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THE GROUPS OF QUASICONFORMAL HOMEOMORPHISMS ON RIEMANN SURFACES

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Abstract. Suppose M is a connected Riemann surface. Let $\mathcal{H}(M)$ denote the homeomorphism group of M with the compact-open topology, and $\mathcal{H}^{\mathrm{QC}}(M)$ denote the subgroup of quasiconformal mappings of M onto itself, and let $\mathcal{H}(M)_0$ and $\mathcal{H}^{\mathrm{QC}}(M)_0$ denote the identity components of $\mathcal{H}(M)$ and $\mathcal{H}^{\mathrm{QC}}(M)$ respectively. In this paper we show that the pair $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0)$ is an (s, Σ)-manifold, and determine their topological types.

1. Introduction

In this paper we will study the local topological type of the group of quasiconformal (QC) mappings on a Riemann surface with *the compact-open topology*. Let $s = (-1, 1)^\infty$ and $\Sigma = \{(x_n) : \sup_n |x_n| < 1\}$. Since the quasiconformality is a sort of boundedness condition, we can expect that this group is a Σ-manifold. The purpose of this paper is to confirm this assertion.

Suppose M is a 2-manifold and let $\mathcal{H}(M)$ denote the homeomorphism group of M with the compact-open topology. In [Ya1], [Ya2] we have investigated some triples consisting of $\mathcal{H}(M)$ and its subgroups. When M is assigned with a PL-structure, $\mathcal{H}^{\mathrm{PL}}(M)$ denotes the subgroup of PL-homeomorphisms of M, and when M has a fixed metric, then $\mathcal{H}^{\mathrm{LIP}}(M)$ denotes the subgroup of locally Lipschitz homeomorphisms of M. The superscript "c" means compact supports, the subscript "+" means orientation preserving (o.p.), and "0" denotes the identity connected components of the corresponding groups. A Euclidean PL-manifold means a PL-manifold which is a subpolyhedron of some Euclidean space $\mathbb{R}^n$ and has the standard metric induced from $\mathbb{R}^n$. Let $\sigma = \{(x_n) \in s : x_n = 0 \ (\forall' \text{ (almost all) } n)\}$, $\sigma_f^\infty = \{(x_n) \in \sigma^\infty : x_n = 0 \ (\forall' n)\}$ and $\Sigma_f^\infty = \{(x_n) \in \Sigma^\infty : x_n = 0 \ (\forall' n)\}$. We have shown that:

1. ([Ya1]) If M is a compact Euclidean PL 2-manifold, then the triple $(\mathcal{H}(M), \mathcal{H}^{\mathrm{LIP}}(M), \mathcal{H}^{\mathrm{PL}}(M))$ is an (s, Σ, σ)-manifold.
2. ([Ya2]) If M is a noncompact connected PL 2-manifold, then the triple $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{PL}}(M)_0, \mathcal{H}_+^{\mathrm{PL,c}}(M))$ is an $(s^\infty, \sigma^\infty, \sigma_f^\infty)$-manifold.
3. ([Ya2]) If M is a noncompact connected Euclidean PL 2-manifold, then the triple $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{LIP}}(M)_0, \mathcal{H}^{\mathrm{LIP,c}}(M)_0)$ is an $(s^\infty, \Sigma^\infty, \Sigma_f^\infty)$-manifold.

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Suppose M is a Riemann surface (without boundary). We can consider the following subgroups of $\mathcal{H}(M)$: (a) $\mathcal{H}^{\mathrm{LQC}}(M)$, the subgroup of locally QC (LQC)-homeomorphisms of M, (b) $\mathcal{H}^{\mathrm{QC}}(M)$, the subgroup of QC-homeomorphisms of M, and (c) $\mathcal{H}^{\mathrm{QC,c}}(M)$, the subgroup of QC-homeomorphisms of M with compact supports. Main results of this paper are the following:

Main Theorem. *Suppose M is a Riemann surface.*

(i) *If M is compact, then the pair $(\mathcal{H}_+(M), \mathcal{H}^{\mathrm{QC}}(M))$ is an (s, Σ)-manifold.*

