ and a Borel isomorphism $f$ from $X \setminus E$ onto $F$ such that $f \subseteq A$. We say that $A$ is $\langle I_1, I_2 \rangle$-tall if there are Borel sets $E \in I_1, F \in I_2$ and a Borel isomorphism $f$ from $E$ onto $X \setminus F$ such that $f \subseteq A$. We say that $A$ is $\langle I_1, I_2 \rangle$-large if there are Borel sets $E \in I_1, F \in I_2$...
and a Borel isomorphism \( f \) from \( X \setminus E \) onto \( X \setminus F \) such that \( f \subseteq A \). It can be easily checked that any of the following notions: \( \langle I_1, I_2 \rangle \)-width, \( -tallness \) and \( -largeness \), is monotonic with respect to each of variables. For instance, if \( I_1 \subseteq I_3 \) and \( A \) is \( \langle I_1, I_2 \rangle \)-wide, then it is \( \langle I_3, I_2 \rangle \)-wide. Obviously, those notions can be generalized to the case where \( A \subseteq Z \times Y \) and \( Z, Y \) are some abstract (e.g. Polish, analytic) topological spaces.

The following lemma shows how one can infer the largeness of a set from its wideness and a kind of tallness. That idea was used several times in the literature. (See [M1], [GM], [S].)

**Lemma 1.** Assume that a Borel set \( A \subseteq X^2 \) satisfies the conditions:

1. \( A \) is \( \langle I_1, I_2 \rangle \)-wide,
2. for each Borel set \( F \in I_2 \) there are Borel sets \( C \subseteq X \), \( D \in I_2 \), \( D \supseteq F \), and a Borel isomorphism \( g \) from \( C \) onto \( X \setminus D \) such that \( g \subseteq A \).

Then \( A \) is \( \langle I_1, I_2 \rangle \)-large.

**Proof.** By (1) we find Borel sets \( E \in I_1 \), \( F \in I_2 \) and a Borel isomorphism \( f \) from \( X \setminus E \) onto \( F \) such that \( f \subseteq A \). Then pick sets \( C, D \) and a function \( g \) satisfying condition (2). Put

\[
    h(x) = \begin{cases} 
        f(x) & \text{for } x \in X \setminus E \setminus C, \\
        g(x) & \text{for } x \in C. 
    \end{cases}
\]

Then \( h \) is a one-to-one Borel function and \( h \subseteq A \). The sets \( \text{dom } h = (X \setminus E) \cup C \) and \( \text{ran } h = (F \setminus f(C)) \cup (X \setminus D) \) are Borel and their complements are in \( I_1 \) and \( I_2 \), respectively. \( \square \)

**Corollary 1.** If a Borel set \( A \subseteq X^2 \) is \( \langle I_1, I_2 \rangle \)-wide and \( A \setminus (X \times F) \) is \( \langle I_3, I_2 \rangle \)-tall for each Borel set \( F \in I_2 \), then \( A \) is \( \langle I_1, I_2 \rangle \)-large.

**Proof.** It suffices to check that condition (2) of Lemma 1 holds true. Let \( F \in I_2 \) be a Borel set. Since \( A \setminus (X \times F) \) is \( \langle I_3, I_2 \rangle \)-tall, there are Borel sets \( C \in I_3 \), \( D \in I_2 \) and a Borel isomorphism \( g \) from \( C \) onto \( X \setminus D \) such that \( g \subseteq A \setminus (X \times F) \). Obviously, \( D \supseteq F \). \( \square \)

In the sequel, if we consider a theorem about a Borel set \( A \subseteq X^2 \), we can formulate its (equivalent) **dual version** where the order of the coordinates in \( X^2 \) is reversed. Then the notions of a horizontal section and a vertical section, a domain and a range, \( \langle I_1, I_2 \rangle \)-wideness and \( \langle I_2, I_1 \rangle \)-tallness, \( \langle I_1, I_2 \rangle \)-largeness and \( \langle I_2, I_1 \rangle \)-largeness are interchanged in both theorems.

