

A CLASS OF 3-DIMENSIONAL MANIFOLDS WITH BOUNDED FIRST EIGENVALUE ON 1-FORMS

GIOVANNI GENTILE

(Communicated by Peter Li)

ABSTRACT. Let (P, g) be the framebundle over an oriented, C^∞ Riemannian surface S . Denote by $\lambda - 1, 1(P, g)$ the first nonzero eigenvalue of the Laplace operator acting on differential forms of degree 1. We prove that $\lambda_{1,1}(P, g) \leq c$ for all (P, g) with canonical metrics g of volume 1.

1. INTRODUCTION

Let (M^n, g) be a compact, connected Riemannian manifold of n dimensions. The Laplacian $\Delta_{g,p}$ acting on differential forms of degree p on M has discrete spectrum. Let $\lambda_{1,p}(g)$ denote the smallest positive eigenvalue of $\Delta_{g,p}$. For functions we set as usual $\lambda_1(g) = \lambda_{1,0}(g)$. Let Ω be the class of all metrics on an orientable closed surface S with given volume. Then we have

$$(1) \quad \lambda_1(g) \leq \frac{8\pi(\gamma + 1)}{\text{Vol}(S, g)}$$

for any Riemannian metric $g \in \Omega$. This was proved by J. Hersch [7] (for S^2) and by P. Yang and S.-T. Yau [14] for surfaces of higher genus.

In connection with this result, M. Berger [1] and S. Tanno [11] asked whether there exists a constant $k(M)$ such that

$$(2) \quad \lambda_{1,p}(g) \leq \frac{k(M)}{\text{Vol}(M^n, g)^{2/n}}$$

for any Riemannian metric g on M .

The answer is negative in the case of functions for $n \geq 3$ (Bleecker [3], Urakawa [12], Xu [13], Colbois and Dodziuk [5]), and also negative in the case of differential forms of degree $2 \leq p \leq n - 2$ and dimension $n \geq 4$ (Gentile and Pagliara [6]).

One can obtain positive answers if one restricts the class of manifolds (M, Ω) . For Ω the class of all Kähler metrics on a complex manifold M , whose Kähler form represents a given cohomology class, an upper bound for $\lambda_1(M, g)$, i.e. functions, $g \in \Omega$ exists and is related to the algebraic-geometric properties of M . This was discovered by P. Li and S.-T. Yau [8] and by J.-P. Bourguignon, P. Li and S.-T. Yau [4]. Another positive answer for an (M, Ω) was given by L. Polterovich [10], when M is a closed symplectic manifold and Ω a special class of Kähler metrics.

Received by the editors December 1, 1997.

1991 *Mathematics Subject Classification*. Primary 53C20; Secondary 58G25.

In the present paper we give a positive answer to the above problem on 1-forms in a special class of 3-dimensional manifolds. They are S^1 -bundles P over 2-dimensional manifolds S with canonical metrics (see below). We estimate $\lambda_{1,1}(g)$ form above in terms of the fibrelength and the volume of the base-space S . In particular $\lambda_{1,1}(g)$ is bounded from above by a constant if we fix the volume of P , in contrast to the case of function, where D. Bleecker [3] showed, that $\lambda_1(g)$ cannot be bound by a constant in this class of manifolds.

2. THE CLASS OF MANIFOLDS

Let P be the framebundle over an oriented closed Riemannian surface (S, h) . This is a principal S^1 -bundle. An S^1 -invariant metric on the fibre and the metric h of the base space S induce a Riemannian metric g on P such that $\pi: (P, g) \rightarrow (S, h)$ is a Riemannian submersion with totally geodesic fibres and horizontal distribution associated with the Levi-Civita connection (A. Besse [2]).

