

VARIATIONAL PRINCIPLES FOR AVERAGE EXIT TIME MOMENTS FOR DIFFUSIONS IN EUCLIDEAN SPACE

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ABSTRACT. Let D be a smoothly bounded domain in Euclidean space and let X_t be a diffusion in Euclidean space. For a class of diffusions, we develop variational principles which characterize the average of the moments of the exit time from D of a particle driven by X_t , where the average is taken over all starting points in D .

1. INTRODUCTION

In this note we study diffusions on \mathbb{R}^d and properties of their corresponding exit times from smoothly bounded, connected, open domains in \mathbb{R}^d with compact closure. We will denote by X_t a diffusion in \mathbb{R}^d with corresponding generator L a uniformly elliptic operator of divergence form. We will write $Lf = \operatorname{div}(a\nabla f)$ where the coefficient matrix $a = a_{ij}(x)$ is smooth and symmetric.

Let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\}$ be the first exit time of X_t from $D(S)$.

We study the average k th moment of the exit time for a particle driven by X_t , starting in D :

$$\mathcal{E}_k = \mathcal{E}_k(D) = \int_D E_x(\tau^k) dx$$

where E_x denotes expectation under the measure P_x satisfying $P_x\{X_0 = x\} = 1$, for all $x \in \mathbb{R}^d$. Note that \mathcal{E}_k is invariant under Euclidean motions.

We give a variational characterization of \mathcal{E}_k for each positive integer value of k in the following theorem:

Theorem 1.1. *Let X_t be a diffusion on \mathbb{R}^d with generator L a uniformly elliptic operator of divergence form, $Lf = \operatorname{div}(a\nabla f)$, where the coefficient matrix a is smooth and symmetric. Let D be a smoothly bounded open domain in \mathbb{R}^d with compact closure, \bar{D} . Define \mathcal{E}_k as above and let \mathcal{F}_k be defined by*

$$\mathcal{F}_k = \left\{ f \in C^\infty(\bar{D}); \int_D f(x) dx \neq 0, f = Lf = \dots = L^{k-1}f = 0 \text{ on } \partial D \right\}.$$

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Let $\llbracket \frac{k}{2} \rrbracket$ be the greatest integer of $\frac{k}{2}$. Then, for k even,

$$(1.1) \quad \mathcal{E}_k = k! \sup_{f \in \mathcal{F}_k} \frac{(\int_D f)^2}{\int_D |L^{\frac{k}{2}} f|^2}$$

and for k odd,

$$(1.2) \quad \mathcal{E}_k = k! \sup_{f \in \mathcal{F}_k} \frac{(\int_D f)^2}{\int_D \left| \nabla L^{\llbracket \frac{k}{2} \rrbracket} f \right|_L^2}$$

where $\langle \nabla f, \nabla g \rangle_L = \sum_{i,j} a_{ij} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i}$ is the inner product associated with L .

The proof of Theorem 1.1 is an application of the generalized Dynkin formula [AK] (cf. also [P1]), followed by an explicit computation. That smooth minimizers for the variational principles cited in Theorem 1.1 exist is explicit in our computations.

Our study of the sequence $\{\mathcal{E}_k\}$ is largely motivated by the now classic work in spectral analysis concerning to what extent a smoothly bounded domain in Euclidean space is determined by its Dirichlet spectrum. More precisely, when the diffusion is standard Brownian motion with generator $L = \frac{1}{2}\Delta$, we are interested in studying to what extent the sequence $\{\mathcal{E}_k\}$ determines the geometry of the underlying domain. There are a number of preliminary results in this direction. For example, in [KMM] the authors prove that among domains of a fixed volume, each element of the sequence is maximized if and only if the underlying domain is a ball of the appropriate volume.

For the case $k = 1$, it is known that the functional \mathcal{E}_1 computes the *torsional rigidity* of a domain. The St. Venant torsion problem, a problem with a long and distinguished history, is to determine those domains of a given volume which maximize torsional rigidity. The problem was settled by Polya (cf. [P2]) who proved that among domains of a fixed volume, the torsional rigidity is maximized by a ball. This result can be recovered using (1.2) and properties of the quotient given in (1.2) under symmetric rearrangement (cf. also [KM1]).

