WEIGHTED CACCIOPPOLI-TYPE ESTIMATES
AND WEAK REVERSE HÖLDERS INEQUALITIES
FOR A-HARMONIC TENSORS

SHUSEN DING

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Abstract. We obtain a local weighted Caccioppoli-type estimate and prove the weighted version of the weak reverse Hölder inequality for A-harmonic tensors.

1. Introduction

Harmonic functions have wide applications in many fields, such as potential theory, partial differential equations, harmonic analysis and the theory of $H^p$-spaces. A-harmonic tensors are interesting and important generalizations of $p$-harmonic tensors. In the meantime, $p$-harmonic tensors are extensions of conjugate harmonic functions and $p$-harmonic functions, $p > 1$. In recent years there have been remarkable advances made in the field of A-harmonic tensors. Many interesting results of A-harmonic tensors and their applications in fields such as potential theory, quasiregular mappings and the theory of elasticity have been found; see [1], [2], [3], [7], [8], [9], [10], [11], [12], [14]. For many purposes, we need to know the integrability of A-harmonic tensors and estimate the integrals for A-harmonic tensors. In this paper we first obtain the local weighted Caccioppoli-type estimate and the weighted version of the weak reverse Hölder inequality for A-harmonic tensors. These integral inequalities can be used to study the integrability of A-harmonic tensors and estimate the integrals for A-harmonic tensors.

We always assume $\Omega$ is a connected open subset of $\mathbb{R}^n$ throughout this paper. Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $l = 0, 1, \cdots, n$, the linear space of $l$-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered $l$-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, is denoted by $\wedge^l = \wedge^l(\mathbb{R}^n)$. The Grassmann algebra $\wedge = \bigoplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all $l$-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the Hodge star operator $*: \wedge \to \wedge$ by the rule $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge * \beta = \beta \wedge * \alpha = \langle \alpha, \beta \rangle (1)$ for all $\alpha, \beta \in \wedge$.
Hence the norm of $\alpha \in \Lambda$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = * (\alpha \wedge * \alpha) \in \Lambda^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \Lambda^l \to \Lambda^{n-l}$ and $\star \star (-1)^{l(n-l)} : \wedge^l \to \wedge^l$. Let $0 < p < \infty$; we denote the weighted $L^p$-norm of a measurable function $f$ over $E$ by
\[
||f||_{p,E,w} = \left( \int_E |f(x)|^p w(x) dx \right)^{1/p}.
\]

A differential $l$-form $\omega$ on $\Omega$ is a Schwartz distribution on $\Omega$ with values in $\Lambda(\mathbb{R}^n)$. We denote the space of differential $l$-forms by $D'(\Omega, \Lambda^l)$. We write $L^p(\Omega, \Lambda^l)$ for the $l$-forms $\omega(x) = \sum_{I} \omega_I(x) dx_I = \sum \omega_{i_1i_2\cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered $l$-tuples $I$. Thus $L^p(\Omega, \Lambda^l)$ is a Banach space with norm
\[
||\omega||_{p,\Omega} = \left( \int_{\Omega} |\omega(x)|^p dx \right)^{1/p} = \left( \int_{\Omega} \left( \sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.
\]

Similarly, $W^l_p(\Omega, \Lambda^l)$ is those differential $l$-forms on $\Omega$ whose coefficients are in $W^1_p(\Omega, \mathbb{R})$. The notations $W^{1}_{p,\text{loc}}(\Omega, \mathbb{R})$ and $W^{1}_{p,\text{loc}}(\Omega, \Lambda^l)$ are self-explanatory. We denote the exterior derivative by $\d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $\d^* : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$ is given by $\d^* = (-1)^{nl+1} \star \d \star$ on $D'(\Omega, \Lambda^{l+1})$, $l = 0, 1, \cdots, n$.

Recently there has been new interest developed in the study of the $A$-harmonic equation for differential forms, largely pertaining to applications in quasiconformal analysis and nonlinear elasticity, that is:

\[
(1.1) \quad d^* A(x, d\omega) = 0,
\]

where $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ satisfies the following conditions:

\[
(1.2) \quad |A(x, \xi)| \leq a|\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p
\]

for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1}_{p,\text{loc}}(\Omega, \Lambda^{l-1})$ such that

\[
\int_{\Omega} \langle A(x, d\omega), d\varphi \rangle = 0
\]

for all $\varphi \in W^{1}_{p}(\Omega, \Lambda^{l-1})$ with compact support.

