VARIATIONAL PRINCIPLES FOR AVERAGE EXIT TIME MOMENTS FOR DIFFUSIONS IN EUCLIDEAN SPACE

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Abstract. Let $D$ be a smoothly bounded domain in Euclidean space and let $X_t$ be a diffusion in Euclidean space. For a class of diffusions, we develop variational principles which characterize the average of the moments of the exit time from $D$ of a particle driven by $X_t$, where the average is taken over all starting points in $D$.

1. Introduction

In this note we study diffusions on $\mathbb{R}^d$ and properties of their corresponding exit times from smoothly bounded, connected, open domains in $\mathbb{R}^d$ with compact closure. We will denote by $X_t$ a diffusion in $\mathbb{R}^d$ with corresponding generator $L$ a uniformly elliptic operator of divergence form. We will write $Lf = \text{div}(a \nabla f)$ where the coefficient matrix $a = a_{ij}(x)$ is smooth and symmetric.

Let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\}$ be the first exit time of $X_t$ from $D(S)$.

We study the average $k$th moment of the exit time for a particle driven by $X_t$, starting in $D$:

$$\mathcal{E}_k = \mathcal{E}_k(D) = \int_D E_x(\tau^k)dx$$

where $E_x$ denotes expectation under the measure $P_x$ satisfying $P_x\{X_0 = x\} = 1$, for all $x \in \mathbb{R}^d$. Note that $\mathcal{E}_k$ is invariant under Euclidean motions.

We give a variational characterization of $\mathcal{E}_k$ for each positive integer value of $k$ in the following theorem:

**Theorem 1.1.** Let $X_t$ be a diffusion on $\mathbb{R}^d$ with generator $L$ a uniformly elliptic operator of divergence form, $Lf = \text{div}(a \nabla f)$, where the coefficient matrix $a$ is smooth and symmetric. Let $D$ be a smoothly bounded open domain in $\mathbb{R}^d$ with compact closure, $D$. Define $\mathcal{E}_k$ as above and let $\mathcal{F}_k$ be defined by

$$\mathcal{F}_k = \left\{ f \in C^\infty(\bar{D}); \int_D f(x)dx \neq 0, \ f = Lf = \cdots = L^{k-1}f = 0 \text{ on } \partial D \right\}.$$

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Let \( k_2 \) be the greatest integer of \( k \). Then, for \( k \) even,

\[
E_k = k! \sup_{f \in F} \frac{(\int_D f)^2}{\int_D |L^2 f|^2}
\]

and for \( k \) odd,

\[
E_k = k! \sup_{f \in F} \frac{(\int_D f)^2}{\int_D \left| \nabla L^{k_2} f \right|^2}
\]

where \( \langle \nabla f, \nabla g \rangle_L = \sum_{i,j} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \) is the inner product associated with \( L \).

The proof of Theorem 1.1 is an application of the generalized Dynkin formula [AK] (cf. also [P1]), followed by an explicit computation. That smooth minimizers for the variational principles cited in Theorem 1.1 exist is explicit in our computations.

Our study of the sequence \( \{E_k\} \) is largely motivated by the now classic work in spectral analysis concerning to what extent a smoothly bounded domain in Euclidean space is determined by its Dirichlet spectrum. More precisely, when the diffusion is standard Brownian motion with generator \( L = \frac{1}{2} \Delta \), we are interested in studying to what extent the sequence \( \{E_k\} \) determines the geometry of the underlying domain. There are a number of preliminary results in this direction. For example, in [KMM] the authors prove that among domains of a fixed volume, each element of the sequence is maximized if and only if the underlying domain is a ball of the appropriate volume.

For the case \( k = 1 \), it is known that the functional \( E_1 \) computes the torsional rigidity of a domain. The St. Venant torsion problem, a problem with a long and distinguished history, is to determine those domains of a given volume which maximize torsional rigidity. The problem was settled by Polya (cf. [P2]) who proved that among domains of a fixed volume, the torsional rigidity is maximized by a ball. This result can be recovered using (1.2) and properties of the quotient given in (1.2) under symmetric rearrangement (cf. also [KM1]).