Let ξ be the Killing vectorfield along the S^1 -action and η its dual. Let $x = (X_1, X_2)$ be a frame at $u = \pi(x)$. Each $x \in P$ gives rise to a linear isomorphism $x: \mathbb{R}^2 \rightarrow T_u, u = \pi(x), x(E_i) = X_i$ where (E_1, E_2) is the standard basis of \mathbb{R}^2 . Let $V \in \mathbb{R}^2, x \in P, B(V)_x :=$ unique horizontal vector at x which projects on $x(V) \in T_u$. $B(V)$ is called a basic vectorfield (K. Nomizu [9]).

Lemma 1. i) $V \neq 0 \rightarrow B(V) \neq 0$.

ii) $(\xi, B(E_1), B(E_2))$ gives rise to a complete parallelisation of P .

iii) $[\xi, B(V)] = B(\xi \cdot V)$, where the dot denotes standard representation of $\xi \in LS^1$ on \mathbb{R}^2 :

$$\xi \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The proof of this and the following lemma can be found in K. Nomizu [9, page 50–52]. Let (ξ, B_1, B_2) be an ONS for $g, B_i = B(E_i)$ and Ω the curvature form of the fibration.

Since the torsion vanishes we have:

Lemma 2. i) $[B_1, B_2] = -\Omega(B_1, B_2)$.

ii) $[\xi, B_1] = B(\xi \cdot E_1) = B(E_2) = B_2$.

iii) $[\xi, B_2] = B(\xi \cdot E_2) = B(-E_1) = -B_1$.

3. RESULTS

Theorem 1. Let (P, g) be as above. Then

$$\lambda_{1,1}(g) \leq \min \left(\frac{1}{L^2}, \frac{8\pi(1 + \gamma)}{\text{Vol}(S, h)} \right)$$

where $2\pi L$ is the length of the fibres. In particular if we fix $\text{Vol}(P, g) = 1$, then

$$\lambda_{1,1}(g) \leq [16\pi^2(1 + \gamma)]^{2/3}.$$

Remarks. If $\text{Vol}(P, g) = 1, \lambda_1(g) = \lambda_{1,0}(g)$ is not bounded as long as the curvature of (S, h) is nowhere 0 (D. Bleecker [3]).

Proof. The metrics on P are determined up to the length of the fibres from the metric on S and the condition imposed on P (A. Besse [2]). We want to examine the dependence of the eigenvalue on the length of the fibres. By rescaling the metric on P along the fibres while keeping it constant on the orthogonal complement

(canonical variation), we obtain a 1-parameter family of metrics g_L on P for which $\pi: (P, g_L) \rightarrow (S, h)$ is a Riemannian submersion and the length of the fibres is $2\pi L$.

If (ξ, B_1, B_2) is an ONS for g and $\bar{\xi} = \frac{1}{L}\xi$, then $(\bar{\xi}, B_1, B_2)$ is an ONS for g_L . Let $(\bar{\eta}, \beta_1, \beta_2)$ be the dual base.

Lemma 3. β_1 is an eigen 1-form of the Laplace operator on M

$$\begin{aligned} d\beta_1 &= -\beta_1([\bar{\xi}, \beta_1])\bar{\eta} \wedge \beta_1 - \beta_1([\bar{\xi}, \beta_2])\bar{\eta} \wedge \beta_2 \\ &= -\frac{1}{L}\beta_1([\xi, \beta_2])\bar{\eta} \wedge \beta_2 \\ &= \frac{1}{L}\bar{\eta} \wedge \beta_2 \end{aligned}$$

where we have used Lemma 2.

$$\begin{aligned} \delta\beta_1 &= *d*\beta_1 \\ &= -*d(\bar{\eta} \wedge \beta_2) \\ &= -* (d\bar{\eta} \wedge \beta_2 - \bar{\eta} \wedge d\beta_2) \\ &= 0 \end{aligned}$$

again from Lemma 2. Therefore we get

$$\begin{aligned} \Delta\beta_1 &= (\delta d + d\delta)\beta_1 \\ &= *d*d\beta_1 \\ &= *d*\left(\frac{1}{L}\bar{\eta} \wedge \beta_2\right) \\ &= -\frac{1}{L}*d\beta_1 \\ &= \frac{1}{L^2}\beta_1. \end{aligned}$$

In the same way one finds that β_2 is also an eigen 1-form with eigenvalue $1/L^2$.