(ii) *If M is noncompact connected, then $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{LQC}}(M)_0, \mathcal{H}^{\mathrm{QC,c}}(M)_0) \cong (\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{LQC}}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0)$, and they are $(s^\infty, \Sigma^\infty, \Sigma^\infty_f)$-manifolds.*

Corollary. *If M is a connected Riemann surface, then $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0) \cong (s, \Sigma) \times N$, where N is defined as follows:*

1. *Compact case: Let g denote the genus of M.*
   (i) $g = 0 \Longrightarrow N = \mathrm{SO}(3)$, (ii) $g = 1 \Longrightarrow N =$ *the torus,* (iii) $g \geq 2 \Longrightarrow N = 1$ *point.*
2. *Noncompact case:* (i) $M \cong \mathbb{R}^2$ *or* $\mathbb{R}^2 \setminus 1$ *point* $\Longrightarrow N =$ *the circle,* (ii) *otherwise $N = 1$ point.*

In [Va] QC-mappings on higher-dimensional manifolds are studied. We conclude this section with the following question.

Problem. When M is a compact Riemannian manifold, is the group $\mathcal{H}^{\mathrm{QC}}(M)$ of QC-homeomorphisms on M a Σ-manifold?

2. Basic facts on infinite-dimensional manifolds

An $(l+1)$-tuple $(X, X_1, \cdots, X_l)$ of a space X and subspaces $X_1 \supset \cdots \supset X_l$ is said to be an $(E, E_1, \cdots, E_l)$-manifold if for every point $x \in X$ there exist an open neighborhood U of x and an open set V of E such that $(U, U \cap X_1, \cdots U \cap X_l) \cong (V, V \cap E_1, \cdots, V \cap E_l)$. A tuple $(X, X_1, \cdots, X_l)$ is said to be $(E, E_1, \cdots, E_l)$-stable if $(X \times E, X_1 \times E_1, \cdots, X_l \times E_l) \cong (X, X_1, \cdots, X_l)$. A space is σ-compact if it is a countable union of compact subsets. We say that a subset A of X has the homotopy negligible (h.n.) complement in X if there exists a homotopy $\phi_t : X \to X$ such that $\phi_0 = id_X$ and $\phi_t(X) \subset A$ $(0 < t \leq 1)$. To prove the Main Theorem, we will apply the following characterizations of (s, Σ)- and $(s^\infty, \Sigma^\infty, \Sigma^\infty_f)$-manifolds.

Theorem ([Ch], cf. [Ya1])**.** *A pair (X, X_1) is an (s, Σ)-manifold iff*

(i) (a) *X is a separable completely metrizable ANR,* (b) *X_1 is σ-compact,*

(ii) *X_1 has the h.n. complement in X,*

(iii) *(X, X_1) is (s, Σ)-stable.*

Theorem ([Ya2, Appendix])**.** *A triple (X, X_1, X_2) is an $(s^\infty, \Sigma^\infty, \Sigma^\infty_f)$-manifold iff*

(i) (a) *X is a separable completely metrizable ANR,* (b) *X_1 is $F_{\sigma\delta}$ in X, X_2 is σ-compact,*

(ii) *X_2 has the h.n. complement in X,*

(iii) *(X, X_1, X_2) is $(s^\infty, \Sigma^\infty, \Sigma^\infty_f)$-stable.*

As for the condition (i)(a) it is known that (1) ([LN]) if M is a compact 2-manifold, then $\mathcal{H}(M)$ is a separable completely metrizable ANR and (2) ([Ya2]) if M is a noncompact connected 2-manifold, then $\mathcal{H}(M)_0$ is a separable completely metrizable ANR. The remaining conditions (i)(b), (ii) will be verified in Sections

3, and the condition (iii) will be proved in Section 4. The remaining assertion and the Corollary follow from the Main Theorem, Homotopy invariance [Ya1, Corollary 2.2(1)] and [Ha, Sc], [Ya2, Corollary 1.1].