2. Basic results and definitions

Let \( (\Omega, \mathcal{B}) \) be a measurable space and \( \{P_x\}_{x \in \mathbb{R}^d} \) a family of probability measures on \( (\Omega, \mathcal{B}) \). Let \( \{X_t\}_{t \geq 0} \) denote a \( d \)-dimensional diffusion with generator \( L \), a uniformly elliptic operator in divergence form and for which \( P_x\{X_0 = x\} = 1 \), for \( x \in \mathbb{R}^d \).

Let \( D \) be a smoothly bounded, connected, open domain with compact closure. As in the introduction, we define the first exit time for a particle driven by \( X_t \) from \( D \) by \( \tau = \tau(\omega) = \inf\{t : X_t(\omega) \notin D\} \). For each \( x \in \mathbb{R}^d \), we will denote the expected value of a random variable \( Y \) under the probability measure \( P_x \) by \( E_x(Y) \).

There is a useful relationship between the solution of a certain Poisson problem on the domain \( D \) and the expected value of the \( k \)th power of the first exit time of a particle driven by \( X_t \) from \( D \) starting at \( x \in D \). Suppose \( u_k \) solves the problem

\[
L^k u_k + (-1)^{k-1} k! = 0 \text{ on } D,
\]

\[
u_k = Lu_k = \cdots L^{k-1} u_k = 0 \text{ on } \partial D.
\]
Note that $u_k$ can be defined inductively by
\begin{align}
Lu_1 + 1 &= 0 \text{ on } D, \\
u_1 &= 0 \text{ on } \partial D 
\end{align}
and
\begin{align}
Lu_k + ku_{k-1} &= 0 \text{ on } D, \\
u_k &= 0 \text{ on } \partial D.
\end{align}

Using the generalized Dynkin formula [H] (cf. also [AK] and [P1]) we have
\begin{align}
E_x[u_k(X_0)] - E_x[u_k(X_\tau)] &= \sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\
&+ \frac{(-1)^k}{(k-1)!} E_x\left[\int_0^{\tau} s^{k-1} L^k u_k(X_s) ds\right].
\end{align}

Using the definition of $u_k$ and $\tau$, this gives $u_k(x) = E_x[\tau^k]$ and $\mathcal{E}_k$ can be expressed in terms of $u_k$ by $\mathcal{E}_k(D) = \int_D u_k(x) dx$.

We will need a number of integral formulæ involving the function $u_1$ and the geometry of the diffusion $L$. To ease notation in the sequel we define, for $\alpha$ and $\beta$ tangent vectors at $x \in D$, a scalar product, $\langle \alpha, \beta \rangle_L$, by
\begin{equation}
\langle \alpha, \beta \rangle_L = \alpha^T a(x) \beta,
\end{equation}
where $\alpha^T$ denotes the transpose of $\alpha$.

Let $u_1$ be as defined in (2.1) and let $f \in \mathcal{F}_k$. Let $\nu$ be the outward pointing unit normal vector to $\partial D$. By the Divergence Theorem,
\begin{align}
\int_D f Lu_1 - u_1 Lf &= \int_{\partial D} f \langle \nabla u_1, \nu \rangle_L - u_1 \langle \nabla f, \nu \rangle_L = 0.
\end{align}

We conclude
\begin{equation}
\int_D f = \int_D u_1 Lf.
\end{equation}

If $X$ is a vectorfield on $D$ and $f \in \mathcal{F}_k$, then $\text{div}(fX) = f \text{div}(X) + \langle \nabla f, X \rangle$ where $\langle \alpha, \beta \rangle$ is the standard scalar product. By the Divergence Theorem,
\begin{align}
\int_D \text{div}(fX) &= \int_{\partial D} f \langle X, \nu \rangle = 0
\end{align}
and we conclude that
\begin{align}
\int_D f \text{div}(X) &= -\int_D \langle \nabla f, X \rangle.
\end{align}

In particular, if $u_1$ is as defined in (2.1) and $X = a_{ij}(x) \nabla u_1$, then
\begin{equation}
\int_D f = \int_D \langle \nabla u_1, \nabla f \rangle_L.
\end{equation}
3. Variational characterizations

Throughout this section let \( D \) be as above and let \( \mathcal{F}_k \) be given as in Theorem 1.1. We begin with a lemma which generalizes (2.4) and (2.5).

