PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 9, Pages 2679–2683
S 0002-9939(99)04899-6

ON THE PRODUCT OF TWO GENERALIZED DERIVATIONS

MOHAMED BARRAA AND STEEN PEDERSEN

(Communicated by Palle E. T. Jorgensen)

Abstract. Two elements A and B in a ring $\mathfrak{R}$ determine a generalized derivation $\delta_{A,B}$ on $\mathfrak{R}$ by setting $\delta_{A,B}(X) = AX - XA$ for any X in $\mathfrak{R}$. We characterize when the product $\delta_{C,D}\delta_{A,B}$ is a *generalized derivation* in the cases when the ring $\mathfrak{R}$ is the algebra of all bounded operators on a Banach space $\mathcal{E}$, and when $\mathfrak{R}$ is a C^*–algebra $\mathfrak{A}$. We use these characterizations to compute the commutant of the range of $\delta_{A,B}$.

1. Introduction

Let $\mathfrak{B}$ be a ring or an algebra. A *derivation* on $\mathfrak{B}$ is an additive (linear) map $\delta : \mathfrak{B} \to \mathfrak{B}$ satisfying $\delta(XY) = \delta(X)Y + X\delta(Y)$ for all X and Y in $\mathfrak{B}$. Consequences of assuming

$$(*) \qquad \text{the product } \delta_1\delta_2 \text{ of two derivations } \delta_1 \text{and } \delta_2 \text{ again is a derivation}$$

have been investigated in a number of papers. Posner [Po] proved that if $\mathfrak{B}$ is a prime ring of characteristic not 2, then $(*)$ implies $\delta_1 = 0$ or $\delta_2 = 0$. This result has been reproved in other papers, e.g., [FS] and [Wi], under the stronger assumption that $\mathfrak{B} = \mathcal{L}(\mathcal{H})$, the algebra of all bounded linear operators on the Hilbert space $\mathcal{H}$. If $\mathfrak{B} = \mathfrak{A}$ is a C^*–algebra, then $(*)$ implies $\delta_1\delta_2 = 0$ [Ma1] (a result of this type is proved in [Ped] for unbounded densely defined derivations), and $(*)$ together with $\delta_1 = \delta_2 = \delta$ implies $\delta = 0$. In [Wi] the range of a derivation on $\mathcal{L}(\mathcal{H})$ is investigated, a major tool in [Wi] is the result mentioned above. It is known (e.g., [Ch], [Ka], [Sa1], [Se]) that a derivation of $\mathcal{L}(\mathcal{E})$ is *inner*, i.e., there exists an element A in $\mathcal{L}(\mathcal{E})$, the algebra of all bounded linear operators on the Banach space $\mathcal{E}$, so that $\delta(X) = \delta_A(X) = AX - XA$ for all X in $\mathcal{L}(\mathcal{E})$. It is also known (a result of Sakai) [Pe], [Sa2] that, if δ is a derivation of a C^*–algebra $\mathfrak{A}$ then, $\delta = \delta_A$ for some A in the enveloping von Neumann algebra of $\mathfrak{A}$.

For two operators A and B in $\mathcal{L}(\mathcal{E})$ (or in the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$), we say that the operator $\delta_{A,B}(X) = AX - XB$ is a *generalized derivation.* It is shown in [Br] that, if the product of two derivations of an prime ring of characteristic not 2 is a generalized derivation, then one of the derivations must be zero. We characterize when the product $\delta_C\delta_{A,B}$ of a derivation and a generalized derivation

---

Received by the editors December 30, 1996 and, in revised form, November 20, 1997.
1991 *Mathematics Subject Classification.* Primary 47B47, 46L40.
Key words and phrases. Derivation, generalized derivation, elementary operator, C^*–algebra.

on $\mathcal{L}(\mathcal{E})$ is a derivation. In contrast to the product of two derivations, the product $\delta_C\delta_{A,B}$ can be a non-zero derivation. As a consequence of the characterization we obtain information about the commutant of the range of a generalized derivation. We also apply the results for $\mathfrak{B} = \mathcal{L}(\mathcal{E})$ to obtain a characterization for the case when $\mathfrak{B} = \mathfrak{A}$ is a C^*–algebra. The later characterization is strong enough to yield the results, mentioned above, characterizing when the products $\delta\delta$ and $\delta_1\delta_2$ are derivations on a C^*–algebra as easy corollaries. In the final section we consider the product of two generalized derivations.