Let \( M, N, C \) denote, respectively, the \( \sigma \)-ideals of all meager, Lebesgue null and countable subsets of \( X \). The following facts are known:

**Fact 1** ([S]). Let \( A \subseteq X^2 \) be a Borel set such that \( \{ x \in X : A_x \in M \} \in M \) and \( \{ y \in X : A_y \in M \} \in M \). Then \( A \) is \( \langle M, M \rangle \)-large.

**Fact 2** (cf. [GM, Th.4.4]). Let \( A \subseteq X^2 \) be a Borel set such that \( \{ x \in X : A_x \in C \} \in N \) and \( \{ y \in X : A_y \in C \} \in N \). Then \( A \) is \( \langle N, N \rangle \)-large.

The exact measure analogue of Fact 1 was proved earlier in [M1], and Fact 2 is its stronger version. Now, it seems natural to ask whether the category analogue of Fact 2 holds true.

**Problem 1.** Let \( A \subseteq X^2 \) be a Borel set such that \( \{ x \in X : A_x \in C \} \in M \) and \( \{ y \in X : A_y \in C \} \in M \). Can one conclude that \( A \) is \( \langle M, M \rangle \)-large? (We do not know.)
Facts 1 and 2 were proved by the use of the scheme presented in Corollary 1. In particular, the proof of Fact 1 was based on the following result and its dual version (which we also formulate to show an example of a dual theorem).

**Fact 3** ([S, Th.2]). Let \( A \subseteq X^2 \) be a Borel set such that \( \{ x \in X : A_x \in \mathcal{M} \} \in \mathcal{M} \). Then \( A \) is \( (\mathcal{M}, \mathcal{M}) \)-wide.

Dual version: Let \( A \subseteq X^2 \) be a Borel set such that \( \{ y \in X : A^y \in \mathcal{M} \} \in \mathcal{M} \). Then \( A \) is \( (\mathcal{M}, \mathcal{M}) \)-tall.

Let us give (in a general fashion taken from [GM]) a tallness criterion needed to derive Fact 2 from Corollary 1. That statement will be useful in Section 3. Namely, we reformulate Theorem 4.1 from [GM] with additional information that the respective domain is \( K_\sigma \) (a countable union of compact sets). That, however, follows at once from [GM, Th.3.1] and from the proof of Theorem 4.1 presented in [GM]. In our version we use finite measures instead of probability measures.

**Fact 4** (cf. [GM, Thms 4.1 and 3.1]). Let \( Z \) and \( Y \) be analytic spaces. Let \( \mu_Z \) and \( \mu_Y \) be finite Borel measures on \( Z \) and \( Y \), respectively, and let \( \mathcal{N}_Z, \mathcal{N}_Y \) be the families of \( \mu_Z \)-, \( \mu_Y \)-null sets. Assume that \( A \subseteq Z \times Y \) is a Borel set such that \( \{ y \in Y : A^y \in \mathcal{C} \} \in \mathcal{N}_Z \). Then there are a Borel set \( E \in \mathcal{N}_Z \) contained in a \( K_\sigma \) set, a Borel set \( F \in \mathcal{N}_Y \) and a Borel isomorphism \( f \) from \( E \) onto \( Y \setminus F \). Thus, if \( Z \) is a Polish space and if \( \mu_Z \) does not vanish on nonvoid open sets in \( Z \), then \( E \in \mathcal{M}_Z \cap \mathcal{N}_Z \) (where \( \mathcal{M}_Z \) stands for the family of all meager sets in \( Z \)) and so, \( A \) is \( (\mathcal{M}_Z \cap \mathcal{N}_Z, \mathcal{N}_Y) \)-tall.

Finally, note that in aiming to solve Problem 1 by the use of Corollary 1 it is enough to answer the following question affirmatively.

**Problem 2.** Let \( A \subseteq X^2 \) be a Borel set such that \( \{ x \in X : A_x \in \mathcal{C} \} \in \mathcal{M} \). Can one conclude that \( A \) is \( (\mathcal{M}, \mathcal{M}) \)-wide?

### 2. Strengthened theorem of Sarbadhikari

In this section we will improve Fact 3. Our auxiliary Proposition 1 strengthens slightly Theorem 1 from [S]. The proof is similar but we give it with details for the reader’s convenience.