**Lemma 3.1.** Let \( u_n \) be as defined by (2.2), let \( k \) be a positive integer, and let \( f \in \mathcal{F}_k \). If \( k = 2n \), then

\[
\int_D f = \frac{(-1)^n}{n!} \int_D u_n L^n f.
\]  

(3.1)

If \( k = 2n + 1 \), then

\[
\int_D f = \frac{(-1)^n}{(n+1)!} \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L.
\]  

(3.2)

where the scalar product is as given in (2.3).

**Proof.** Suppose that \( k = 2n \) and for \( 0 \leq l \leq n - 1 \), define

\[
P_l = (L^l u_n)(L^{n-l} f) - (L^{l+1} u_n)(L^{n-(l+1)} f).
\]

Then

\[
\sum_{l=0}^{n-1} P_l = u_n L^n f - f L^n u_n.
\]  

(3.3)

Let \( \nu \) be the outward pointing unit normal vector along \( \partial D \). By the Divergence Theorem and the fact that \( L^l u_n = 0 \) on \( \partial D \), for \( l = 0, \ldots, n - 1 \),

\[
\int_D P_l = \int_{\partial D} (L^l u_n) \langle \nabla L^{n-l-1} f, \nu \rangle_L - (L^{n-(l+1)} f) \langle \nabla L^l u_n, \nu \rangle_L = 0.
\]  

(3.4)

Combining (3.3) and (3.4) and using that \( L^n u_n = (-1)^n n! \), we have established (3.1).

Suppose \( k = 2n + 1 \) and for \( 0 \leq l \leq n \), define

\[
R_l = (L^l u_{n+1})(L^{n+1-l} f) - (L^{l+1} u_{n+1})(L^{n+1-(l+1)} f).
\]

Then

\[
\sum_{l=0}^{n} R_l = u_{n+1} L^{n+1} f - f L^{n+1} u_{n+1}.
\]

As above, we use the Divergence Theorem to see that

\[
\int_D R_l = \int_{\partial D} (L^l u_{n+1}) \langle \nabla L^{n-l} f, \nu \rangle_L - (L^{n-l} f) \langle \nabla L^l u_{n+1}, \nu \rangle_L = 0.
\]

Since \( L^{n+1} u_{n+1} = (-1)^{n+1}(n+1)! \), we conclude

\[
\int_D f = \frac{(-1)^{n+1}}{(n+1)!} \int_D u_{n+1} L(L^n f).
\]

If \( X \) is the vectorfield given by \( X = a \nabla (L^n f) \), then following the argument used to establish (2.5),

\[
\int_D u_{n+1} L(L^n f) = \int_D u_{n+1} \text{div}(X)
\]

\[
= - \int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L
\]