On the other hand, if Ψ is a nontrivial eigenfunction of (S, h) , then, using the result of Yang and Yau, we have (1) and $\Delta \circ d = d \circ \Delta$

$$\Delta^S(d\Psi) \leq \frac{8\pi(1+\gamma)}{\text{Vol}(S, h)}d\Psi.$$

We lift the 1-form $d\Psi$ of S to P . This gives (since $\dim S = 2$)

$$\Delta^P(\pi^*(d\Psi)) = \pi^*(\Delta^S(d\Psi)) \leq \frac{8\pi(1+\gamma)}{\text{Vol}(S, h)}\pi^*(d\Psi).$$

Putting it together, we get

$$\lambda_{1,1}(g) \leq \min\left(\frac{1}{L^2}, \frac{8\pi(1+\gamma)}{\text{Vol}(S, h)}\right).$$

If we fix $\text{Vol}(P, g) = 1$, then $\text{Vol}(S, h) = \frac{1}{2\pi L}$ and the second assertion follows from

$$\min\left(\frac{1}{L^2}, 16\pi^2(1+\gamma)L\right) \leq [16\pi^2(1+\gamma)]^{2/3} \quad \forall L > 0. \quad \square$$

Remarks. If one takes as class (P, Ω) the framebundles over manifolds M of arbitrary dimension n , with certain canonical metrics on $SO(N)$, one can still find an analogue of β_1 , but there is no result which corresponds to the equality of Hersch-Yang-Yau (1). Also one does not know whether or not the first eigenvalue on functions is bounded in that class.

REFERENCES

- [1] M. Berger, *Sur les premières valeurs propres des variétés riemanniennes*, Compositio Math. **26** (1973), 129–149. MR **47**:5461
- [2] A. Besse, *Einstein manifolds*, Springer 1987. MR **88f**:53087
- [3] D. Bleeker, *The spectrum of a Riemannian manifold with a unit Killing vector field*, Transactions Amer. Math. Soc. **275** (1983), 409–416. MR **84c**:53037
- [4] J.-P. Bourguignon, P. Li and S.-T. Yau, *Upper bound for the first eigenvalue of algebraic submanifolds*, Comm. Math. Helv. **69** (1994), 199–207. MR **95j**:58168
- [5] B. Colbois and J. Dodziuk, *Riemannian metrics with large λ_1* , Proc. Amer. Math. Soc. **122** (1994), 905–906. MR **95a**:58130
- [6] G. Gentile and V. Pagliara, *Riemannian metrics with large first eigenvalue on forms of degree p* , Proc. Amer. Math. Soc. **123** (1995), 3855–3858. MR **96b**:58115
- [7] J. Hersch, *Quatre propriétés isopérimétriques des membranes sphériques homogènes*, C. R. Acad. Sci. Paris Sér. A **270** (1970), 139–144.
- [8] P. Li and S.-T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), 269–291. MR **84f**:53049
- [9] K. Nomizu, *Lie groups and differential geometry*, The Math. Soc. Japan 1956. MR **18**:821d
- [10] L. Polterovich, *Symplectic aspects of the first eigenvalue*, to appear.
- [11] S. Tanno, *Geometric expressions of eigen 1-forms of the Laplacian on spheres*, Spec. Riem. manifolds, Kaigai Publ. Kyoto (1983), 115–128.
- [12] H. Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan **31** (1979), 209–226. MR **80e**:58046
- [13] Y. Xu, *Diverging Eigenvalues and Collapsing Riemannian Metrics*, Institute for Advanced Study, October 1992.
- [14] P. Yang and S.-T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa **7** (1980), 55–63. MR **81m**:58084

DEPARTMENT OF MATHEMATICS, ETH-ZENTRUM, HG G34, CH 8092 ZURICH, SWITZERLAND