**Proposition 1.** Let \( B \subseteq X^2 \) be a Borel set such that \( \{ x \in X : B_x \text{ is comeager} \} \) is comeager in \( X \). Then there is a comeager Borel set \( E \subseteq X \), a nowhere dense, Lebesgue null Borel set \( F \subseteq X \), and a Borel isomorphism \( f \) from \( E \) onto \( F \) such that \( f \subseteq B \).

**Proof.** Fix a countable base \( \{ U_n : n \geq 1 \} \) of nonempty intervals open in \( X \). Note that \( B \) is comeager in \( X^2 \) by the assumption and by the converse of the Kuratowski-Ulam theorem [O, Th.15.4]. Hence, there exist dense open sets \( V_1 \subseteq V_2 \subseteq \ldots \) in \( X^2 \) with \( \bigcap V_n \subseteq B \).

By induction on \( n \), we shall define a sequence \( \{ B_n \}_{n=1}^{\infty} \) of subsets of \( X \) and a sequence \( \{(a_{ni}, b_{ni})\}_{i=1}^{\infty} \) of nonvoid open intervals such that for all \( n \) the following conditions hold:

1. for \( H_n = \bigcup_{i=1}^{\infty} B_{ni} \times (a_{ni}, b_{ni}) \) we have \( H_{n+1} \subseteq H_n \subseteq V_n \) and \( \text{cl}((H_{n+1})_x) \subseteq (H_n)_x \) for all \( x \in X \);
(2) the sets \( B_{ni} \) \((i = 1, 2, \ldots)\) are pairwise disjoint nonmeager \( G_\delta \) with \( B_{ni} \subseteq (k_{ni}/2^n, (k_{ni} + 1)/2^n) \) for a positive integer \( k_{ni} \), and \( \bigcup_{i=1}^{\infty} B_{ni} \) is comeager in \( X \);

(3) the intervals \([a_{ni}, b_{ni}] \) \((i = 1, 2, \ldots)\) are pairwise disjoint, \( \sum_{i=1}^{\infty} (b_{ni} - a_{ni}) < 1/2^n \), and there exists a nonempty open set \( W_n \subseteq U_n \setminus \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}] \).

Before the construction we shall show that \( H = \bigcap_{n=1}^{\infty} H_n \) is the graph of the required function \( f \).

Since \( \text{dom} \, H \subseteq \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} \), we have \( H_x = \emptyset \) for \( x \notin \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} \). Let \( x \in \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} \). We shall prove that there is a unique \( y \in X \) such that \( y \in H_x \) (then \( \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} = \text{dom} \, f \) for our function \( f \)). Observe that by (2) there exists a unique sequence \( \{i_n(x)\}_{n=1}^{\infty} = \{i_n\}_{n=1}^{\infty} \) such that \( x \in \bigcap_{n=1}^{\infty} B_{n,i_n} \). Hence, for every \( n \), by (1), we have

\[
\text{cl} ((H_{n+1})_x) \subseteq (H_n)_x \subseteq \text{cl} ((H_n)_x) = [a_{n,i_n}, b_{n,i_n}].
\]

Since (3) implies that \( b_{n,i_n} - a_{n,i_n} < 1/2^n \), there is a unique \( y \in X \) such that \( \{y\} = \bigcap_{n=1}^{\infty} [a_{n,i_n}, b_{n,i_n}] \). Then it is not hard to check that \( (x, y) \in H \). In fact, \( H_x = \{y\} \), and we put \( y = f(x) \). Next, note that \( \text{dom} \, f = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} \) is a comeager Borel set, by (2).