3. Basic facts on QC-mappings

3.1. **Modules of quadrilaterals.** First we recall some basic facts on the modules of quadrilaterals [LV, I §2.4]. Suppose M is a Riemann surface. A quadrilateral in M is a topological closed disk Q with four distinguished points z_1, z_2, z_3, z_4 in ∂Q, where these points sit on ∂Q in the positive order with respect to the orientation on ∂Q induced from M. Given a quadrilateral (Q, z_i), there exists a rectangle $R = \{z = x + iy \in \mathbb{C} : 0 \le x \le a, 0 \le y \le b\}$ $(a, b > 0)$ and a homeomorphism $h : (Q; z_1, z_2, z_3, z_4) \to (R; 0, a, a + ib, ib)$ which is conformal on $\overset{\circ}{Q}$. The value a/b depends only on the quadrilateral (Q, z_i) and is independent of the choice of (R, h) as above. Thus we can define the module of (Q, z_i), $M(Q, z_i)$, as a/b.

We refer to [LV, I, IV] and [Leh, I, V §1] for the geometric and analytic definitions of QC-mappings and related topics. Let $f : M \to N$ be an o.p. homeomorphism between Riemann surfaces. We say that

(i) f is a K-QC homeomorphism for $K \ge 1$ if for every quadrilateral (Q, z_i) in M

$$\frac{1}{K} M(Q, z_i) \le M(f(Q), f(z_i)) \le K M(Q, z_i),$$

(ii) f is a QC-homeomorphism if it is a K-QC homeomorphism for some $K \ge 1$, and

(iii) f is a locally QC (LQC)-homeomorphism if each point of M has an open neighborhood U such that $f : U \to f(U) \subset N$ is a QC-homeomorphism.

Every LQC-homeomorphism is QC over a neighborhood of any compact subset. The next lemmas will be used in Section 4. Suppose Q is a disk in M, z_1, z_3, z_4 are three distinct points in ∂Q which lie on ∂Q in positive order, and $e : [0, 1] \to \partial Q$ is an o.p. embedding with $e(0) = z_1$, $e(1) = z_3$. For each $t \in (0, 1)$ we can consider the module $M(Q, z_1, e(t), z_3, z_4) \in (0, \infty)$.

Lemma 1. (i) *The function* $\alpha : \mathcal{H}_+(M) \times (0, 1) \ni (f, t) \mapsto M(f(Q, z_1, e(t), z_3, z_4)) \in (0, \infty)$ *is continuous.*

(ii) *The function* $\beta : (0, 1) \ni t \mapsto M(Q, z_1, e(t), z_3, z_4) \in (0, \infty)$ *is an increasing homeomorphism.*

Proof. (i) The continuity of α follows from the continuity of module [LV, I §4.9].

(ii) Suppose $0 < s \le t < 1$. There exists a homeomorphism $h : (Q, z_1, e(t), z_3, z_4) \to R_0 = (R; 0, a, a + i, i)$ which is conformal on $\overset{\circ}{Q}$, where $R = [0, a] \times [0, 1]$ and $a = \beta(t)$. Let $a_1 = h(e(s)) \in (0, a]$ and $R_1 = (R; 0, a_1, a + i, i)$. Then by [LV, I §4.6, Lemma 4.2] (a) $\beta(s) = M(R_1) \le M(R_0) = \beta(t)$ and (b) if $\beta(s) = \beta(t)$, then R_1 is a rectangle. This means that $a_1 = a$ and $s = t$. By [LV, I §4.5] $\beta(t) \to 0$ $(t \to 0)$ and $\beta(t) \to \infty$ $(t \to 1)$. This implies the conclusion. □

By Lemma 1 (ii), for each $f \in \mathcal{H}_+(M)$ there exists a unique $t(f)$, $0 < t(f) < 1$, such that $M(f(Q, z_1, e(t(f)), z_3, z_4)) = 1$.

Lemma 2. *The function* $\mathcal{H}_+(M) \ni f \mapsto t(f) \in (0, 1)$ *is continuous.*

3.2. **σ-compactness of $\mathcal{H}^{\mathrm{QC}}(M)$.** When M and N are Riemann surfaces, then $\mathcal{E}^{(\mathrm{K}\text{-})\mathrm{QC}}(M,N)$ denotes the space of (K-)QC-homeomorphisms $f : M \to f(M) \subset N$ with the compact-open topology. The next lemmas follow from the normality of K-QC-mappings.