**Definition 1.3.** We call $u$ an $A$-harmonic tensor in $\Omega$ if $u$ satisfies the $A$-harmonic equation (1.1) in $\Omega$.

Let us mention some basic terms for harmonic tensors as follows. A differential $l$-form $u \in D'(\Omega, \Lambda^l)$ is called a closed form if $du = 0$ in $\Omega$. A differential form $u$ is called a $p$-harmonic tensor if

\[
d^* |du|^p - 2 du = 0 \quad \text{and} \quad d^* u = 0,
\]

where $1 < p < \infty$. See [7] for more results about $p$-harmonic tensors. In order to formulate some estimates it is required first of all that the equation be written in the form of a first order differential system:

\[
(1.4) \quad A(x, du) = d^* v.
\]
In this way we obtain a pair \((u,v)\) of \((l-1)\)-form \(u\) and \((l+1)\)-form \(v\), called the conjugate \(A\)-harmonic fields. Example: \(du = d^*v\) is an analogue of a Cauchy-Riemann system in \(\mathbb{R}^n\). Clearly, the \(A\)-harmonic equation is not affected by adding a closed form to \(u\) and coclosed form to \(v\). Therefore, any type of estimates between \(u\) and \(v\) must be modulo such forms. Suppose that \(u\) is a solution to (1.1) in \(\Omega\). Then, at least locally in a ball \(B\), there exists a form \(v \in W^1_q(B, \Lambda^{l+1})\), \(\frac{1}{p} + \frac{1}{q} = 1\), such that (1.4) holds.

**Definition 1.5.** When \(u\) and \(v\) satisfy (1.4) in \(\Omega\), and \(A^{-1}\) exists in \(\Omega\), we call \(u\) and \(v\) conjugate \(A\)-harmonic tensors in \(\Omega\).

**Definition 1.6.** We call \(u\) a \(p\)-harmonic function if \(u\) satisfies the \(p\)-harmonic equation

\[
\text{div}(\nabla u \mid \nabla u|^{p-2}) = 0
\]

with \(p > 1\). Its conjugate in the plane is a \(q\)-harmonic function \(v\), \(p^{-1} + q^{-1} = 1\), which satisfies

\[
\nabla u \mid \nabla u|^{p-2} = \left( \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \right).
\]

Note that if \(p = q = 2\), we get the usual conjugate harmonic functions.

We write \(R = \mathbb{R}^1\). Balls are denoted by \(B\) and \(\sigma B\) is the ball with the same center as \(B\) and with \(\text{diam}(\sigma B) = \sigma \text{diam}(B)\). The \(n\)-dimensional Lebesgue measure of a set \(E \subseteq \mathbb{R}^n\) is denoted by \(|E|\). We call \(w\) a weight if \(w \in L^1_{\text{loc}}(\mathbb{R}^n)\) and \(w > 0\) a.e. Also in general \(d\mu = wdx\) where \(w\) is a weight. The following result appears in [8]: Let \(Q \subset \mathbb{R}^n\) be a cube or a ball. To each \(y \in Q\) there corresponds a linear operator \(K_y : C^\infty(Q, \Lambda^l) \rightarrow C^\infty(Q, \Lambda^{l-1})\) defined by

\[
(K_y \omega)(x; \xi_1, \cdots, \xi_l) = \int_0^1 t^l-1 \omega(tx + ty; x - y, \xi_1, \cdots, \xi_{l-1}) dt
\]

and the decomposition

\[
\omega = d(K_y \omega) + K_y(d\omega).
\]

We define another linear operator \(T_Q : C^\infty(Q, \Lambda^l) \rightarrow C^\infty(Q, \Lambda^{l-1})\) by averaging \(K_y\) over all points \(y\) in \(Q\)

\[
T_Q \omega = \int_Q \varphi(y) K_y \omega dy,
\]

where \(\varphi \in C^\infty_0(Q)\) is normalized by \(\int_Q \varphi(y) dy = 1\). We define the \(l\)-form \(\omega_Q \in D'(Q, \Lambda^l)\) by

\[
\omega_Q = |Q|^{-1} \int_Q \omega(y) dy, \quad l = 0, \quad \text{and} \quad \omega_Q = d(T_Q \omega), \quad l = 1, 2, \cdots, n,
\]

for all \(\omega \in L^p(Q, \Lambda^l), \ 1 \leq p < \infty\).

