

HARDY'S INEQUALITY FOR $W_0^{1,p}$ -FUNCTIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove that for every Riemannian manifold \mathcal{X} with the isoperimetric profile of particular type there holds an inequality of Hardy type for functions of the class $W_0^{1,p}(\mathcal{X})$. We also study manifolds satisfying Hardy's inequality and, in particular, we establish an estimate for the rate of growth of the weighted volume of the noncompact part of such a manifold.

1. MAIN RESULTS

Let \mathcal{X} be a connected noncompact Riemannian C^2 -manifold without boundary. We denote by $\rho(x', x'')$ the distance between two points $x', x'' \in \mathcal{X}$. For an arbitrary point $x \in \mathcal{X}$ we set

$$\varepsilon(x) = \inf_{(y_k)} \liminf \rho(x, y_k),$$

where the infimum is taken over all sequences of points $(y_k) \subset \mathcal{X}$, which do not have points of accumulation in \mathcal{X} .

Let $\alpha(t), \beta(t) : [0, \infty) \rightarrow [0, \infty)$ be positive continuous functions. We fix constants p, q such that $1 < p \leq q < \infty$. Let $\Omega \subset \mathcal{X}$ be an arbitrary domain, and let $\partial\Omega$ be its boundary. We consider a weighted volume

$$V_{\mathcal{X}}(\Omega) = \int_{\Omega} \alpha(\varepsilon(x))^q dv.$$

Here, dv is the element of volume on the manifold \mathcal{X} . Below we shall assume that $I_{\mathcal{X}} = V_{\mathcal{X}}(\mathcal{X}) \leq \infty$.

We denote a weighted area

$$A_{\mathcal{X}}(\partial\Omega) = \int_{\partial\Omega} \beta(\varepsilon(x)) \alpha(\varepsilon(x))^{(p-1)q/p} dH^{n-1},$$

where dH^{n-1} is the element of the $(n-1)$ -dimensional Hausdorff measure on $\partial\Omega$.

We consider *isoperimetric profiles* of the Riemannian manifold \mathcal{X} with weight functions α and β . An isoperimetric profile of the manifold \mathcal{X} is the function

$$(1.1) \quad \theta_{\mathcal{X}} : [0, I_{\mathcal{X}}] \rightarrow R_+, \quad \theta(0) = 0,$$

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defined by

$$\theta_{\mathcal{X}}(\tau) = \inf\{A_{\mathcal{X}}(\partial\Omega) : \Omega \subset \mathcal{Y} \text{ a compact domain} \\ \text{with } H^{n-1}(\partial\Omega) < \infty, \quad V_{\mathcal{X}}(\Omega) = \tau\},$$

i.e. the isoperimetric profile $\theta_{\mathcal{X}}$ is the best function among the functions θ satisfying

$$(1.2) \quad \theta(V_{\mathcal{X}}(\Omega)) \leq A_{\mathcal{X}}(\partial\Omega).$$

In the special case of surfaces this definition goes back to Ahlfors [1, p.188]. In the general case see, for example, [4].

In general, the isoperimetric profile $\theta_{\mathcal{X}}(\tau)$ is difficult to compute. It is also difficult to estimate the isoperimetric profile in terms of curvature and other geometric data. We give some of these cases below.

1.3. Example. Let \mathcal{X} be a complete, simply connected, n -dimensional Riemannian manifold with nonpositive sectional curvature. We write

$$V_{\mathcal{X}}(\Omega) = \text{vol}(\Omega) \quad \text{and} \quad A_{\mathcal{X}}(\partial\Omega) = \text{area}(\partial\Omega).$$

By [12], [7] for every n there is a constant $c_n > 0$ such that

$$c_n (\text{vol}(\Omega))^{(n-1)/n} \leq \text{area}(\partial\Omega)$$

and it follows that

$$\theta_{\mathcal{X}}(\tau) \geq c_n \tau^{(n-1)/n}.$$

1.4. Example. Let \mathcal{X} be a complete, simply connected Riemannian manifold, $\dim \mathcal{X} = n$. If the sectional curvature $K_{\mathcal{X}}$ of \mathcal{X} satisfies $K_{\mathcal{X}} \leq k < 0$, $k = \text{const}$, then

$$(n-1)\sqrt{-k} \text{vol}(\Omega) \leq \text{area}(\partial\Omega)$$

([19], p.504; [6], 34.2.6) and thus

$$\theta_{\mathcal{X}}(\tau) \geq (n-1)\sqrt{-k} \tau.$$

Furthermore, we shall consider only isoperimetric profiles $\theta = \theta_{\mathcal{X}}(t)$ of a special type. In fact, we shall assume that

$$(1.5) \quad B = \sup_{r \in (0, I_{\mathcal{X}})} r^{1/q} \left(\int_r^{I_{\mathcal{X}}} \frac{dt}{\theta(t)^{p/(p-1)}} \right)^{(p-1)/p} < \infty.$$