Lemma 3. *If M and N are domains of the Riemann sphere $\hat{\mathbb{C}}$, then* (i) $\mathcal{E}^{\mathrm{QC}}(M,N)$ *and $\mathcal{H}^{\mathrm{QC}}(M)$ are σ-compact, and* (ii) $\mathcal{H}^{\mathrm{LQC}}(M)$ *is $F_{\sigma\delta}$ in $\mathcal{H}(M)$.*

Proof. (i) It suffices to show that for each $K \geq 1$ the subspace $\mathcal{E}^{K\text{-}\mathrm{QC}}(M,N)$ (respectively $\mathcal{H}^{K-\mathrm{QC}}(M)$) is σ-compact. We fix three distinct points $x_1, x_2, x_3 \in M$ and choose a countable open base $\mathcal{U}$ of N (respectively M). Consider a triple $U_1, U_2, U_3 \in \mathcal{U}$ such that the closures in $\hat{\mathbb{C}}$, $cl_{\hat{\mathbb{C}}} U_i$, are disjoint. Set $\mathcal{F} = \{f \in \mathcal{E}^{K\text{-}\mathrm{QC}}(M,N)$ (respectively $\mathcal{H}^{K-\mathrm{QC}}(M)$) $: f(x_i) \in cl_{\hat{\mathbb{C}}} U_i$ $(i = 1,2,3)\}$. Since $\mathcal{E}^{K-\mathrm{QC}}(M,N)$ (respectively $\mathcal{H}^{K-\mathrm{QC}}(M)$) is separable metrizable and it is the countable union of $\mathcal{F}$'s, it suffices to show that $\mathcal{F}$ is sequentially compact. By the normality of K-QC mappings ([LV, II.Theorem 5.1]) every sequence $f_n \in \mathcal{F}$ has a subsequence which converges to a map $f : M \to \hat{\mathbb{C}}$ uniformly on each compact subset. Since $f(x_i) \in cl_{\hat{\mathbb{C}}} U_i$, the limit map f is not constant, and hence f is K-QC [LV, II.Theorem 5.3] and $f(M) \subset N$ (respectively $f(M) = M$) [LV, II §5.6 Theorems 5.4, 5.5, §5.7]. This implies the conclusion.

(ii) We can find an open cover $\{U_i\}_{i\geq 1}$ of M such that each U_i is connected and $cl\, U_i$ is compact. Consider the map

$$\Phi : \mathcal{H}(M) \to \prod_{i=1}^{\infty} \mathcal{E}(U_i, M), \quad \Phi(f) = (f|_{U_i})_i.$$

By condition (i) each $\mathcal{E}^{\mathrm{QC}}(U_i, M)$ is σ-compact, so it is F_σ in $\mathcal{E}(U_i, M)$. Hence $\prod_{i=1}^{\infty} \mathcal{E}^{\mathrm{QC}}(U_i, M)$ is $F_{\sigma\delta}$ in $\prod_{i=1}^{\infty} \mathcal{E}(U_i, M)$. Hence

$$\mathcal{H}^{\mathrm{LQC}}(M) = \Phi^{-1}(\prod_{i=1}^{\infty} \mathcal{E}^{\mathrm{QC}}(U_i, M))$$

is also $F_{\sigma\delta}$ in $\mathcal{H}(M)$. □

Lemma 4. *If M is a connected Riemann surface, then* (i) $\mathcal{H}^{\mathrm{QC}}(M)$ *and* $\mathcal{H}^{\mathrm{QC,c}}(M)$ *are σ-compact and* (ii) $\mathcal{H}^{\mathrm{LQC}}(M)$ *is $F_{\sigma\delta}$ in $\mathcal{H}(M)$.*