2. **The Local Weighted Caccioppoli-type Estimate**

**Definition 2.1.** We say the weight \(w(x)\) satisfies the \(A_r\) condition, \(r > 1\), written \(w \in A_r\), if \(w(x) > 0\) a.e., and, for any ball \(B \subset \mathbb{R}^n\),

\[
\sup_B \left( \frac{1}{|B|} \int_B w dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)} < \infty.
\]
See [5] and [6] for the basic properties of $A_r$-weights. We need the following lemma [5].

**Lemma 2.2.** If $w \in A_r$, then there exist constants $\beta > 1$ and $C$, independent of $w$, such that
\[
\| w \|_{\beta, B} \leq C|B|^{(1-\beta)/\beta} \| w \|_{1, B}
\]
for all balls $B \subset \mathbb{R}^n$.

We will also need the following generalized Hölder’s inequality.

**Lemma 2.3.** Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, then
\[
\| fg \|_s, \Omega \leq \| f \|_{\alpha, \Omega} \cdot \| g \|_{\beta, \Omega}
\]
for any $\Omega \subset \mathbb{R}^n$.

In [10], C. A. Nolder obtains the following local Caccioppoli-type estimate.

**Theorem A.** Let $u$ be an $A$-harmonic tensor in $\Omega$ and let $\sigma > 1$. Then there exists a constant $C$, independent of $u$ and $du$, such that
\[
\| du \|_{s, B} \leq C|B|^{-1}\| u - c \|_{s, \sigma B}
\]
for all balls or cubes $B$ with $\sigma B \subset \Omega$ and all closed forms $c$. Here $1 < s < \infty$.

The following weak reverse Hölder inequality appears in [10].

**Theorem B.** Let $u$ be an $A$-harmonic tensor in $\Omega$, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant $C$, independent of $u$, such that
\[
\| u \|_{s, B} \leq C|B|^{(t-s)/st}\| u \|_{t, \sigma B}
\]
for all balls or cubes $B$ with $\sigma B \subset \Omega$.

We now generalize Theorem A into the following local weighted Caccioppoli-type estimate for $A$-harmonic tensors.

**Theorem 2.5.** Let $u \in D'(\Omega, \Lambda^l)$, $l = 0, 1, \ldots, n$, be an $A$-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $A$-harmonic equation and $w \in A_r$ for some $r > 1$. Then there exists a constant $C$, independent of $u$ and $du$, such that
\[
\| du \|_{s, B, w} \leq C|B|^{-1}\| u - c \|_{s, \rho B, w},
\]
for all balls $B$ with $\rho B \subset \Omega$ and all closed forms $c$.

Note that (2.6) can be written as
\[
(2.6') \quad \left( \int_B |du|^s w dx \right)^{1/s} \leq \frac{C}{|B|} \left( \int_{\rho B} |u - c|^s w dx \right)^{1/s},
\]
or
\[
(2.6'') \quad \left( \int_B |du|^s \mu \right)^{1/s} \leq \frac{C}{|B|} \left( \int_{\rho B} |u - c|^s \mu \right)^{1/s},
\]
where the measure $\mu$ is defined by $d\mu = w(x)dx$ and $w \in A_r$. 

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Proof. Since \( w \in A_r \), for some \( r > 1 \), by Lemma 2.2, there exist constants \( \beta > 1 \) and \( C_1 > 0 \), such that

\[
\| w \|_{\beta,B} \leq C_1 |B|^{(1-\beta)/\beta} \| w \|_{1,B}
\]

for any cube or any ball \( B \subset \mathbb{R}^n \). Choose \( t = s\beta/(\beta - 1) \); then \( 1 < s < t \) and \( \beta = t/(t-s) \). Since \( 1/s = 1/t + (t-s)/st \), by Hölder’s inequality, Theorem A and (2.7), we have