To show that \( f \) is one-to-one, consider \( x_1, x_2 \in \text{dom} \, f \), \( x_1 \neq x_2 \). Let \( \{i_n(x_j)\}_{n=1}^{\infty} = \{i_n^{(j)}\}_{n=1}^{\infty} \) be the sequence chosen for \( x_j \) \((j = 1, 2)\) as above. From \( x_1 \neq x_2 \) it follows that \( \{i_n^{(1)}\}_{n=1}^{\infty} \neq \{i_n^{(2)}\}_{n=1}^{\infty} \) and consequently,

\[
\bigcap_{n=1}^{\infty} [a_{n,i_n^{(1)}}, b_{n,i_n^{(1)}}] \neq \bigcap_{n=1}^{\infty} [a_{n,i_n^{(2)}}, b_{n,i_n^{(2)}}],
\]

by (3). Hence, \( f(x_1) \neq f(x_2) \).

Now, let \( U \) be open in \( X \). The set \( A = (X \times U) \cap H \) is Borel and \( \text{dom} \, A \) is Borel as the image of the Borel set \( A \) by the one-to-one continuous function \( \text{pr}_1 : H \to X \). (See [Ke, Th.15.1].) Since \( \text{dom} \, A = f^{-1}[U] \), it follows that \( f \) is Borel measurable. Similarly, we show that the converse function \( f^{-1} \) is Borel. Thus, in particular,

\[
F = f \left( \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_{ni} \right)
\]

is a Borel set.

The condition (3) implies that \( \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}] \) is a nowhere dense set. Moreover,

\[
\lambda \left( \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}] \right) \geq \sum_{i=1}^{\infty} (b_{ni} - a_{ni}) < \frac{1}{2^n}
\]

for each positive integer \( n \). So \( F \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} [a_{ni}, b_{ni}] \) is a nowhere dense, Lebesgue null set.
The construction. Assume that \( n \) is a positive integer and that we have defined sequences \( \{B_{ni}\}_{i=1}^{\infty} \) and \( \{(a_{ni}, b_{ni})\}_{i=1}^{\infty} \) fulfilling conditions (1), (2), (3).

Fix \( i \). Note that \( V_{n+1} \cap \{B_{ni} \times (a_{ni}, b_{ni})\} \) is comeager in \( B_{ni} \times (a_{ni}, b_{ni}) \). Hence, by the Kuratowski-Ulam theorem [O, Th.15.1], the set

\[
\{ y \in (a_{ni}, b_{ni}) : V_{n+1}^{a_{ni}} \cap B_{ni} \text{ is comeager in } B_{ni} \}
\]

is comeager in \( (a_{ni}, b_{ni}) \). Pick \( y_n^{(i)}, z_n^{(i)} \) from this set with \( y_n^{(i)} < z_n^{(i)} \). If \( U_{n+1} \cap (a_{ni}, b_{ni}) \neq \emptyset \), let \( l_{ni} \) denote the length of the interval \( U_{n+1} \cap (a_{ni}, b_{ni}) \), and, in the opposite case, let \( l_{ni} = 1 \). Let \( M_n^{(i)} \) be a positive integer such that

\[
\frac{1}{M_n^{(i)}} < \min \left\{ z_n^{(i)} - y_n^{(i)}, l_{ni}, \frac{1}{2} (b_{ni} - a_{ni}) \right\}
\]

and put

\[
A_n^{ij} = \left\{ x \in B_{ni} : \left[ y_n^{(i)} - \frac{1}{4j}, y_n^{(i)} + \frac{1}{4j} \right] \cup \left[ z_n^{(i)} - \frac{1}{4j}, z_n^{(i)} + \frac{1}{4j} \right] \subseteq (V_{n+1})_{a_{ni}} \cap (a_{ni}, b_{ni}) \right\}
\]

for \( j \geq M_n^{(i)} \).

Then every set \( A_n^{ij} \) is of type \( G_\delta \). Indeed, \( X \setminus A_n^{ij} \) is the union of an \( F_\sigma \) set \( X \setminus B_{ni} \) and a closed set in \( X \) (being the projection of the respective compact set in \( X^2 \)).

Next, observe that

\[
\bigcup_{j=M_n^{(i)}} A_n^{ij} = V_{n+1}^{a_{ni}} \cap B_{ni} \cap V_{n+1}^{z_n^{(i)}} \cap B_{ni}
\]

which follows from the definition of \( A_n^{ij} \). We also have that \( V_{n+1}^{a_{ni}} \cap V_{n+1}^{z_n^{(i)}} \cap B_{ni} \) is comeager in \( B_{ni} \).