2. BASIC RESULTS AND DEFINITIONS

Let (Ω, \mathcal{B}) be a measurable space and $\{P_x\}_{x \in \mathbb{R}^d}$ a family of probability measures on (Ω, \mathcal{B}) . Let $\{X_t\}_{t \geq 0}$ denote a d -dimensional diffusion with generator L , a uniformly elliptic operator in divergence form and for which $P_x\{X_0 = x\} = 1$, for $x \in \mathbb{R}^d$.

Let D be a smoothly bounded, connected, open domain with compact closure. As in the introduction, we define the first exit time for a particle driven by X_t from D by $\tau = \tau(\omega) = \inf\{t : X_t(\omega) \notin D\}$. For each $x \in \mathbb{R}^d$, we will denote the expected value of a random variable Y under the probability measure P_x by $E_x(Y)$.

There is a useful relationship between the solution of a certain Poisson problem on the domain D and the expected value of the k th power of the first exit time of a particle driven by X_t from D starting at $x \in D$. Suppose u_k solves the problem

$$\begin{aligned} L^k u_k + (-1)^{k-1} k! &= 0 \text{ on } D, \\ u_k = L u_k = \dots L^{k-1} u_k &= 0 \text{ on } \partial D. \end{aligned}$$

Note that u_k can be defined inductively by

$$(2.1) \quad \begin{aligned} Lu_1 + 1 &= 0 \text{ on } D, \\ u_1 &= 0 \text{ on } \partial D \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} Lu_k + ku_{k-1} &= 0 \text{ on } D, \\ u_k &= 0 \text{ on } \partial D. \end{aligned}$$

Using the generalized Dynkin formula [H] (cf. also [AK] and [P1]) we have

$$\begin{aligned} E_x[u_k(X_0)] - E_x[u_k(X_\tau)] &= \sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\ &\quad + \frac{(-1)^k}{(k-1)!} E_x \left[\int_0^\tau s^{k-1} L^k u_k(X_s) ds \right]. \end{aligned}$$

Using the definition of u_k and τ , this gives $u_k(x) = E_x[\tau^k]$ and \mathcal{E}_k can be expressed in terms of u_k by $\mathcal{E}_k(D) = \int_D u_k(x) dx$.

We will need a number of integral formulae involving the function u_1 and the geometry of the diffusion L . To ease notation in the sequel we define, for α and β tangent vectors at $x \in D$, a scalar product, $\langle \alpha, \beta \rangle_L$, by

$$(2.3) \quad \langle \alpha, \beta \rangle_L = \alpha^T a(x) \beta$$

where α^T denotes the transpose of α .

Let u_1 be as defined in (2.1) and let $f \in \mathcal{F}_k$. Let ν be the outward pointing unit normal vector to ∂D . By the Divergence Theorem,

$$\int_D f Lu_1 - u_1 Lf = \int_{\partial D} f \langle \nabla u_1, \nu \rangle_L - u_1 \langle \nabla f, \nu \rangle_L = 0.$$

We conclude

$$(2.4) \quad \int_D f = - \int_D u_1 Lf.$$

If X is a vectorfield on D and $f \in \mathcal{F}_k$, then $\text{div}(fX) = f \text{div}(X) + \langle \nabla f, X \rangle$ where $\langle \alpha, \beta \rangle$ is the standard scalar product. By the Divergence Theorem,

$$\int_D \text{div}(fX) = \int_{\partial D} f \langle X, \nu \rangle = 0$$

and we conclude that

$$\int_D f \text{div}(X) = - \int_D \langle \nabla f, X \rangle.$$

In particular, if u_1 is as defined in (2.1) and $X = a_{ij}(x) \nabla u_1$, then

$$(2.5) \quad \int_D f = \int_D \langle \nabla u_1, \nabla f \rangle_L.$$

3. VARIATIONAL CHARACTERIZATIONS

Throughout this section let D be as above and let \mathcal{F}_k be given as in Theorem 1.1. We begin with a lemma which generalizes (2.4) and (2.5).