We refer the reader to [Pe] for background information about C^*–algebras, and to [Pe] and [Sa2] for the theory of derivations in operator algebras. Generalized derivations have been studies extensively, see e.g., [CF], [FS], and [Ma2].

2. THE COMMUTANT OF THE RANGE

In this part of the paper we consider bounded linear operators on a Banach space $\mathcal{E}$, we show that there exist C in $\mathcal{L}(\mathcal{E}) \setminus \mathbb{C}I$ such that $\delta_C\delta_{A,B}$ is a derivation, if and only if $A + B \in \mathbb{C}I$. We use this to compute the commutant, in $\mathcal{L}(\mathcal{E})$, of the range of $\delta_{A,B}$. The first result is an immediate consequence of Theorem 5 below.

Theorem 1. *Let $\mathcal{E}$ be a Banach space and let A and B be in $\mathcal{L}(\mathcal{E})$. There exist C in $\mathcal{L}(\mathcal{E}) \setminus \mathbb{C}I$ so that $\delta_C\delta_{A,B}$ is a derivation if and only if $A + B \in \mathbb{C}I$. More precisely, when $A + B \in \mathbb{C}I$, the possible choices for C are given by*

1. *if $A - B \in \mathbb{C}I$, then C can be any element of $\mathcal{L}(\mathcal{E}) \setminus \mathbb{C}I$*
2. *if $A - B \notin \mathbb{C}I$, then $C = a(A - B) + bI$, where $a, b \in \mathbb{C}$ and $a \neq 0$.*

Remark. Theorem 1 remains true for standard algebras. A *standard algebra* is a subalgebra of $\mathcal{L}(\mathcal{E})$ containing all the finite rank operators.

Corollary 2. *If A and B are in $\mathcal{L}(\mathcal{E})$, then the commutant in $\mathcal{L}(\mathcal{E})$ of the range of $\delta_{A,B}$ is:*

1. *$\mathbb{C}I$, if $A + B \notin \mathbb{C}I$ or if $A + B \in \mathbb{C}I$ and either $A - B \in \mathbb{C}I \setminus \{0\}$ or $(A - B)^2 \notin \mathbb{C}I$;*
2. *$\mathcal{L}(\mathcal{E})$, if $A + B \in \mathbb{C}I$ and $A - B = 0$;*
3. *$\{\ a(A - B) + bI\ :\ a, b \in \mathbb{C}\ \}$, if $A + B \in \mathbb{C}I$, $A - B \notin \mathbb{C}I$ and $(A - B)^2 \in \mathbb{C}I$.*

Proof. First observe that an element C of $\mathcal{L}(\mathcal{E})$ is in the commutant $(Ran\,\delta_{A,B})'$ if and only if $\delta_C\delta_{A,B} = 0$. Hence, if C is in the commutant of the range of $\delta_{A,B}$ then $\delta_C\delta_{A,B}$ is a derivation. The proof of the corollary is divided into three cases.