Find pairwise disjoint Borel sets \( G_n^{ij} \subseteq A_n^{ij} \), \( j \geq M_n^{(i)} \), with \( \bigcup_{j=M_n^{(i)}} A_n^{ij} = \bigcup_{j=M_n^{(i)}} G_n^{ij} \). Put

\[
C_n^{ij} = G_n^{ij} \cap \left( k_{ni} \frac{2}{2n+1}, k_{ni} + 1 \frac{2}{2n+1} \right), \quad D_n^{ij} = G_n^{ij} \cap \left( k_{ni} \frac{2}{2n+1}, -k_{ni} \frac{2}{2n} \right).
\]

We modify sets \( C_n^{ij}, D_n^{ij}, j \geq M_n^{(i)} \), as follows. First, we ignore those which are meager. Next, we throw out, from the remaining sets \( C_n^{ij}, D_n^{ij} \), meager parts to get \( G_\delta \) sets. In that way, we obtain nonmeager \( G_\delta \) sets. For simplicity, we suppose that all our sets \( C_n^{ij}, D_n^{ij}, j \geq M_n^{(i)} \), have these properties.

Finally, we arrange sequences \( \{C_n^{ij}\}_{i \geq 1, j \geq M_n^{(i)}} \) and \( \{D_n^{ij}\}_{i \geq 1, j \geq M_n^{(i)}} \) into one sequence \( \{B_{n+1,i}^{(i)}\}_{i \geq 1} \), and by the same method we arrange sequences

\[
\left\{ \left( y_n^{(i)} - \frac{1}{4j}, z_n^{(i)} - \frac{1}{4j+1} \right) \right\}_{i \geq 1, j \geq M_n^{(i)}} \text{ and } \left\{ \left( z_n^{(i)} - \frac{1}{4j}, z_n^{(i)} - \frac{1}{4j+1} \right) \right\}_{i \geq 1, j \geq M_n^{(i)}}
\]

into one sequence \( \{(a_{ni} + i, b_{ni} + 1)\}_{i \geq 1} \).

Now, it is not difficult to check that conditions (1), (2), (3) are satisfied for \( n+1 \).

The construction for \( n = 1 \) is similar; we use \( V_1 \cap (X \times (0, 1)) \) instead of \( V_{n+1} \cap (B_{ni} \times (a_{ni}, b_{ni})) \) applied in the above proof.
Corollary 2. The statement of Proposition 1 remains true if $B \subseteq Y \times Z$ where $Y$ is an uncountable dense-in-itself Polish space and $Z$ is a closed nondegenerate interval.

Proof. Suppose first that $Z = X$ and $Y$ equals the set $X_0$ of all irrationals from $X$. Since $X_0$ is a comeager $G_δ$ set in $X$, we can apply Proposition 1 to our set $B \subseteq Y \times Z$. We obtain the respective function $f$ which is good.

Now, consider a general assumption about $Y$ and $Z$. We remove from $Y$ the boundaries of open sets from a fixed countable base in $Y$. Thus, we get a zero-dimensional Polish space $Y_c$ comeager in $Y$. Next, we consider a $G_δ$ set $Y_0 \subseteq Y_c$ which is dense and boundary in $Y_c$. Obviously, $Y \setminus Y_0$ is meager. By the theorem of Mazurkiewicz [Ku, §36, II, Th.3], there is a homeomorphism $h$ from $Y_0$ onto $X_0$. Let $l$ be a linear function from $Z$ onto $X$. We apply Proposition 1 to the set

$$g(B \cap (Y_0 \times Z))$$

where $g(s,t) = (h(s), l(t))$ for $s,t \in Y_0 \times Z$.

Then we obtain the respective function $f$ from $E$ onto $F$. It is easy to check that the function $(l^{-1} \circ f \circ h)[h^{-1}[E]]$ from$h^{-1}[E]$ onto $l^{-1}[F]$ is good. □

Proposition 2. Let $B \subseteq X^2$ be a Borel set such that $\{x \in X : B_x \in \mathcal{M}\} \in \mathcal{M}$. Then $B$ is $(\mathcal{M}, \mathcal{M} \cap \mathcal{N})$-wide.