\[
\| du \|_{s,B,w} = \left( \int_B \left( |du|^{1/s} \right)^s \, dx \right)^{1/s} \\
\leq \left( \int_B |du|^t \, dx \right)^{1/t} \left( \int_B \left( w^{1/s} \right)^{st/(t-s)} \, dx \right)^{(t-s)/st} \\
\leq C_2 \| du \|_{t,B} \cdot \| w \|_{\beta,B}^{1/s} \\
\leq C_3 |B|^{-1} \| u - c \|_{t,\sigma B} \cdot \| w \|_{\beta,B}^{1/s} \\
\leq C_4 |B|^{-1} \| |B|^{(1-\beta)/\beta} \| w \|_{1,B}^{1/s} \| u - c \|_{t,\sigma B} \\
= C_4 |B|^{-1} \| |B|^{-1/s} \| w \|_{1,B}^{1/s} \| u - c \|_{t,\sigma B}
\]

for all balls \( B \) with \( \sigma B \subset \Omega \) and all closed forms \( c \). Since \( c \) is a closed form and \( u \) is an \( A \)-harmonic tensor, then \( u - c \) is still an \( A \)-harmonic tensor. Taking \( m = s/r \), we find that \( m < s < t \). Applying Theorem B yields

\[
\| u - c \|_{t,\sigma B} \leq C_5 |B|^{(m-1)/mt} \| u - c \|_{m,\sigma^2 B} \\
\leq C_5 |B|^{(m-1)/mt} \| u - c \|_{m,\rho B}
\]

where \( \rho = \sigma^2 \). Substituting (2.9) in (2.8), we have

\[
\| du \|_{s,B,w} \leq C_6 |B|^{-1} \| |B|^{-1/m} \cdot \| w \|_{1,B}^{1/s} \| u - c \|_{m,\rho B}.
\]

Now \( 1/m = 1/s + (s-m)/sm \); by Hölder’s inequality again, we obtain

\[
\| u - c \|_{m,\rho B} = \left( \int_{\rho B} |u - c|^{m} \, dx \right)^{1/m} \\
= \left( \int_{\rho B} \left( |u - c|^{1/s} w^{1-s} \right)^m \, dx \right)^{1/m} \\
\leq \left( \int_{\rho B} |u - c|^{s} w \, dx \right)^{1/s} \left( \int_{\rho B} \left( \frac{1}{w} \right)^{m/(s-m)} \, dx \right)^{(s-m)/sm} \\
\leq \| u - c \|_{s,\rho B \cdot w} \cdot \| 1/w \|_{m/(s-m),\rho B}^{1/s}
\]

for all balls \( B \) with \( \rho B \subset \Omega \) and all closed forms \( c \). Combining (2.10) and (2.11), we obtain

\[
\| du \|_{s,B,w} \leq C_6 |B|^{-1} \| |B|^{-1/m} \cdot \| w \|_{1,B}^{1/s} \cdot \| 1/w \|_{m/(s-m),\rho B}^{1/s} \| u - c \|_{s,\rho B \cdot w}.
\]
Since $w \in A_r$, we then have
\[
\|w\|_{1,B}^{1/s} \cdot \|1/w\|_{m/(s-m),\rho B}^{1/s} = \left(\int_B \frac{w}{\rho B} \right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{m/(s-m)} dx\right)^{(s-m)/sm}
\]
\[
\leq \left(\left(\int_{\rho B} wdx\right) \left(\int_{\rho B} \left(\frac{1}{w}\right)^{1/(s/m-1)} \right)^{s/m-1} \right)^{1/s}\]
\[
= \left|\rho B\right|^{s/m} \left(\frac{1}{|\rho B|} \int_{\rho B} wdx\right) \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w}\right)^{1/(r-1)} \right)^{r-1} \right)^{1/s}\]
\[
\leq C_7 |B|^{1/m}.
\]
(2.13)

Substituting (2.13) in (2.12), we find that
\[
\|du\|_{s,B,w} \leq C|B|^{-1} \|u - c\|_{s,\rho B,w}
\]
for all balls $B$ with $\rho B \subset \Omega$ and all closed forms $c$. This ends the proof of Theorem 2.5.

3. The weighted version of the weak reverse Hölder inequality

We now generalize Theorem B into the following weighted form.