and we have established (3.2). \( \square \)
We now prove Theorem 1.1. Suppose $k = 2n$ and, for $f \in F_k$, consider the quotient
\[
Q_k(f) = \frac{(\int_D f)^2}{\int_D |L^n f|^2}.
\]
From (3.1)
\[
Q_k(f) = \frac{(\frac{1}{n!})^2 \left(\int_D u_n L^n f\right)^2}{\int_D |L^n f|^2}.
\]
Let $G_k = \{ g \in F_n : g = L^n f \text{ for some } f \in F_k \}$. Let $\mathcal{H}_k$ be the completion of $G_k$ in the Hilbert space, $L^2$, of square integrable functions on $D$. If we denote the inner product of $g$ and $h \in L^2$ by $(g, h)$ and by $\|g\|$ the $L^2$ norm of $g$, then we can view $Q_k$ as a map $Q_k : G_k \subset \mathcal{H}_k \rightarrow \mathbb{R}$,
\[
Q_k(g) = \left(\frac{1}{n!}\right)^2 \left(\frac{(u_n, g)}{\| g \|}\right)^2.
\]
Clearly, the domain of $Q_k$ can be extended to nonzero elements of $\mathcal{H}_k$ and $Q_k(cg) = Q_k(g)$ for every nonzero scalar $c$. It follows that $Q_k$ is maximized when $g \in \mathcal{H}_k$ is in the direction of $u_n \in \mathcal{H}_k$. If $g = cu_n$ we have $L^n(c'u_2n) = g$, and computing $Q_k(cu_n)$ we see that
\[
\sup_{g \in \mathcal{H}_k} Q_k(g) = Q_k(cu_n)
= \frac{(\int_D u_{2n})^2}{\int_D (L^n u_{2n})^2}
\]
where we have applied (3.1) of Lemma 3.1 to the numerator. Note that $(L^n u_{2n})^2 = (-1)^n (2n)! u_n u_{2n}$. Applying Lemma 3.1 to the denominator we obtain
\[
\sup_{g \in \mathcal{H}_k} Q_k(g) = \frac{(\int_D u_{2n})^2}{(2n)! \int_D u_{2n}}
= \frac{1}{k!} \mathcal{E}_k(D)
\]
which establishes (1.1) of Theorem 1.1.

The proof of (1.2) of Theorem 1.1 is similar. Suppose $k = 2n+1$ and, for $f \in F_k$, consider the quotient
\[
\tilde{Q}_k(f) = \frac{(\int_D \nabla f)^2}{\int_D |\nabla L^n f|^2}.
\]
From (3.2) of Lemma 3.1,
\[
\tilde{Q}_k(f) = \frac{1}{(n+1)!} \left(\int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L \right)^2.
\]
Let $C^\infty(\bar{D}, \mathbb{R}^d)$ be the space of smooth vectorfields on $\bar{D}$. Let
\[
\tilde{G}_k = \{ X \in C^\infty(\bar{D}, \mathbb{R}^d) : X = \nabla g \text{ for some } g \in F_{n+1} \text{ with } g = L^n f \text{ for some } f \in F_k \}.
\]
Let $\tilde{\mathcal{H}}_k$ be the completion of $\tilde{\mathcal{G}}_k$ in the space of vectorfields square integrable with respect to the inner product $\langle \alpha, \beta \rangle_L$. We can view $\tilde{Q}_k$ as a map $\tilde{Q}_k : \tilde{\mathcal{G}}_k \subset \tilde{\mathcal{H}}_k \to \mathbb{R}$,

$$\tilde{Q}_k(g) = \left( \frac{1}{(n+1)!} \right)^2 \left( \frac{\langle \nabla u_{n+1}, g \rangle_L}{\|g\|_L} \right)^2.$$ 

It is clear that the domain of $\tilde{Q}_k$ extends to nonzero vectors in the space $\tilde{\mathcal{H}}_k$ and that for all nonzero scalars $c$, $\tilde{Q}_k(cg) = \tilde{Q}_k(g)$. It follows that $\tilde{Q}_k$ is maximized when $g = c \nabla u_{n+1}$ where $c$ is some nonzero constant. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that

$$\sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) = \tilde{Q}_k(c \nabla u_{n+1})$$

$$= \left( \frac{\int_D u_{2n+1}}{(2n+1)!} \right)^2 \frac{\int_D \|\nabla L^n u_{2n+1}\|_L^2}{\|\nabla u_{n+1}\|_L^2}$$

where we have used (3.2) on the numerator.

Note that $\|\nabla L^n u_{2n+1}\|_L^2 = \frac{(-1)^n}{n+1} (2n+1)! \langle \nabla u_{n+1}, \nabla L^n u_{2n+1} \rangle_L$. Applying (3.2) of Lemma 3.1 to the denominator we obtain

$$\sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) = \left( \frac{\int_D u_{2n+1}}{2n+1} \right)^2 \frac{\int_D u_{2n+1}}{\|\nabla u_{n+1}\|_L^2}$$

$$= \frac{1}{k!} E_k(D)$$

which establishes (1.2) of Theorem 1.1.

REFERENCES