Proof. We take the universal covering $\tilde{M} \xrightarrow{\pi} M$ ([Leh, IV §§2,3]). Let $\mathcal{H}(\tilde{M})^\Gamma = \{\tilde{f} \in \mathcal{H}(\tilde{M}) : \tilde{f}$ is a lift of some $f \in \mathcal{H}(M)$ (i.e. $\pi\tilde{f} = f\pi)\}$. This subset is closed in $\mathcal{H}(\tilde{M})$. The natural map $\pi_* : \mathcal{H}(\tilde{M})^\Gamma \to \mathcal{H}(M)$, $\pi_*(\tilde{f}) = f$, is a covering projection. In fact, if $x_0 \in M$ and U is an open disk in M, then the open set $\mathcal{U} = \{f \in \mathcal{H}(M) : f(x_0) \in U\}$ is evenly covered by π_*. Since π is locally conformal, $\tilde{f}$ is (L)QC iff f is also (cf. [LV, I §3.2], [Leh, I §2.1]) and $\mathcal{H}^{(\mathrm{L})\mathrm{QC}}(\tilde{M})^\Gamma \equiv \mathcal{H}(\tilde{M})^\Gamma \cap \mathcal{H}^{(\mathrm{L})\mathrm{QC}}(\tilde{M}) = \pi_*^{-1}(\mathcal{H}^{(\mathrm{L})\mathrm{QC}}(M))$. By the Uniformization theorem ([Leh, IV §3.2, Theorem 3.3]) and Lemma 3, $\mathcal{H}^{\mathrm{QC}}(\tilde{M})$ is σ-compact and $\mathcal{H}^{\mathrm{LQC}}(\tilde{M})$ is $F_{\sigma\delta}$ in $\mathcal{H}(\tilde{M})$. Hence $\mathcal{H}^{\mathrm{QC}}(\tilde{M})^\Gamma$ is σ-compact since it is closed in $\mathcal{H}^{\mathrm{QC}}(\tilde{M})$, and $\mathcal{H}^{\mathrm{LQC}}(\tilde{M})^\Gamma$ is $F_{\sigma\delta}$ in $\mathcal{H}(\tilde{M})^\Gamma$.

(i) Thus $\mathcal{H}^{\mathrm{QC}}(M) = \pi_*(\mathcal{H}^{\mathrm{QC}}(\tilde{M})^\Gamma)$ is σ-compact and so is $\mathcal{H}^{\mathrm{QC,c}}(M)$ since it is F_σ in $\mathcal{H}^{\mathrm{QC}}(M)$.

(ii) Since π_* is a covering projection, we can find an open cover $\{\mathcal{U}_i\}_{i\geq 1}$ of $\mathcal{H}(M)$ and open sets $\{\tilde{\mathcal{U}}_i\}_{i\geq 1}$ of $\mathcal{H}(\tilde{M})^\Gamma$ such that $\pi_* : \tilde{\mathcal{U}}_i \to \mathcal{U}_i$ is a homeomorphism for

each $i \geq 1$. Since $\mathcal{F}_i = \mathcal{H}^{\mathrm{LQC}}(\tilde{M})^\Gamma \cap \tilde{\mathcal{U}}_i$ is $F_{\sigma\delta}$ in $\tilde{\mathcal{U}}_i$, $\mathcal{H}^{\mathrm{LQC}}(M) \cap \mathcal{U}_i = \pi_*(\mathcal{F}_i)$ is $F_{\sigma\delta}$ in $\mathcal{U}_i$, and it is also $F_{\sigma\delta}$ in $\mathcal{H}(M)$ since $\mathcal{U}_i$ is F_σ in $\mathcal{H}(M)$. Since each $\mathcal{H}_i = (\mathcal{H}^{\mathrm{LQC}}(M) \cap \mathcal{U}_i) \cup (\mathcal{H}(M) \setminus \mathcal{U}_i)$ is $F_{\sigma\delta}$ in $\mathcal{H}(M)$, so is $\mathcal{H}^{\mathrm{LQC}}(M) = \bigcap_i \mathcal{H}_i$. $\square$

Note that $\mathcal{H}^{\mathrm{QC}}(M)_0$ and $\mathcal{H}^{\mathrm{QC,c}}(M)_0$ are σ-compact and $\mathcal{H}^{\mathrm{LQC}}(M)_0$ is $F_{\sigma\delta}$ in $\mathcal{H}(M)_0$ since every connected component is closed in the ambient space.