Proof. Let $\{U_n\}_{n \geq 1}$ be a countable open base for $X$. For each $n \geq 1$ we denote

$$D_n = \{x \in X : B_x \cap U_n \text{ is comeager in } U_n\}.$$

Then put

$$A_1 = D_1, \quad A_n = D_n \setminus \bigcup_{i=1}^{n-1} D_i \quad (n > 1).$$

Since every $D_n$ is Borel (cf. [Ke, Exercise 22.22]), so is $A_n$. Notice that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} D_n = \{x \in X : B_x \notin \mathcal{M}\}$, thus $\bigcup_{n=1}^{\infty} A_n$ is comeager in $X$. Next we ignore those sets $A_n$ which are meager, and we throw out from the remaining sets $A_n$ their meager parts to get nonmeager dense-in-itself $G_δ$ sets. So, we may assume that each $A_n$ is a nonmeager dense-in-itself $G_δ$ set.

We use induction on $n$ to define the required function $f$. For $k \leq n$ suppose a Borel isomorphism $f_k$ from $E_k \subseteq A_k$ onto $F_k \subseteq U_k$ has been defined so that $E_k$ is a Borel comeager set in $A_k$ and $F_k$ is null and nowhere dense. Since $\bigcup_{i=1}^{n} F_i$ is nowhere dense, we can pick a closed nondegenerate interval $V_{n+1} \subseteq U_{n+1} \setminus \bigcup_{i=1}^{n} F_i$. Put

$$B_{n+1} = (A_{n+1} \times V_{n+1}) \cap B.$$ 

Notice that $B_{n+1}$ is a Borel subset of the nonmeager set $A_{n+1} \times V_{n+1}$ and $\{x \in A_{n+1} : (B_{n+1})_x \text{ is comeager in } V_{n+1}\} = A_{n+1}$. By applying Corollary 2, we find a comeager Borel set $E_{n+1}$ in $A_{n+1}$ and a Borel isomorphism $f_{n+1}$ from $E_{n+1}$ onto $F_{n+1} \subseteq V_{n+1}$ such that $F_{n+1}$ is a meager null set and $f_{n+1} \subseteq B$.

Since $E_n$, $n \geq 1$, are pairwise disjoint, we can define a function $f$ from $\bigcup_{i=1}^{\infty} E_i$ onto $\bigcup_{i=1}^{\infty} F_i$ by $f(x) = f_n(x)$ for $x \in E_n$. Thus, $f$ witnesses that $B$ is $(\mathcal{M}, \mathcal{M} \cap \mathcal{N})$-wide. □
3. Applications

**Theorem 1.** Assume that $A \subseteq X^2$ is a Borel set such that

(a) \( \{ x \in X : A_x \in \mathcal{M} \} \in \mathcal{M} \),

(b) \( \{ y \in X : A^y \in \mathcal{C} \} \in \mathcal{N} \).

Then $A$ is \( \langle \mathcal{M}, \mathcal{N} \rangle \)-large.

**Proof.** Method 1. By (a) and Proposition 2, the set $A$ is \( \langle \mathcal{M}, \mathcal{M} \cap \mathcal{N} \rangle \)-wide; thus, also \( \langle \mathcal{M}, \mathcal{N} \rangle \)-wide. Let $F \in \mathcal{N}$ be a Borel set. By (b) we have

\[
\{ y \in X : (A \setminus (X \times F))^y \in \mathcal{C} \} = F \cup \{ y \in X : A^y \in \mathcal{C} \} \in \mathcal{N}.
\]

Hence, $A \setminus (X \times F)$ is \( \langle \mathcal{M} \cap \mathcal{N}, \mathcal{N} \rangle \)-tall by Fact 4. Now, the assertion follows from Corollary 1.

Method 2. We will prove the dual theorem. So, instead of (a) and (b) we assume that:

(a*) \( \{ y \in X : A^y \in \mathcal{M} \} \in \mathcal{M} \),

(b*) \( \{ x \in X : A_x \in \mathcal{C} \} \in \mathcal{N} \).