Below by functions of the class $W_0^{1,p}(\mathcal{X})$, $p > 1$, we shall understand the closure of the class of the $C_0^\infty(\mathcal{X})$ -functions in the norm

$$\|u\|_{W^{1,p}} = \|u\|_p + \|\nabla u\|_p$$

where $\|\cdot\|_p$ stands for the standard norm in $L^p(\mathcal{X})$.

We have the following theorem.

1.6. Theorem. *Let \mathcal{X} be an n -dimensional connected noncompact Riemannian C^2 -manifold without boundary. If the manifold \mathcal{X} has an isoperimetric profile with the properties (1.1), (1.2) and (1.5), then for an arbitrary function $u \in W_0^{1,p}(\mathcal{X})$, $p > 1$, we have*

$$(1.7) \quad \left(\int_{\mathcal{X}} |\alpha(\varepsilon(x))u(x)|^q dv \right)^{1/q} \leq \lambda \left(\int_{\mathcal{X}} (\beta(\varepsilon(x))|\nabla u(x)|)^p dv \right)^{1/p},$$

where $q \geq p$ is arbitrary and λ is such that

$$(1.8) \quad B \leq \lambda \leq B q^{1/q} \left(\frac{q}{q-1}\right)^{(p-1)/p}.$$

In the case when $\mathcal{X} = D$ is a proper subdomain of R^n , the function

$$\varepsilon(x) = \rho(x, \partial D)$$

is the distance from a point $x \in D$ to the boundary ∂D , the inequality (1.7) yields a generalized Hardy inequality. We record a widely known variant of it ($\alpha(t) = 1/t, \beta(t) = 1$ and $p = q$)

$$(1.9) \quad \int_D \left(\frac{u(x)}{\rho(x, \partial D)}\right)^p dv \leq \lambda^p \int_D |\nabla u(x)|^p dv, \quad u \in W_0^{1,p}(D), \quad p > 1.$$

A one-dimensional version of Hardy's classical inequality can be found in [10, p. 175]; for its generalization to the case of domains $D \subset R^n$ see [2], [15], [14], [17], [18], and [13].

For $\alpha(t) = \beta(t) \equiv 1$ and $p = q$ the inequality (1.7) can be written as the well-known Poincaré-Sobolev inequality (cf. [19], [11, p. 48], and the literature cited there)

$$(1.10) \quad \int_D |u(x)|^p dv \leq \lambda^p \int_D |\nabla u(x)|^p dv, \quad u \in W_0^{1,p}(D), \quad p > 1.$$

1.11. Example. Let $\mathcal{X} = D$ be a domain in R^2 , and let $p = q = 2, \beta \equiv 1$. We use the metric

$$ds^2 = \alpha^2(\rho(x, \partial D)) (dx_1^2 + dx_2^2),$$

and consider the manifold $\mathcal{X} = (D, ds)$. We have for an arbitrary domain $\Omega \subset D$ with $H^1(\partial\Omega) < \infty$:

$$V_{(D, ds)}(\Omega) = \int_{\Omega} \alpha(\rho(x, \partial D))^2 dx_1 dx_2, \quad A_{(D, ds)}(\partial\Omega) = \int_{\partial\Omega} \alpha(\rho(x, \partial D)) dH^1.$$

The Gaussian curvature K of $\mathcal{X} = (D, ds)$ is equal (see, for example, [8, §13 of Chapter 2])

$$K = -\frac{1}{\alpha^2(\rho)} \Delta \log \alpha(\rho).$$

It is nonpositive if and only if

$$\Delta \log \alpha(\rho(x, \partial D)) \geq 0, \quad \forall x \in D.$$

Since $|\nabla \rho(x, \partial D)| \equiv 1$, this inequality is equivalent to the following inequality

$$(1.12) \quad \frac{\alpha'(\rho)}{\alpha(\rho)} \Delta \rho + \left(\frac{\alpha'(\rho)}{\alpha(\rho)}\right)' \geq 0, \quad \forall x \in D.$$

Therefore, if the condition (1.12) is true then we have for the isoperimetric profile of the manifold (D, ds) (see Example 1.3)

$$\theta_{(D, ds)}(\tau) \geq c_2 \tau^{1/2},$$

and the property (1.5) holds.