3.3. **QC-triangulations.** We will introduce the notion of QC-triangulations of Riemann surfaces. Suppose M is a Riemann surface. Let Δ be a 2-simplex in $\mathbb{C}$. We say an embedding $f : \Delta \to M$ is quasiconformal (QC) if it admits a QC extension to an open neighborhood. More generally, suppose σ is an abstract (or Euclidean) 2-simplex and $f : \sigma \to M$ is an embedding. Then σ inherits an orientation from M by f. We say that f is QC if for some 2-simplex Δ in $\mathbb{C}$ and some o.p. simplicial isomorphism $\phi : \Delta \to \sigma$, the embedding $f\phi : \Delta \to M$ is QC. Since any o.p. affine isomorphism of $\mathbb{C}$ is QC, this definition is independent of the choice of Δ and ϕ. A triangulation of a space X is a pair (L, f), where L is an abstract simplicial complex (or a Euclidean complex) and $f : |L| \cong X$ is a homeomorphism from the realization of L onto X. We say a triangulation (L, f) of X is quasiconformal (QC) if for each 2-simplex σ of L the restriction $f : \sigma \to M$ is a QC embedding. For example any C^1-triangulation of M is QC (cf. [Wh]).

Lemma 5. *Suppose (L, f) is a QC-triangulation of a Riemann surface M.*

(i) *For any 1-simplex τ of L, $f(\tau)$ is a QC-arc (i.e., there exists a QC-mapping from an open neighborhood of $f(\tau)$ onto an open set of $\mathbb{C}$ which maps $f(\tau)$ onto a straight segment).*

(ii) *For any subdivision L' of L the pair (L', f) is also a QC-triangulation.*

(iii) *An o.p. homeomorphism $h : M \to M$ is K-QC (respectively LQC) iff for any 2-simplex σ in L the restriction*

$$h : f(\mathring{\sigma}) \longrightarrow M$$

is K-QC (respectively QC).

The statements (i) and (ii) follow from the definition and (iii) follows from removability of analytic arcs [LV, I.Theorem 8.3] and isolated boundary points [LV, I.Theorem 8.1], and analytic definition [LV, IV §2.3].

Lemma 6. *Suppose (L, f) is a QC-triangulation of a Riemann surface M and let $\mathcal{H}^{\mathrm{PL}}(M)$ denote the subgroup of PL-homeomorphisms of M with respect to this PL-structure. Then $\mathcal{H}_+^{\mathrm{PL,c}}(M) \subset \mathcal{H}^{\mathrm{QC,c}}(M)$ and $\mathcal{H}_+^{\mathrm{PL}}(M) \subset \mathcal{H}^{\mathrm{LQC}}(M)$.*

In particular, (i) when M is compact, $\mathcal{H}^{\mathrm{QC}}(M)(\supset \mathcal{H}_+^{\mathrm{PL}}(M))$ has the h.n. complement in $\mathcal{H}_+(M)$ [GH] and (ii) when M is noncompact connected, $\mathcal{H}^{\mathrm{QC,c}}(M)_0(\supset \mathcal{H}_+^{\mathrm{PL,c}}(M))$ has the h.n. complement in $\mathcal{H}(M)_0$ [Ya2].

4. Stability property of $(\mathcal{H}(M), \mathcal{H}^{\mathrm{QC}}(M))$

In this final section we will investigate the stability property of $\mathcal{H}^{\mathrm{QC}}(M)$. The argument is a modification of the LIP-case [SW], [Ya1, §3.3]. Let $s_b^\infty = \{((x_{n,m})_m)_n \in s^\infty : \sup_{n,m} |x_{n,m}| < 1\}$. In [Ya2, Corollary A.1] it is shown that $(s^\infty, \Sigma^\infty, s_b^\infty) \cong (s^\infty, \Sigma^\infty, \Sigma_f^\infty)$.

Lemma 7. (I) *If M is a Riemann surface, then the pair $(\mathcal{H}_+(M), \mathcal{H}^{\mathrm{QC}}(M))$ is (s, Σ)-stable.*

(II) *If M is a noncompact Riemann surface, then the quadruple $(\mathcal{H}_+(M), \mathcal{H}^{\mathrm{LQC}}(M), \mathcal{H}^{\mathrm{QC}}(M), \mathcal{H}^{\mathrm{QC,c}}(M))$ is $(s^\infty, \Sigma^\infty, s_b^\infty, \Sigma_f^\infty)$-stable.*

Proof. We will prove (II). (1) First we work on $\mathbb{C}$.

(i) By $\langle v_0, v_1, v_2\rangle$ we denote the triangle in $\mathbb{C}$ with vertices v_0, v_1, v_2. Consider the triangle $T = \langle -1, 1, i\rangle$. Let $T^* = \langle -1, -i, 1\rangle$ (the reflection of T in the x-axis) and $Q = T \cup T^*$.