From (b*) and the dual version of Fact 4 we infer that $A$ is \( \langle \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle \)-wide.

Let $F \in \mathcal{M}$ be Borel. Then similarly as in (*) we check that

\[
\{ y \in X : (A \setminus (X \times F))^y \in \mathcal{M} \} \in \mathcal{M}.
\]

Hence, $A \setminus (X \times F)$ is \( \langle \mathcal{M}, \mathcal{M} \rangle \)-tall by the dual version of Fact 3. Now, by Corollary 1, the set $A$ is \( \langle \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle \)-large. So, the dual version of Theorem 1 has been proved. Hence, Theorem 1 is also true. \( \square \)

**Proposition 3.** Let $A \subseteq X^2$ be a Borel set such that \( \{ x \in X : A_x \in \mathcal{M} \} \in \mathcal{M} \cap \mathcal{N} \). Then $A$ is \( \langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle \)-wide.

**Proof.** Since \( \{ x \in X : A_x \in \mathcal{M} \} \in \mathcal{M} \), the set $A$ is \( \langle \mathcal{M}, \mathcal{M} \cap \mathcal{N} \rangle \)-wide, by virtue of Proposition 2. Thus, there are Borel sets $E \in \mathcal{M}$ and $F \in \mathcal{M} \cap \mathcal{N}$ and a Borel isomorphism $f$ from $X \setminus E$ onto $F$ such that $f \subseteq A$. We may assume that $E$ is of type $F_\sigma$. If $E \in \mathcal{N}$, the set $A$ is \( \langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle \)-wide. Thus, assume that $E \notin \mathcal{N}$ and apply the dual Fact 4 to the set $A \cap (E \times (X \setminus F))$ in the space $E \times X$ where in $E$ we consider Lebesgue measure restricted to subsets of $E$. The respective assumptions are satisfied since

\[
\{ x \in X : (A \cap (E \times (X \setminus F)))_x \in \mathcal{C} \} = E \cap \{ x \in X : A_x \setminus F \in \mathcal{C} \} \subseteq \{ x \in X : A_x \setminus F \in \mathcal{M} \} = \{ x \in X : A_x \in \mathcal{M} \} \in \mathcal{N}.
\]

Thus, there are Borel sets $U \in \mathcal{N}$, $U \subseteq E$, and $V \in \mathcal{M} \cap \mathcal{N}$, and a Borel isomorphism $g$ from $E \setminus U$ onto $V$ such that $g \subseteq A \cap (E \times (X \setminus F))$. Put

\[
h(x) = \begin{cases} f(x) & \text{for } x \in X \setminus E, \\ g(x) & \text{for } x \in E \setminus U. \end{cases}
\]

Then $h$ is Borel one-to-one included in $A$ with dom $h = X \setminus U$, where $X \setminus \text{dom } h = U \in \mathcal{M} \cap \mathcal{N}$, and $\text{ran } h = F \cup V \in \mathcal{M} \cap \mathcal{N}$. Hence, $A$ is \( \langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle \)-wide. \( \square \)

**Theorem 2.** If a Borel set $A \subseteq X^2$ satisfies the conditions

\[
\{ x \in X : A_x \in \mathcal{M} \} \in \mathcal{M} \cap \mathcal{N}, \quad \{ y \in X : A^y \in \mathcal{M} \} \in \mathcal{M} \cap \mathcal{N},
\]

then $A$ is \( \langle \mathcal{M} \cap \mathcal{N}, \mathcal{M} \cap \mathcal{N} \rangle \)-large.

**Proof.** It suffices to apply Proposition 3, its dual version and Corollary 1. \( \square \)
Remark (added in proof). Roman Pol has shown us the following short proof of Proposition 1: Find a $G_δ$ dense set $E \subseteq X$ and a perfect null set $F \subseteq X$ such that $E \times F \subseteq B$; then a Borel isomorphism $f$ from $E$ onto $F$ is as desired. In a recent note by Roman Pol and the authors, submitted for publication, Problem 1 has been solved in the negative.

References


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