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PONCELET'S THEOREM IN SPACE

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ABSTRACT. A plane polygon $\mathcal{P}$ inscribed in a conic C and circumscribed to a conic D can be continuously 'rotated', as it were. One of the many proofs consists in viewing each side of $\mathcal{P}$ as translation by a torsion point of an elliptic curve. In the n-space version, involving torsion points of hyperelliptic Jacobians, there is a $g = (n-1)$-dimensional family of rotations, where $g =$ genus of the hyperelliptic curve; the polygon is now inscribed in one and circumscribed to $n-1$ quadrics.

This note presents a generalization of Poncelet's theorem from plane configurations to configurations in projective space $\mathbf{P}^{g+1}$ (we work over the complex numbers throughout). In the plane, one way to state the theorem produces a 'rotation' of any n-gon inscribed in a conic C and circumscribed to a conic D. The key to the theorem is a point of order n of a suitable elliptic curve; each point of the curve corresponds to a value of the 'rotation'; cf. [GH1], [GH2]. In $\mathbf{P}^{g+1}$, the analogous theorem involves a point of finite order on the Jacobian of a hyperelliptic curve X of genus g; the corresponding configuration is an n-gon tangent to g fixed (confocal) quadrics and inscribed in a $(g+1)$-st one; again the vertices can be continuously 'rotated' corresponding to the points of the Jacobian Jac$X$, giving rise to a g-dimensional family of n-gons.

The configuration of common tangents to g (confocal) ellipsoids in $\mathbf{R}^{g+1}$ has been studied in connection with (integrable) billiards; indeed, integrability of the billiard with elliptical boundary gives one of the many proofs of Poncelet's theorem. A generalization of the theorem is thus given by one or the other of the two (equivalent) dynamical problems: geodesic motion on the g-dimensional ellipsoid, or bounces on a g-dimensional elliptical boundary (in the Appendix, we explain the reason why billiard motion was associated to the geodesic equations on the ellipsoid in two essentially different ways, either as the limit of geodesic motion in one-higher dimension when one axis is shrunk to a point, or as discretized geodesic motion—a stroboscope in the language of [DLT]—on an ellipsoid of the same dimension as the billiard boundary). What is new in this note is the identification of the point of finite order as $\mathcal{O}(P - \iota P)$, where P is a point on X corresponding to the billiard boundary, and ι is the hyperelliptic involution. This provides an algebraic statement analogous to the elliptic case, as well as generalizations of other versions of the theorem which involve adding several points successively, hence possibly any point of Jac$X$ of the form $\mathcal{O}(P_1 + ... + P_g - \iota P_1 + ... + \iota P_g)$. We obtain our result by

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going through the model of Jac$X$ given in [Re] which uses the geometry of $\mathbf{P}^{2g+1}$, and the dictionary between this model and the phase space for the geodesic flow given in [K1].

By providing the abelian-variety interpretation of the configuration, we can bring together several versions of Poncelet-type results, including two 3-dimensional: one by Griffiths and Harris which we reduce to Darboux's theorem in the plane, and one by Weyr (1870, I owe this reference to [BB]) which is a translation of Poncelet's theorem into the Reid model in $\mathbf{P}^{2g+1}$ (here $g = 1$).

We conclude by mentioning three deep directions in which the generalized Poncelet theorem could be applied (arithmetic questions, vector bundles, and compactifications). Poncelet's theorem has an extraordinary permanence and versatility which the focus of this note prevents me from acknowledging; I must content myself with thanking W. Barth, D. Goroff, J.L. King, G. Trautman, B. Weiss; and also Paula Carey of the BU Libraries, who generously shared information and materials for a survey-to-be. I also wish to express my gratitude for the hospitality of the Bunting Institute of Radcliffe College (1995–96).

§1. Poncelet and Darboux

1.1 Two equivalent versions of Poncelet.

Version I ([GH1])**.** Let C, D be two plane conics in general position (i.e., the 4 points of intersection are distinct). If there exists a nondegenerate n-gon Π_n which is inscribed in C and circumscribed to D (i.e., Π_n is a polygon with n distinct vertices belonging to C and n distinct sides tangent to D), then there is a 1-parameter family of such n-gons, which can be viewed as 'rotations' of Π_n (i.e., any vertex p can be moved continuously over C and the n-gon will follow).