1. $A + B \notin \mathbb{C}I$. In this case Theorem 1 implies any C in the commutant of the range of $\delta_{A,B}$ is in $\mathbb{C}I$. Clearly, $\mathbb{C}I$ is a subset of the commutant.
2. $A + B \in \mathbb{C}I$ and $A - B \in \mathbb{C}I$. In this case $A = aI$ and $B = bI$, hence $\delta_{A,B}(X) = (a - b)X$ for all X in $\mathcal{L}(\mathcal{E})$, the remaining details are left for the reader.
3. $A + B \in \mathbb{C}I$ and $A - B \notin \mathbb{C}I$. If follows from Theorem 1 that

$$(Ran\,\delta_{A,B})' \subseteq \{a(A - B) + bI\ :\ a, b \in \mathbb{C}\}.$$

It follows from (4), (5) and (7) below that $A = aC + bI$ and $B = -aC + cI$ for some scalars a, b, and c, hence we have

$$\begin{aligned} (Ran\, \delta_{A,B})' \ni C \text{ iff} \\ \delta_C \delta_{A,B} = \delta_{aC^2+(b-c)C} = 0 \text{ iff} \\ aC^2 + (b-c)C \in \mathbb{C}I \text{ iff} \\ (2aC)^2 + 2(b-c) \cdot 2aC \in \mathbb{C}I \text{ iff} \\ (A - B - (b-c)I)^2 + 2(b-c)(A - B - (b-c)I \in \mathbb{C}I \text{ iff} \\ (A-B)^2 \in \mathbb{C}I. \end{aligned}$$

Again a few easy details are left for the reader. □

3. Applications to C^*–algebras

Let $\mathfrak{A}$ be a C^*–algebra, and let A and B be in the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$. Let π be the reduced atomic representation of $\mathfrak{A}$, that is, $\pi = \sum_{t\in\hat{A}}^{\otimes} \pi_t$ on the Hilbert space $\mathcal{H} = \sum_{t\in\hat{A}}^{\otimes} \mathcal{H}_t$. It is well known that π is faithful, and that the double commutant, $\pi(\mathfrak{A})''$, of $\pi(\mathfrak{A})$ satisfies

$$\pi(\mathfrak{A})'' = \prod_{t\in\hat{A}} \mathcal{L}(\mathcal{H}_t).$$

See, for example, [Pe] for more details about the reduced atomic representation. Let C in the enveloping von Neumann algebra of $\mathfrak{A}$ be so that $\delta_C\delta_{A,B}$ is a derivation of $\mathfrak{A}$, and pick D in the enveloping von Neumann algebra, so that $\delta_C\delta_{A,B} = \delta_D$. Then for X in $\mathfrak{A}$ we have

$$C(AX - XB) - (AX - XB)C = DX - XD. \tag{1}$$

For $t \in \hat{A}$ we can apply π_t to (1) and get

$$\begin{aligned} &\pi_t(C)(\pi_t(A)\pi_t(X) - \pi_t(X)\pi_t(B)) \\ &\qquad -(\pi_t(A)\pi_t(X) - \pi_t(X)\pi_t(B))\pi_t(C) \\ &\quad = \pi_t(D)\pi_t(X) - \pi_t(X)\pi_t(D). \end{aligned}$$

If $\pi_t(C)$ is not a scalar multiple of the identity on $\mathcal{H}_t$, then we can apply Theorem 1 to the operators $\delta_{\pi_t(C)}$ and $\delta_{\pi_t(A),\pi_t(B)}$ to conclude that $\pi_t(A+B) = \pi_t(A)+\pi_t(B)$ is a scalar times the identity on $\mathcal{H}_t$. It follows from the faithfulness of the reduced atomic representation that $A+B$ is a scalar times the identity in $M(\mathfrak{A})$. It is clear that if $\pi_t(C)$ is a scalar multiple of the identity on $\mathcal{H}_t$, then $\pi_t(A)$ and $\pi_t(B)$ can be arbitrary operators on $\mathcal{H}_t$. We have shown:

Corollary 3. *If A, B and C are operators in the multiplier algebra $M(\mathfrak{A})$ of some C^*–algebra $\mathfrak{A}$, then $\delta_C\delta_{A,B}$ is a derivation on $\mathfrak{A}$ if and only if, for each $t \in \hat{A}$, either $\pi_t(A+B)$ or $\pi_t(C)$ is a scalar multiple of the identity on $\mathcal{H}_t$.*

Immediate consequences of Corollary 3 are the known results that if δ is a derivation on a C^*–algebra so that $\delta\delta$ also is a derivation, then $\delta = 0$; and if δ_1, δ_2, and $\delta_1\delta_2$ all are derivations on a C^*–algebra, then $\delta_1\delta_2 = 0$.