For each $t \in (-1, 1)$ we have a canonical $\alpha(t) \in \mathcal{H}_\partial^{\mathrm{PL}}(T)$ which maps simplicially $\langle -1, 0, i\rangle$ onto $\langle -1, t, i\rangle$ and $\langle 0, 1, i\rangle$ onto $\langle t, 1, i\rangle$. We extend $\alpha(t)$ to the $\beta(t) \in \mathcal{H}_\partial^{\mathrm{PL}}(Q)$ by the reflection in the x-axis. By a simple calculation (cf. [LV, IV §2.3]), we can see that $\beta(t)$ is $K(t)$-QC on $\overset{\circ}{Q}$, where

$$k(t) = \frac{|t|}{\sqrt{(|t|-1)^2+1}} \quad (-1 < t < 1) \qquad \text{and} \qquad K(t) = \frac{1+k(t)}{1-k(t)}.$$

(ii) Choose a sequence of triangles of the form : $T_n = \langle a_n - b_n, a_n + b_n, a_n + ib_n\rangle$, where $a_n > 0$, $a_n \to 0$ and $b_n > 0$ is sufficiently small so that T_n's are disjoint. Let $Q_n = T_n \cup T_n^*$, where $T_n^* = \langle a_n - b_n, a_n - ib_n, a_n + b_n\rangle$ (the reflection of T_n in the x-axis).

Consider the conformal mapping $\psi_n : T \to T_n,\ \psi_n(z) = b_n z + a_n$ and define $\varphi_n(t) \in \mathcal{H}(Q_n)$ by $\varphi_n(t) = \psi_n \beta(t) \psi_n^{-1}$, which is $K(t)$-QC on $\overset{\circ}{Q}_n$.

(iii) Let E be a disk in $\mathbb{C}$ with $Q_n \subset \operatorname{Int} E$ $(n \geq 1)$. For each $\mathbf{t} = (t_n) \in s = (-1, 1)^\infty$, we define $\varphi(\mathbf{t}) \in \mathcal{H}_\partial(E)$ by

$$\varphi(\mathbf{t})(z) = \begin{cases} \varphi_n(t_n)(z), & z \in Q_n, \\ z, & z \in E \setminus \bigcup_n Q_n. \end{cases}$$

Then the map $s \ni \mathbf{t} \mapsto \varphi(\mathbf{t}) \in \mathcal{H}_\partial(E)_0$ is continuous and $\varphi(0) = id_E$. If $0 \leq a < 1$ and $|t_n| \leq a$ $(n \geq 1)$, then each $\varphi_n(t_n) : \overset{\circ}{Q}_n \to \overset{\circ}{Q}_n$ is $K(a)$-QC and by removability $\varphi(\mathbf{t})$ is also $K(a)$-QC on $\overset{\circ}{E}$.

(2) Since M is noncompact, we can find local complex coordinate neighborhoods (U_k, γ_k) $(k \geq 1)$ such that $E \subset \gamma_k(U_k)$, $cl\, U_k$ is compact and $\{U_k\}_{k\geq 1}$ is discrete in M (i.e., each point has a neighborhood which meets at most one U_k). We define a map $F : s^\infty \to \mathcal{H}(M)_0$ by

$$F(\mathbf{t})(z) = \begin{cases} \gamma_k^{-1}\phi(\mathbf{t}_k)\gamma_k(z), & z \in \gamma_k^{-1}(E),\ k \geq 1, \\ z, & z \notin \bigcup_k \gamma_k^{-1}(E) \end{cases} \qquad (\mathbf{t} = (\mathbf{t}_k)_k \in s^\infty).$$

By (1-iii) F is continuous and it defines a map of quadruples:

$$F : (s^\infty, \Sigma^\infty, s_b^\infty, \Sigma_f^\infty) \to (\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{LQC}}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0, \mathcal{H}^{\mathrm{QC,c}}(M)_0).$$