Consequently, we obtain for such domains

$$\int_D \alpha(\rho(x, \partial D))^2 u^2(x) dx \leq \lambda^2 \int_D |\nabla u(x)|^2 dx, \quad u \in W_0^{1,2}(D).$$

In the special case $\alpha(\tau) = \frac{1}{\tau}$, we can conclude that if the domain $D \subset R^2$ has the property

$$\Delta\rho(x, \partial D) \leq \frac{1}{\rho(x, \partial D)},$$

then the Hardy’s inequality (1.9) (with $p = 2$) is correct for this domain. For convex domains the function $\rho(x, \partial D)$ is superharmonic ([3]) and hence the above inequality holds.

The next theorem characterizes manifolds, on which a generalized Hardy’s inequality is valid.

1.13. Theorem. *Let \mathcal{X} be an n -dimensional connected noncompact Riemannian C^2 -manifold without boundary, satisfying the condition*

$$(1.14) \quad \forall t \in (0, \varepsilon_0), \quad \{x \in \mathcal{X} : \varepsilon(x) \geq t\} \subset\subset \mathcal{X}, \quad \varepsilon_0 = \sup_{x \in \mathcal{X}} \varepsilon(x).$$

If for every function $u \in W_0^{1,p}(\mathcal{X})$, $p > 1$, the inequality (1.7) holds with the constant $\lambda < \infty$, then

$$(1.15) \quad \sup_{r \in (0, \varepsilon_0)} \left(\int_r^{\varepsilon_0} \alpha(t)^q S(t) dt \right)^{1/q} \left(\int_0^r \left(\frac{1}{\beta(t)^p S(t)} \right)^{1/(p-1)} dt \right)^{(p-1)/p} \leq \lambda,$$

where

$$S(t) = \int_{\varepsilon(x)=t} dH^{n-1}, \quad t \in (0, \varepsilon_0),$$

is the $(n - 1)$ -dimensional Hausdorff measure of the set $\{x \in \mathcal{X} : \varepsilon(x) = t\}$.

A relation of the type of (1.15) contains important information on the structure of the manifold \mathcal{X} . For the purpose of illustration we give two particular cases of Theorem 1.13.

In the case $\alpha(t) = 1/t, \beta(t) \equiv 1, p = q$, we have:

1.16. Corollary. *Let \mathcal{X} be an n -dimensional connected noncompact Riemannian C^2 -manifold without boundary, satisfying the condition (1.14). If for all functions $u \in W_0^{1,p}(\mathcal{X}), p > 1$, the inequality (1.9) holds, then*

$$(1.17) \quad \limsup_{r \rightarrow 0} \left(\log \frac{1}{r} \right)^{p/(p-1)} \int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt < \infty.$$

and for each pair of numbers $r', r'', 0 < r' < r'' < \varepsilon_0$,

$$(1.18) \quad \int_{\varepsilon(x) > r''} \frac{dv}{\varepsilon(x)^p} \bigg/ \int_{r' < \varepsilon(x) < r''} \frac{dv}{\varepsilon(x)^p} \leq \frac{\lambda^p}{(\log \frac{r''}{r'})^p}.$$

Setting $\alpha(t) = \beta(t) \equiv 1, p = q$, we arrive at the following characterization of noncompact manifolds, for which the Poincaré-Sobolev inequality is fulfilled.

1.19. Corollary. *Let \mathcal{X} be an n -dimensional noncompact connected Riemannian C^2 -manifold without boundary, satisfying condition (1.14). If for all $u \in W_0^{1,p}(\mathcal{X}), p > 1$, the inequality (1.10) holds, then*

$$(1.20) \quad \int_{\varepsilon(x)>r} dv \bigg/ \int_{\varepsilon(x)<r} dv \leq \frac{\lambda^p}{r^p}, \quad \forall r \in (0, \varepsilon_0).$$

2. PROOF OF THEOREM 1.6

We shall apply a standard scheme of proof which originates from a paper of Federer and Fleming [9, Remark 6.6] and which later found frequent applications to similar questions (cf., for instance, [16], [5]).