Corollary 4. *Let $\mathfrak{A}$ be a C^*–algebra and let A and B be in the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$. Then $(Ran\,\delta_{A,B})' \neq \mathbb{C}I$ if and only if there is an irreducible representation π of $\mathfrak{A}$ so that either $\pi(A) = \pi(B) \in \mathbb{C}I$ or $\pi(A+B) \in \mathbb{C}I$, $\pi(A-B) \notin \mathbb{C}I$ and $\pi(A-B)^2 \in \mathbb{C}I$.*

Proof. The proof is based on Corollary 2 instead of Theorem 1, otherwise it is similar to the proof of Corollary 3. □

When defining $\delta_{A,B}$ on a C^*–algebra $\mathfrak{A}$ is it not really necessary that A and B are in the multiplier algebra $M(\mathfrak{A})$. We only need that A and B are in the enveloping von Neumann algebra and that $\delta_{A,B}$ maps $\mathfrak{A}$ into itself.

4. The product on $\mathcal{L}(\mathcal{E})$

In this section we investigate when the product of two generalized derivations again is a generalized derivation. In particular we prove the following generalization of Theorem 1.

Theorem 5. *Let A, B, C, and D be bounded operators on a Banach space $\mathcal{E}$.*

1. *If $A \notin \mathbb{C}I$ and $B \notin \mathbb{C}I$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if $C = aA + cI$ and $D = -aB + dI$ for some scalars a, c, and d;*
2. *If $A \in \mathbb{C}I$ and $B \notin \mathbb{C}I$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if $C \in \mathbb{C}I$;*
3. *If $A \notin \mathbb{C}I$ and $B \in \mathbb{C}I$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation if and only if $D \in \mathbb{C}I$;*
4. *If $A \in \mathbb{C}I$ and $B \in \mathbb{C}I$, then $\delta_{C,D}\delta_{A,B}$ is a generalized derivation.*

Proof. We will only prove part 1, the proofs of parts 2 and 3 are similar, but simpler, and part 4 is trivial. Suppose A, $B \notin \mathbb{C}I$ and $\delta_{C,D}\delta_{A,B} = \delta_{E,F}$. Let $G = CA - E$ and $H = BD - F$; then

$$CXBy + AXDy = GXy - XHy \tag{2}$$

for all $X \in \mathcal{L}(\mathcal{E})$. Consider the element $X = x \otimes f$ of $\mathcal{L}(\mathcal{E})$ determined by $Xy = f(y)x$. Then (2) becomes

$$f(By)Cx + f(Dy)Ax = f(y)Gx - f(Hy)x \tag{3}$$

for all x and y in $\mathcal{E}$ and all f in $\mathcal{E}'$. Since $B \notin \mathbb{C}I$ we can find y in $\mathcal{E}$ so that y and By are linearly independent, hence we can pick f in $\mathcal{E}'$ so that $f(y) = 0$ and $f(By) = -1$. Using these choices in (3) we see that $Cx = f(Dy)Ax + f(Hy)x$ for all x in $\mathcal{E}$. It follows that

$$C = aA + cI \tag{4}$$

for some scalars a and c. Considering the adjoint of (2) it follows similarly that

$$D = bB + dI \tag{5}$$

for some scalars b and d. It remains to show that $a + b = 0$. Let $K = G - dA$ and $L = H + cB$. Substituting (4) and (5) into (2) we get