Lemma 3.1. *Let u_n be as defined by (2.2), let k be a positive integer, and let $f \in \mathcal{F}_k$. If $k = 2n$, then*

$$(3.1) \quad \int_D f = \frac{(-1)^n}{n!} \int_D u_n L^n f.$$

If $k = 2n + 1$, then

$$(3.2) \quad \int_D f = \frac{(-1)^n}{(n+1)!} \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L$$

where the scalar product is as given in (2.3).

Proof. Suppose that $k = 2n$ and for $0 \leq l \leq n-1$, define

$$P_l = (L^l u_n)(L^{n-l} f) - (L^{l+1} u_n)(L^{n-(l+1)} f).$$

Then

$$(3.3) \quad \sum_{l=0}^{n-1} P_l = u_n L^n f - f L^n u_n.$$

Let ν be the outward pointing unit normal vector along ∂D . By the Divergence Theorem and the fact that $L^l u_n = 0$ on ∂D , for $l = 0, \dots, n-1$,

$$(3.4) \quad \int_D P_l = \int_{\partial D} (L^l u_n) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-(l+1)} f) \langle \nabla L^l u_n, \nu \rangle_L = 0.$$

Combining (3.3) and (3.4) and using that $L^n u_n = (-1)^n n!$, we have established (3.1).

Suppose $k = 2n + 1$ and for $0 \leq l \leq n$, define

$$R_l = (L^l u_{n+1})(L^{n+1-l} f) - (L^{l+1} u_{n+1})(L^{n+1-(l+1)} f).$$

Then

$$\sum_{l=0}^n R_l = u_{n+1} L^{n+1} f - f L^{n+1} u_{n+1}.$$

As above, we use the Divergence Theorem to see that

$$\int_D R_l = \int_{\partial D} (L^l u_{n+1}) \langle \nabla L^{n-l} f, \nu \rangle_L - (L^{n-l} f) \langle \nabla L^l u_{n+1}, \nu \rangle_L = 0.$$

Since $L^{n+1} u_{n+1} = (-1)^{n+1} (n+1)!$, we conclude

$$\int_D f = \frac{(-1)^{n+1}}{(n+1)!} \int_D u_{n+1} L(L^n f).$$

If X is the vectorfield given by $X = a \nabla(L^n f)$, then following the argument used to establish (2.5),

$$\begin{aligned} \int_D u_{n+1} L(L^n f) &= \int_D u_{n+1} \operatorname{div}(X) \\ &= - \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L \end{aligned}$$

and we have established (3.2). \square

We now prove Theorem 1.1. Suppose $k = 2n$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$Q_k(f) = \frac{(\int_D f)^2}{\int_D |L^n f|^2}.$$

From (3.1)

$$Q_k(f) = \frac{\left(\frac{1}{n!}\right)^2 (\int_D u_n L^n f)^2}{\int_D |L^n f|^2}.$$

Let $\mathcal{G}_k = \{g \in \mathcal{F}_n : g = L^n f \text{ for some } f \in \mathcal{F}_k\}$. Let \mathcal{H}_k be the completion of \mathcal{G}_k in the Hilbert space, \mathcal{L}^2 , of square integrable functions on D . If we denote the inner product of g and $h \in \mathcal{L}^2$ by $\langle g, h \rangle$ and by $\|g\|$ the \mathcal{L}^2 norm of g , then we can view Q_k as a map $Q_k : \mathcal{G}_k \subset \mathcal{H}_k \rightarrow \mathbb{R}$,

$$Q_k(g) = \left(\frac{1}{n!}\right)^2 \left(\frac{\langle u_n, g \rangle}{\|g\|}\right)^2.$$