Note. The word rotation originates with the picture over $\mathbf{R}$. Indeed, two conics can be mapped by a projective transformation of the plane to a pair of circles, C surrounding D, say. The n-gon can actually be rotated if the circles are concentric, but this corresponds to the case of bitangent conics, which we excluded and which can be viewed as a degenerate limit; the elliptic curve defined below in 1.2 acquires a singularity.

Version II ([BKOR, 4.1.1])**.** Let $C, D_1, \ldots, D_{n-1}$ be conics from one pencil. Consider an n-gon inscribed in C whose first vertex is p and whose first $n-1$ sides are tangent to the successive D_i. Then, as p moves along C, the nth side l of the polygon will envelop a curve which is again a conic belonging to the same pencil.

One of the most satisfying explanations for both statements is given by introducing an elliptic curve (we follow [GH1]) and observing that the process of going from one side of the polygon to the next corresponds to the addition of a particular point on the curve, which is independent of the choice of vertex p on the conic C; the polygon closes when the result of the successive additions is zero.

1.2 The curve and the point. The incidence correspondence $I = \{(p, l) : \ p \in l\} \subset C \times D^*$ is a curve of genus 1 (here D^* is the curve of tangents to D). Both involutions, $i_1 : (p, l) \mapsto (p', l)$ interchanging the points of contact $\{p, p'\} = l \cap C$ and $i_2 : (p, l) \mapsto (p, l')$ interchanging the tangents to D which meet at $p = l \cap l'$, have fixed points, so their composition is a translation on an elliptic curve $E = \mathbf{C}/\Lambda$

(obtained by a choice of origin on I), $i_2 \circ i_1 : z \mapsto z + \tau$, with $z, \tau \in \mathbf{C}$. The configuration of Poncelet (Version I) corresponds to a point τ of order n.

The elegance of this explanation somewhat suppresses the subtlety involved in encoding the moduli of the situation. In order to translate from transcendental data to algebraic we want a geometric way to: (i) add a given point τ; (ii) add any number of points of E. We achieve this by introducing different models of the curve and its Jacobian, which generalize to any hyperelliptic curve.

To have a model for adding other points of the curve, we consider the whole pencil spanned by C and D; but in order to make contact with two beautiful dynamical problems studied respectively by C.G.J. Jacobi (geodesics on an ellipsoid) and G.D. Birkhoff (billiards inside an ellipse), we dualize the picture and consider the confocal family $Q_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1$ where $\frac{x^2}{a} + \frac{y^2}{b} = 1$ ($a > b$) is an ellipse to which we refer for the classical problems studied over $\mathbf{R}$.

1.3 Remarks. (i) The conics Q_λ^* are given by equations $(a-\lambda)u^2 + (b-\lambda)v^2 = 1$ and l is tangent to Q_λ at p iff p^* is tangent to Q_λ^* at l^* [K1, §1].

(ii) Every pair of conics can be brought into a confocal pair by a projective transformation of the plane. A linear pencil can be brought into a confocal family (quadratic pencil) by a Veronese map. Correspondingly, a point p belongs to one (resp. two) element(s) of a pencil (resp. confocal family), etc. (Readers of [BM] will see that this is the reason behind the fact that the number of n-inscribed conics to a given conic is twice the number of n-circumscribed ones.)

(iii) For two confocal conics C, D, the following maps are the same: $i : (p, l) \mapsto (p, l')$ as in 1.2, and $r : (p, l) \mapsto (p, l'')$ where l'' is defined by a perfect reflection (l, l' make equal angles with the tangent line t_C to C at p) when all these objects are defined over $\mathbf{R}$. To express this condition in projective geometry, we ask that the line t_C, the line u through p and the D-pole of t_C, and the pair l, l'', be a harmonic quadruple (cf. [CCS1]).

Proof. The dual condition defining i says that $\{l^*, l'^*\} = p^* \cap D^*$; the point $t_C^* \in p^*$ and $u^* = p^* \cap$ (D^*-polar line of t_C^*).

(iv) In this paper we only consider generic situations, in particular the four base points of the pencil are distinct. When that is the case, it is known (cf. [J], Ch. 11) that the 3×3 matrices of two generators of the pencil can be simultaneously diagonalized. This is not always true; the singular situation gives rise to four different configurations (cf. [BKOR, §7]) according to the "Segre symbols" [J] for the elementary divisors: [21], [(11)1], [(3)], [(21)], while [(111)] doesn't correspond to a pencil. Notice that in $\mathbf{P}^{2g+1}$ (for general genus g, cf. §2 below) there are many more possibilities for singular pencils of quadrics; while in the ellipsoidal model we give $g+1$ branchpoints of the curve separate treatment, it may be possible to obtain Poncelet-type theorems for the other Segre symbols as well.