$$(a+b)AXBy = KXy - XLy \tag{6}$$

for all y in $\mathcal{E}$ and all $X \in \mathcal{L}(\mathcal{E})$. If y and f are so that $f(y) = 1$ and $f(By) = 0$, then letting $X = x \otimes f$ in (6) we get $K \in \mathbb{C}I$. Similarly, considering adjoints, we get $L \in \mathbb{C}I$. Hence, if $X = x \otimes f$ and $f(y) = 0$, and $f(By) = 1$ then it follows from (6) that $(a+b)A \in \mathbb{C}I$. Since, $A \notin \mathbb{C}I$ we conclude that

$$a + b = 0. \tag{7}$$

The converse is trivial. □

Remark. It is easy to obtain a variety of special cases of Theorem 5 by asking when $\delta_{C,D}\delta_{A,B} = \delta_{E,F}$ and one or more of the following hold: $A = B$, $C = D$, or $E = F$. For example

1. if $\delta_{C,C}\delta_{A,A} = \delta_{E,F}$, then $C \in \mathbb{C}I$ or $A \in \mathbb{C}I$ and $E = F \in \mathbb{C}I$ (this generalization of Posner's theorem was also observed in [Br]);
2. if $\delta_{C,C}\delta_{A,B} = \delta_{E,F}$, then $A + B \in \mathbb{C}I$ and $E = F$ (this is a generalization of Theorem 1 above);
3. if $\delta_{C,D}\delta_{A,A} = \delta_{E,F}$, then $C + D \in \mathbb{C}I$ and $E = F$.

As in section 3 one can extend the results in this section to obtain a characterization of when the product of two generalized derivations of a C^*–algebra is a generalized derivation.

References

[Br] M. Brešar, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Math. J. **33** (1991), pp. 89–93. MR **92b:**46071

[Ch] P. R. Chernoff, *Representations, automorphisms, and derivations of some operator algebras*, J. Funct. Anal. **12** (1972), pp. 275–289. MR **50:**2934

[CF] R. E. Curto and L. A. Fialkow, *The spectral picture of* (L_A, R_B), J. Funct. Anal. **71** (1987), pp. 371–392. MR **88c:**47006

[FS] C. K. Fong and A. R. Sourour, *On the operator identity* $\sum A_k X B_k \equiv 0$, Canad. J. Math. **31** (1979), pp. 845–857. MR **80h:**47020

[Ka] R. V. Kadison, *Derivations on operator algebras*, Ann. of. Math. **83** (1966), pp. 280–293. MR **33:**1747

[Ma1] M. Mathieu, *Properties of the product of two derivations*, Canad. Math. Bull. **32** (1989), pp. 490–497. MR **90k:**46140

[Ma2] M. Mathieu, *Elementary operators on prime C^*–algebras, I*, Math. Ann. **284** (1989), pp. 223–244. MR **90h:**46092

[Pe] G. K. Pedersen, *C^*–algebras and their automorphism groups*, Academic Press, London, 1979.

[Ped] S. Pedersen, *The product of two unbounded derivations*, Canad. Math. Bull. **33** (1990), pp. 345–348. MR **92a:**46081

[Po] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), pp. 1093–1100. MR **20:**2361

[Sa1] S. Sakai, *Derivations on W^*–algebras*, Ann. of. Math. **83** (1966), pp. 273–279. MR **33:**1748

[Sa2] S. Sakai, *Operator algebras in dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 41, Cambridge University Press, Cambridge, 1991. MR **92h:**46099

[Se] P. Semrl, *Ring derivations on standard operator algebras*, J. Funct. Anal. **112** (1993), pp. 318–324. MR **94h:**47084

[Wi] J. P. Williams, *On the range of a derivation*, Pacific J. Math. **38** (1971), pp. 273–279. MR **46:**7923

Departement de Mathematiques, Faculte des Sciences–Semlalia, University Cadi Ayyad, B.P.: S. 15, 40000 Marrakech, Marocco

Department of Mathematics, Wright State University, Dayton, Ohio 45435
E-mail address: `steen@math.wright.edu`