Clearly, the domain of Q_k can be extended to nonzero elements of \mathcal{H}_k and $Q_k(cg) = Q_k(g)$ for every nonzero scalar c . It follows that Q_k is maximized when $g \in \mathcal{H}_k$ is in the direction of $u_n \in \mathcal{H}_k$. If $g = cu_n$ we have $L^n(c'u_{2n}) = g$, and computing $Q_k(cu_n)$ we see that

$$\begin{aligned} \sup_{g \in \mathcal{H}_k} Q_k(g) &= Q_k(cu_n) \\ &= \frac{(\int_D u_{2n})^2}{\int_D (L^n u_{2n})^2} \end{aligned}$$

where we have applied (3.1) of Lemma 3.1 to the numerator. Note that $(L^n u_{2n})^2 = \frac{(-1)^n}{n!} (2n)! u_n L u_{2n}$. Applying Lemma 3.1 to the denominator we obtain

$$\begin{aligned} \sup_{g \in \mathcal{H}_k} Q_k(g) &= \frac{(\int_D u_{2n})^2}{(2n)! \int_D u_{2n}} \\ &= \frac{1}{k!} \mathcal{E}_k(D) \end{aligned}$$

which establishes (1.1) of Theorem 1.1.

The proof of (1.2) of Theorem 1.1 is similar. Suppose $k = 2n + 1$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$\tilde{Q}_k(f) = \frac{(\int_D f)^2}{\int_D |\nabla L^n f|_L^2}.$$

From (3.2) of Lemma 3.1,

$$\tilde{Q}_k(f) = \frac{\left(\frac{1}{(n+1)!}\right)^2 (\int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L)^2}{\int_D |\nabla L^n f|_L^2}.$$

Let $C^\infty(\bar{D}, \mathbb{R}^d)$ be the space of smooth vectorfields on \bar{D} . Let

$$\begin{aligned} \tilde{\mathcal{G}}_k &= \{X \in C^\infty(\bar{D}, \mathbb{R}^d) : X = \nabla g \text{ for some } g \in \mathcal{F}_{n+1} \\ &\quad \text{with } g = L^n f \text{ for some } f \in \mathcal{F}_k\}. \end{aligned}$$

Let $\tilde{\mathcal{H}}_k$ be the completion of $\tilde{\mathcal{G}}_k$ in the space of vectorfields square integrable with respect to the inner product $\langle \alpha, \beta \rangle_L$. We can view \tilde{Q}_k as a map $\tilde{Q}_k : \tilde{\mathcal{G}}_k \subset \tilde{\mathcal{H}}_k \rightarrow \mathbb{R}$,

$$\tilde{Q}_k(g) = \left(\frac{1}{(n+1)!} \right)^2 \left(\frac{\langle \nabla u_{n+1}, g \rangle_L}{\|g\|_L} \right)^2.$$

It is clear that the domain of \tilde{Q}_k extends to nonzero vectors in the space $\tilde{\mathcal{H}}_k$ and that for all nonzero scalars c , $\tilde{Q}_k(cg) = \tilde{Q}_k(g)$. It follows that \tilde{Q}_k is maximized when $g = c\nabla u_{n+1}$ where c is some nonzero constant. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that

$$\begin{aligned} \sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) &= \tilde{Q}_k(c\nabla u_{n+1}) \\ &= \frac{\left(\int_D u_{2n+1} \right)^2}{\int_D \|\nabla L^n u_{2n+1}\|_L^2} \end{aligned}$$

where we have used (3.2) on the numerator.

Note that $\|\nabla L^n u_{2n+1}\|_L^2 = \frac{(-1)^n}{(n+1)!} (2n+1)! \langle \nabla u_{n+1}, \nabla L^n u_{2n+1} \rangle_L$. Applying (3.2) of Lemma 3.1 to the denominator we obtain

$$\begin{aligned} \sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) &= \frac{\left(\int_D u_{2n+1} \right)^2}{(2n+1)! \int_D u_{2n+1}} \\ &= \frac{1}{k!} \mathcal{E}_k(D) \end{aligned}$$

which establishes (1.2) of Theorem 1.1.

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