We fix an arbitrary function $u \in C_0^\infty(\mathcal{X})$. We can assume that $|\nabla u(x)| > 0$ everywhere in the set $\{x \in \mathcal{X} : |u(x)| > 0\}$ except possibly at finitely many points. This can always be achieved on the expense of a small perturbation of the function u (see, for example [8, Theorem 5, p.488]).

We set

$$E_t = \{x \in \mathcal{X} : |u(x)| = t\},$$

$$\mathcal{X}_t = \{x \in \mathcal{X} : |u(x)| > t\}.$$

On the basis of the Kronrod-Federer co-area formula we can write

$$I(t) \equiv \int_{\mathcal{X}_t} \alpha(\varepsilon(x))^q dv = \int_t^\infty d\tau \int_{E_\tau} \alpha(\varepsilon(x))^q \frac{dH^{n-1}}{|\nabla u|}.$$

Therefore for almost all $t \geq 0$ we have

$$I'(t) = - \int_{E_t} \alpha(\varepsilon(x))^q \frac{dH^{n-1}}{|\nabla u|}.$$

Using similar arguments we have

$$(2.1) \quad \begin{aligned} \int_{\mathcal{X}} |\alpha(\varepsilon(x))u(x)|^q dv &= \int_0^\infty \tau^q d\tau \int_{E_\tau} \alpha(\varepsilon(x))^q dH^{n-1} \\ &= - \int_0^\infty \tau^q I'(\tau) d\tau = \int_0^{I_{\mathcal{X}}} \tau^q(I) dI, \end{aligned}$$

where $\tau : [0, I_{\mathcal{X}}] \rightarrow [0, \infty)$ is the inverse function of $I(\tau)$.

Furthermore, we have

$$\begin{aligned} &\left(\int_{E_t} \beta(\varepsilon(x)) \alpha(\varepsilon(x))^{(p-1)q/p} dH^{n-1} \right)^p \\ &\leq \left(\int_{E_t} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1} \right) \left(\int_{E_t} \alpha(\varepsilon(x))^q \frac{dH^{n-1}}{|\nabla u|} \right)^{p-1}, \end{aligned}$$

and therefore

$$(2.2) \quad \int_{E_t} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1} \geq \frac{1}{|I'(t)|^{p-1}} \left(\int_{E_t} \beta(\varepsilon)\alpha(\varepsilon(x))^{(p-1)q/p} dH^{n-1} \right)^p.$$

Because u has a compact support in \mathcal{X} , the open set $\mathcal{X}_t \subset\subset \mathcal{X}$. On the basis of the isoperimetric relation (1.2) we can write

$$\theta(I(t)) \leq \int_{E_t} \beta(\varepsilon)\alpha(\varepsilon(x))^{q(p-1)/p} dH^{n-1}.$$

Thus we get by (2.2)

$$(2.3) \quad \frac{\theta(I(t))^p}{|I'(t)|^{p-1}} \leq \int_{E_t} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1}.$$

Now we can transform the integral on the right side of (1.7). Applying once again the Kronrod-Federer co-area formula, we have

$$\int_{\mathcal{X}} (\beta(\varepsilon(x)) |\nabla u|)^p dv = \int_0^\infty d\tau \int_{E_\tau} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1}.$$

Therefore the inequality (2.3) yields

$$(2.4) \quad \int_{\mathcal{X}} (\beta(\varepsilon(x)) |\nabla u|)^p dv \geq \int_0^\infty \frac{\theta(I(\tau))^p}{|I'(\tau)|^{p-1}} d\tau = \int_0^{I_{\mathcal{X}}} \theta(I)^p |\tau'(I)|^p dI.$$

Combining the relations (2.1) and (2.4), we get

$$\frac{\left(\int_{\mathcal{X}} (\beta(\varepsilon(x)) |\nabla u|)^p dv \right)^{1/p}}{\left(\int_{\mathcal{X}} (\alpha(\varepsilon(x)) |\nabla u|)^q dv \right)^{1/q}} \geq \inf \frac{\left(\int_0^{I_{\mathcal{X}}} \theta(I)^p |\tau'(I)|^p dI \right)^{1/p}}{\left(\int_0^{I_{\mathcal{X}}} \tau(I)^p dI \right)^{1/q}} \equiv \frac{1}{\lambda},$$

where the infimum is taken over all nonincreasing AC -functions τ with $\tau(I), \tau(I_{\mathcal{X}}) = 0$.