1.4 The curve. We give three models of the same elliptic curve. The only data we need is a confocal family together with a fixed conic D of the family which, to fix ideas, we can think of as an ellipse: $\frac{x^2}{a-\beta} + \frac{y^2}{b-\beta} = 1$.

(a) The elliptic curve E is given in Weierstrass form by the equation:

$$\mu^2 = 4(\lambda - a)(\lambda - b)(\lambda - \beta)$$

where $k = (\frac{a-b}{a-\beta})^{1/2}$ is the eccentricity of D; the corresponding abelian variety $\mathrm{Jac}E = \mathbf{C}/\Lambda \cong E$ is determined by choosing the point at infinity as the origin on E. Note that the j-invariant of the curve E appears more naturally in the dual model; indeed, the confocal family $\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}=1$ corresponds to a confocal pencil spanned by $D_1=C^*:=(\frac{x^2}{a}+\frac{y^2}{b}-1)^*$ and $C_1=D^*$; the curve is given by $I\subset C_1\times D_1^* \stackrel{2:1}{\to} C_1$ and a choice of origin can be made so that the curve can be identified with the Riemann surface of the rational function $\sqrt{\det(tC_1+D_1)}$ where C_1, D_1 also denote the 3×3 matrices of the quadratic forms.

Note. The curve changes, however, if one views it as the double cover of different C_λ's branched over the base locus of the pencil; cf. [BM]. This is why it is more profitable to define the curve as in (b) or (c) below.

(b) While in (a) the curve was presented as the double-cover of a conic, we can view it as the double-cover of the $\mathbf{P}^1$ parametrizing the family as follows:

(b_1) fix a generic point in D (any point such that the tangent line $t\in D^*$ at that point isn't one of the 4 base points of the dual pencil). For any λ except: $\lambda=a, \lambda=b$, $\lambda=\beta$ or $\lambda=\infty$, the tangent line t meets $C_\lambda: \frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}=1$ in two points, which we can regard as P_λ and ιP_λ. Alternatively,

(b_2) for any λ, as a tangent to D runs through D^*, it has two intersections with C_λ except for the 4 base points given by $D^*\cap C_\lambda^*$; thus, for $\lambda\neq a,b,\beta,\infty$ we can view the two points $P_\lambda, \iota P_\lambda$ as the points of the elliptic curve that cover D and is branched at the 4 [base points]*.

(c) The curve is the base locus of the pencil of quadrics in $\mathbf{P}^3$ spanned by:

$$x_1^2+x_2^2-x_3^2=0, \quad ax_1^2+bx_2^2+\beta x_3^2=x_0^2.$$

Proof. The proof is a picture that links the billiard and the geodesic system via elliptic coordinates: $(x,y)\mapsto(\lambda_1,\lambda_2)$ such that $\frac{x^2}{a-\lambda_i}+\frac{y^2}{b-\lambda_i}-1=0$ $(i=1,2)$. The trajectory of a particle that bounces on C and is always tangent to D is then described by the abelian integrals:

$$\text{(1.4.1)}\qquad \int_0^{z_1}\frac{dz}{\sqrt{4z(1-z)(1-zk^2)}}-\int_\chi^{z_2}\frac{dz}{\sqrt{4z(1-z)(1-zk^2)}}=\text{ const.},$$

$$\text{(1.4.2)}\qquad \int_0^{z_1}\frac{zdz}{\sqrt{4z(1-z)(1-zk^2)}}-\int_\chi^{z_2}\frac{zdz}{\sqrt{4z(1-z)(1-zk^2)}}=(\text{const.})'t$$

where $z_i=\frac{a-\lambda_{3-i}}{a-b}$, $\chi=\frac{1}{k^2}=\frac{a-\beta}{a-b}$ change the branchpoints into $0,1,\chi,\infty$; cf. [CCS2].