**Theorem 3.1.** Let $u \in D'(\Omega, \mathcal{A}_l)$, $l = 0, 1, \cdots, n$, be an $A$-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$. Assume that $0 < s, t < \infty$ and $w \in A_r$ for some $r > 1$. Then there exists a constant $C$, independent of $u$, such that
\[
\left(\int_B |u|^s wdx\right)^{1/s} \leq C|B|^{(t-s)/st} \left(\sigma B \int_{\sigma B} |u|^{t/s} w^{t/s} dx\right)^{1/t}
\]
for all balls $B$ with $\sigma B \subset \Omega$.

The proof of Theorem 3.1 is similar to that of Theorem 2.5. For completion of the paper, we prove Theorem 3.1 as follows.

**Proof.** Since $w \in A_r$ for some $r > 1$, by Lemma 2.2, there exist constants $\beta > 1$ and $C_1 > 0$, such that
\[
\|w\|_{\beta,B} \leq C_1 |B|^{(1-\beta)/\beta} \|w\|_{1,B}
\]
for any cube or any ball $B \subset \mathbb{R}^n$. Choose $k = \sigma/(\beta - 1)$; then $s < k$ and $\beta = k/(k-s)$. By (3.3) and Hölder’s inequality, we have
\[
\|u\|_{s,B,w} \leq \left(\int_B |u|^k dx\right)^{1/k} \left(\int_B \left|w^{1/s}\right|^{sk/(k-s)} dx\right)^{(k-s)/sk}
\]
\[
= \|u\|_{k,B} \cdot \|w\|_{1,B}^{1/s} \leq C_2 |B|^{-1/k} \|u\|_{1,B}^{1/s} \cdot \|u\|_{k,B}
\]
(3.4)
for all balls $B$ with $\sigma B \subset \Omega$. Choosing $m = st/(s + t(r - 1))$, by Theorem B we obtain
\begin{equation}
\|u\|_{k,B} \leq C_3|B|^{(m-k)/km}\|u\|_{m,\sigma B}.
\end{equation}
Combining (3.4) and (3.5) yields
\begin{equation}
\|u\|_{s,B,w} \leq C_4|B|^{-1/m}\|w\|_{1,B}^{1/s} \cdot \|u\|_{m,\sigma B}.
\end{equation}
Since $m < t$, by Hölder’s inequality, we have
\begin{equation}
\|u\|_{m,\sigma B} = \left(\int_{\sigma B} \left(|u|^s w^{1/s} w^{-1/s}\right)^m dx\right)^{1/m} \leq \left(\int_{\sigma B} |u|^t w^{t/s} dx\right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{mt/(s(t-m))} dx\right)^{(t-m)/mt} \leq \|1/w\|_{mt/(s(t-m)),\sigma B}^{1/s} \left(\int_{\sigma B} |u|^t w^{t/s} dx\right)^{1/t}.
\end{equation}

By the choice of $m$, we find that $r-1 = s(t-m)/mt$. Since $w \in A_r$, we then obtain
\begin{equation}
\|w\|_{1,B}^{1/s} \cdot \|1/w\|_{mt/(s(t-m)),\sigma B}^{1/s} = \left(\int_B w dx\right)^{1/s} \left(\int_{\sigma B} \left(\frac{1}{w}\right)^{mt/(s(t-m))} dx\right)^{s(t-m)/mt} \leq \left|\sigma B\right|^{1+s(t-m)/tm} \left(\frac{1}{|\sigma B|} \int_{\sigma B} w dx\right)^{1/s} \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1} \leq C_5|B|^{1/s+1/m-1/t}.
\end{equation}

From (3.6), (3.7) and (3.8), we have
\begin{equation}
\|u\|_{s,B,w} \leq C_4|B|^{-1/m}\|w\|_{1,B}^{1/s} \cdot \|1/w\|_{mt/(s(t-m)),\sigma B}^{1/s} \left(\int_{\sigma B} |u|^t w^{t/s} dx\right)^{1/t} \leq C_6|B|^{1/s-1/t} \left(\int_{\sigma B} |u|^t w^{t/s} dx\right)^{1/t}.
\end{equation}
It is easy to see that (3.9) is equivalent to (3.2). This ends the proof of Theorem 3.1.

**REFERENCES**


Department of Mathematics and Statistics, University of Minnesota at Duluth, Duluth, Minnesota 55812-2496

E-mail address: sd ding@d.umn.edu

Current address, after 9-1-99: Department of Mathematics, Seattle University, Seattle, Washington 98122