The one-dimensional functional on the right side of the above inequality has been extensively studied. Applying Theorem 4 from [16, the inequality (13) §1.3.1], we arrive at the dual inequality (1.8), where the quantity B is defined by the relation (1.5).

What was said above proves the validity of the inequality (1.7) for $C_0^\infty(\mathcal{X})$ -functions. For the class $W_0^{1,p}(\mathcal{X})$ the proof follows after a passage to limit in the W_p^1 -norm.

3. PROOFS OF THEOREM 1.13 AND COROLLARIES

The proofs are very simple. The condition (1.14) guarantees the precompactness of the open sets

$$\mathcal{X}_t = \{x \in \mathcal{X} : \varepsilon(x) > t\}, \quad t \in (0, \varepsilon_0).$$

Let $\varphi(t)$ be an arbitrary AC -function on $[0, \varepsilon_0]$ which vanishes for all $t \in [0, t_0]$ for some $t_0 > 0$. Each of the functions $\varphi^*(x) = \varphi(\varepsilon(x))$ has a support contained in one of the sets \mathcal{X}_t and is therefore a member of the class $W_0^{1,p}(\mathcal{X})$.

Choosing in the inequality (1.7) the function $u(x) = \varphi^*(x)$, we obtain

$$\left(\int_{\mathcal{X}} |\alpha(\varepsilon(x))\phi(\varepsilon(x))|^q dv \right)^{1/q} \leq \lambda \left(\int_{\mathcal{X}} (\beta(\varepsilon(x))|\phi'(\varepsilon(x))| |\nabla\varepsilon(x)|)^p dv \right)^{1/p}.$$

Integrating along the level sets of $\varepsilon(x)$, we get

$$\begin{aligned} & \left(\int_0^{\varepsilon_0} \alpha(\tau)^q \phi(\tau)^q d\tau \int_{\varepsilon(x)=\tau} \frac{dH^{n-1}}{|\nabla\varepsilon(x)|} \right)^{1/q} \\ & \leq \lambda \left(\int_0^{\varepsilon_0} \beta(\tau)^p |\phi'(\tau)|^p d\tau \int_{\varepsilon(x)=\tau} |\nabla\varepsilon(x)|^{p-1} dH^{n-1} \right)^{1/p}. \end{aligned}$$

Because $\varepsilon(x)$ is the intrinsic distance function on \mathcal{X} we have $|\nabla\varepsilon(x)| \equiv 1$ almost everywhere.

Thus we get for $1 < p \leq q < \infty$

(3.1)

$$\left(\int_0^{\varepsilon_0} \alpha(\tau)^q S(\tau) \phi(\tau)^q d\tau \right)^{1/q} \leq \lambda \left(\int_0^{\varepsilon_0} \beta(\tau)^p S(\tau) |\phi'(\tau)|^p d\tau \right)^{1/p},$$

valid for every AC -function $\varphi(t)$ on $(0, \varepsilon_0)$ vanishing close to 0.

This relation between one-dimensional integrals has been extensively studied in the literature, see e.g. [16, p. 40]. The validity of the inequality (3.1) for all φ with the given properties implies the validity of the inequality (1.15).

We next prove Corollary 1.16. We choose in (1.15) the weight functions $\alpha(t) = 1/t, \beta(t) \equiv 1$ and $p = q$. We arrive at the estimate

(3.2)

$$\sup_{r \in (0, \varepsilon_0)} \left(\int_r^{\varepsilon_0} \frac{S(t)}{t^p} dt \right)^{1/p} \left(\int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{(p-1)/p} \leq \lambda.$$

However, by Hölder's inequality we get for all $r', r'', 0 < r' < r'' < \varepsilon_0$,

(3.3)

$$\left(\log \frac{r''}{r'} \right)^p = \left(\int_{r'}^{r''} \frac{dt}{t} \right)^p \leq \left(\int_{r'}^{r''} \frac{S(t)}{t^p} dt \right) \left(\int_{r'}^{r''} \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{p-1}.$$