This says that the billiard system with boundary any conic of the confocal family is governed by the same elliptic curve (a) provided the trajectories are tangent to the same "caustic" D. Now, the geodesic flow on the ellipse D has the same integrals (cf. [K2]), due to the fact that the tangent lines to a geodesic remain tangent to g (=1 here) confocal quadrics. This brings in (c) via Knörrer's work [K1], in which Reid's model [Re] for Jac E, namely the set of $\mathbf{P}^{g-1}$'s contained in the base locus M of the pencil of quadrics (c), plus a choice of origin, is mapped $2^g:1$ to the lines tangent to the fixed g confocal quadrics. We give more detail in §2 for the $g>1$ case, but indicate here how the picture projects from $\mathbf{P}^3$ to (b). The map $[x_0,\dots,x_3]\mapsto[x_0,x_1,x_2]\stackrel{*}{\mapsto}[w_0,w_1,w_2]$ where $*$ is duality, sends the points of M to D^*. The curve is given, in terms of the pencil of quadrics in $\mathbf{P}^3$, by choosing an origin for Jac E, namely a point m of M. The two points of the curve over λ are the two lines of Q_λ containing m, or equivalently, their intersections p_λ and ιp_λ

with M. Their images in D^*, which we just described, viewed as tangent lines to D, meet at a point of C_λ. $\square$

1.5 Remark. Model (c) gives Weyr's theorem (cf. [BB, 1.1]), namely the closure statement for the following configuration: pick rulings a_1 and a_2 for generic quadrics Q_1 and Q_2 in $\mathbf{P}^3$ and alternate a line in a_s with the line in a_t ($s \neq t = 1$ or 2) which meets it on the intersection $Q_1 \cap Q_2$.

1.6 The point. Adding a point of the curve in the various representations of 1.4 corresponds to the following operations:

(a) for (p, l) to change to (p', l'), where $p \in C_\lambda$ and the points of l have coordinates (z_1, z_2) given in (1.4.1), the constant $2\int_\chi^\lambda \frac{dz}{\sqrt{4z(1-z)(1-zk^2)}}$ gets added to (1.4.1).

(b) The divisor that we add as a point of Jac E is $P_\lambda - \iota P_\lambda$.

(c) In order to add $\pm(P_\lambda - \iota P_\lambda)$ to a point of Jac E viewed as a $\mathbf{P}^{g-1}$ in $\mathbf{P}^{2g+1}$, we take the other intersection with M of one of the two $\mathbf{P}^g$'s contained in Q_λ that go through the given $\mathbf{P}^{g-1}$ and again the intersection with M of the other $\mathbf{P}^g$ through it.

Proof. Calculation (a) is worked out in [CCS2]. However, (b) and (c) require some choices, since Jac E and the various configurations only correspond when we fix an origin—which we pick to correspond to a branch point of the curve; the result is then obtained in §4 of [K1] and §2 of [D]. $\square$

In view of these models for the curve and the point, the algebraic proof of Poncelet's result recalled in 1.2 yields the following:

1.7 Theorem. *Poncelet's polygon inscribed in the conic C_λ closes after n steps iff the point $P_\lambda - \iota P_\lambda$ has order n in E, independently of the starting point. The same is true for the closure of Poncelet's polygons obtained by bouncing off any choice of successive confocal conics $C_{\lambda_1}, ..., C_{\lambda_n}$, iff the sum $(P_1 - \iota P_1) + (P_2 - \iota P_2) + \ldots + (P_n - \iota P_n)$ is zero. These statements are versions I and II, respectively.*

1.8 Corollary (G. Darboux, cf. [T])**.** *Assume that there is an n-gon inscribed in C and circumscribed to D. There is a unique curve N of degree $n-1$ that contains all of the $\binom{n}{2}$ intersections of the sides of this n-gon; N splits into $[\frac{n-1}{2}]$ conics and $2(\frac{n-1}{2} - [\frac{n-1}{2}])$ lines.*

Proof. Version I of Poncelet gives a pencil of linearly equivalent divisors on D, namely its intersections with the sides of the polygon as one vertex moves along C. This pencil determines N uniquely [T, Proposition 1.3] by a determinantal condition. The reducibility statement follows from 1.4 and 1.5; indeed, the polygon corresponds to a cycle $(1, \ldots, n)$ in the group E (where numbers stand for successive vertices). By joining $1 \to 3 \to 5$, etc., $1 \to 4 \to 7$, etc. $\ldots$, one obtains $[\frac{n-1}{2}]$ distinct conics inscribed in the polygons corresponding to these cycles; there is a degenerate one, namely a line, iff n is even (transposition $1 \mapsto \frac{n}{2} + 1 \mapsto 1$). These configurations correspond to subgroups of the same elliptic curve; to obtain the model of 1.4-1.5 the situation must be dualized and C becomes the caustic of whose n tangents we take the $\binom{n}{2}$ intersections. $\square$