(3) Let $f \in \mathcal{H}_+(M)$. Consider the quadrilateral $(T; -1, t, 1, i)$ $(-1 < t < 1)$ and for each $k, n \geq 1$, consider the quadrilateral $T_{k,n}^f(t) \equiv f\gamma_k^{-1}\psi_n(T; -1, t, 1, i)$. As in Lemma 2 there exists a unique $t_{k,n} = t_{k,n}(f) \in (-1, 1)$ with the module $M(T_{k,n}^f(t_{k,n})) = 1$, and the map $\mathcal{H}_+(M) \ni f \mapsto t_{k,n}(f) \in (-1, 1)$ is continuous. For each $K \geq 1$, let $a(K), 0 \leq a(K) < 1$ be the unique real number such that $M(T, -1, a(K), 1, i) = K$. By the symmetry, $M(T, -1, -a(K), 1, i) = 1/K$. Suppose $f : U_k \to f(U_k)$ is K-QC. Since $f^{-1} : f(U_k) \to U_k$ is also K-QC and $\gamma_k^{-1}\psi_n$ is

conformal in $\overset{\circ}{T}$, it follows that

$$\begin{aligned}\frac{1}{K} &= \frac{1}{K}M(f\gamma_k^{-1}\psi_n(T;-1,t_{k,n},1,i)) \le M(\gamma_k^{-1}\psi_n(T;-1,t_{k,n},1,i))\\ &= M(T;-1,t_{k,n},1,i) \le KM(f\gamma_k^{-1}\psi_n(T_n;-1,t_{k,n},1,i)) = K.\end{aligned}$$

Hence $|t_{k,n}| \le a(K)$ and $\mathbf{t}_k(f) \in \Sigma$. Also note that if $f = id$ on U_k, then $\mathbf{t}_k(f) = 0$. Thus we have the map $G : (\mathcal{H}_+(M), \mathcal{H}^{\mathrm{LQC}}(M), \mathcal{H}^{\mathrm{QC}}(M), \mathcal{H}^{\mathrm{QC,c}}(M)) \to (s^\infty, \Sigma^\infty, s_b^\infty, \Sigma_f^\infty)$, $G(f) = (\mathbf{t}_k(f))_k$.

Let $\mathcal{F} = G^{-1}(0)$ and $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (\mathcal{F}\cap\mathcal{H}^{\mathrm{LQC}}(M), \mathcal{F}\cap\mathcal{H}^{\mathrm{QC}}(M), \mathcal{F}\cap\mathcal{H}^{\mathrm{QC,c}}(M))$. By the definition of F and G it follows that

$$\text{(i)} \qquad G(f \circ F(G(f))) = 0 \qquad (f \in \mathcal{H}_+(M)),$$

$$\text{(ii)} \qquad G(f \circ F(\mathbf{t})^{-1}) = \mathbf{t} \qquad (f \in \mathcal{F} \text{ and } \mathbf{t} \in s^\infty).$$

From these facts we have the reciprocal homeomorphisms

$$(\mathcal{H}_+(M), \mathcal{H}^{\mathrm{LQC}}(M), \mathcal{H}^{\mathrm{QC}}(M), \mathcal{H}^{\mathrm{QC,c}}(M)) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} (\mathcal{F}\times s^\infty, \mathcal{F}_1\times\Sigma^\infty, \mathcal{F}_2\times s_b^\infty, \mathcal{F}_3\times\Sigma_f^\infty),$$

$$\Phi(f) = (f\circ F(G(f)), G(f)), \qquad \Psi(f,\mathbf{t}) = f\circ F(\mathbf{t})^{-1}.$$

Since $((s^\infty)^2, (\Sigma^\infty)^2, (s_b^\infty)^2, (\Sigma_f^\infty)^2) \cong (s^\infty, \Sigma^\infty, s_b^\infty, \Sigma_f^\infty)$, this implies the conclusion. □

Since the above homeomorphism of the quadruples preserves the identity component parts, it follows that $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{LQC}}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0, \mathcal{H}^{\mathrm{QC,c}}(M)_0)$ is also $(s^\infty, \Sigma^\infty, s_b^\infty, \Sigma_f^\infty)$-stable. In particular, $(\mathcal{H}(M)_0, \mathcal{H}^{\mathrm{LQC}}(M)_0, \mathcal{H}^{\mathrm{QC}}(M)_0)$ is stable for $(s^\infty, \Sigma^\infty, s_b^\infty) \cong (s^\infty, \Sigma^\infty, \Sigma_f^\infty)$.

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