Let $r' < r'' < \varepsilon_0$. Then we have from (3.2)

$$\begin{aligned} (\log \frac{r''}{r'})^p / \left(\int_{r'}^{r''} (\frac{1}{S(t)})^{1/(p-1)} dt \right)^{p-1} &\leq \int_{r'}^{r''} \frac{S(t)}{t^p} dt \\ &\leq \int_{r'}^{\varepsilon_0} \frac{S(t)}{t^p} dt \leq \lambda^p / \left(\int_0^{r'} (\frac{1}{S(t)})^{1/(p-1)} dt \right)^{p-1} \end{aligned}$$

and

$$(\log \frac{r''}{r'})^p \left(\int_0^{r'} (\frac{1}{S(t)})^{1/(p-1)} dt \right) / \left(\int_{r'}^{r''} (\frac{1}{S(t)})^{1/(p-1)} dt \right)^{p-1} \leq \lambda^p.$$

Therefore,

$$\left(\log \frac{r''}{r'} \right)^{-p} \left(\frac{\int_0^{r''} (\frac{1}{S(t)})^{1/(p-1)} dt - \int_0^{r'} (\frac{1}{S(t)})^{1/(p-1)} dt}{\int_0^{r'} (\frac{1}{S(t)})^{1/(p-1)} dt} \right)^{p-1} \geq \lambda^{-p}$$

and further

$$\left(\log \frac{r''}{r'} \right)^{-p/(p-1)} \left(\frac{\int_0^{r''} (\frac{1}{S(t)})^{1/(p-1)} dt}{\int_0^{r'} (\frac{1}{S(t)})^{1/(p-1)} dt} - 1 \right) \geq \lambda^{-p/(p-1)}$$

or, equivalently,

(3.4)

$$\int_0^{r''} (\frac{1}{S(t)})^{1/(p-1)} dt \geq [1 + \lambda^{-\frac{p}{p-1}} (\log \frac{r''}{r'})^{p/(p-1)}] \int_0^{r'} (\frac{1}{S(t)})^{1/(p-1)} dt.$$

The inequality (3.4) holds for all $r' < r''$. Passing to the limit with $r' \rightarrow 0$, we arrive at (1.17).

Further from (3.2) and (3.3)

(3.5)

$$\left(\log \frac{r''}{r'} \right)^p \int_{r''}^{\varepsilon_0} \frac{S(t)}{t^p} dt / \int_{r'}^{r''} \frac{S(t)}{t^p} dt \leq \lambda^p.$$

We observe that

$$\int_{r'}^{r''} \frac{S(t)}{t^p} dt = \int_{r'}^{r''} \frac{dt}{t^p} \int_{\varepsilon(x)=t} \frac{dH^{n-1}}{|\nabla \varepsilon|} = \int_{r' < \varepsilon(x) < r''} \frac{dv}{\varepsilon(x)^p}$$

and

$$\int_{r''}^{\varepsilon_0} \frac{S(t)}{t^p} dt = \int_{\varepsilon(x) > r''} \frac{dv}{\varepsilon(x)^p}.$$

On the basis of the inequality (3.5) we convince ourselves of the validity of the inequality (1.18).

The proof of Corollary 1.19 is similar. Choose in (1.15) the weight functions $\alpha(t) = \beta(t) \equiv 1$ and $p = q$. Then we have

$$\sup_{r \in (0, \varepsilon_0)} \int_r^{\varepsilon_0} S(t) dt \left(\int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{p-1} \leq \lambda^p.$$

Because

$$r^p = \left(\int_0^r dr \right)^p \leq \left(\int_0^r S(t) dt \right) \left(\int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{p-1},$$

we have

$$\sup_{r \in (0, \varepsilon_0)} r^p \int_r^{\varepsilon_0} S(t) dt \Big/ \int_0^r S(t) dt \leq \lambda^p.$$

Using the formulas

$$\int_0^r S(t) dt = \int_0^r dt \int_{\varepsilon(x)=t} \frac{dH^{n-1}}{|\nabla \varepsilon|} = \int_{\varepsilon < r} dv$$

and

$$\int_r^{\varepsilon_0} S(t) dt = \int_{\varepsilon(x) > r} dv,$$

we arrive at (1.20).

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