1.9 Remark. Pascal's theorem on the mystic hexagon is the case $n = 6$ of 1.7.

1.10 Corollary ([GH1])**.** *Let Q_1, Q_2 be (generic) quadrics in $\mathbf{P}^3$ and let K be a cone of the pencil $\langle Q_1, Q_2 \rangle$ with vertex O. Project Q_i from O and call C_1, C_2 the* *branch loci of Q_1, Q_2 and D the projection of K. Then, the plane Poncelet n-gons of tangents to D, each of whose sides has a vertex on C_1 and one on C_2, correspond exactly to the Griffiths-Harris polyhedra, namely planes bitangent to Q_i along the lines that project to the vertices of the polygon.*

Indeed, it is pointed out in [GH1] that this closure configuration corresponds to two points of finite order of the elliptic curve given by the intersection of the two quadrics and a choice of origin. (A more complete polyhedron for quadrics $Q_1, \dots, Q_{[\frac{n-1}{2}]}$ of the pencil could be obtained using Darboux's theorem (1.7).) In particular, the genus 1 curve corresponding to the plane configuration is the same as the curve $Q_1 \cap Q_2$.

1.11 Remark. In [GH1], a specially symmetric configuration is described (by explicit equations), in order to construct examples of Poncelet polyhedra, in which the quadrics Q_1 and Q_2 are interchanged by an automorphism of $\mathbf{P}^3$. A special case of this occurs when C_1 and C_2 are the same conic C. An example of this is Pascal's theorem (1.9), where the hexagon is circumscribed to D and is to be viewed as the union of two triangles ($n = 3$ for 1.10).

§2. The theorems in space

2.1 Two equivalent versions. We consider a confocal family of quadrics $\mathcal{E}_\lambda$: $\frac{x_1^2}{a_1-\lambda} + \dots + \frac{x_{g+1}^2}{a_{g+1}-\lambda} = 1$ in $\mathbf{P}^{g+1}$ (the physical interpretation takes place on the ellipsoid $\mathcal{E}_0,\ a_1 > \dots > a_{g+1}$). We denote by T the variety of lines that are tangent to g fixed confocal quadrics $\mathcal{E}_{b_j}$ ($j = 1, \dots, g$). We give a projective definition of a geodesic $\alpha(t)$ on $\mathcal{E}_k = \mathcal{E}_{b_k}$ ([K1], 4.3): $l(t) = \langle \alpha(t), \dot\alpha(t) \rangle \in T$ for all t, and $\ddot\alpha(t) \in \bigcap_{j \neq k} T_j(t)$ where $T_j(t)$ is the tangent hyperplane to $\mathcal{E}_k$ at the point of contact with $l(t)$. We give a projective definition of reflection [CCS1]: a line $l \in T$ reflects to a line $l' \in T$ at the point $p = l \cap l' \in \mathcal{E}_0$ if all of the quadruples l, l', m, t_j are harmonic where: m is the intersection of the plane spanned by l, l' with the tangent hyperplane $T_p\mathcal{E}_0$; and t_j is the line joining p with the $\mathcal{E}_j$-pole of $T_p\mathcal{E}_0$. Notice that for $l \in T$ there are 2^g possible choices of $l' \in T$; the above definition of reflection picks l' so that the tangent hyperplanes to $\mathcal{E}_k$ at the points of contact with l, l' meet in a space which is tangent to $\mathcal{E}_0$ at p ([K1], 4.9). Over the reals, reflection corresponds to a perfect bounce on $\mathcal{E}_0$ and it is true that such billiard motion remains tangent to g confocal quadrics, so that the system is completely integrable (cf. [CF]); likewise, the tangent line $l(t)$ to a geodesic on $\mathcal{E}_k$ will remain in T if $l(0) \in T$ (cf. [K1]).

Version I. If there is a polygon inscribed in $\mathcal{E}_0$ and circumscribed to $\mathcal{E}_k$ ($k = 1, \dots, g$) so that its successive sides are linked by a reflection on $\mathcal{E}_0$, then there is a g-dimensional family of such polygons obtained by moving any vertex on $\mathcal{E}_0$.

Version II. By fixing $\mathcal{E}_k$ ($k = 1, \dots, g$) and choosing any number of confocal quadrics $\mathcal{E}_{\lambda_0}, \dots, \mathcal{E}_{\lambda_n}$, one can assume to have a Poncelet polygon whose successive sides are obtained by bouncing on $\mathcal{E}_{\lambda_j}$, in which case there is a g-dimensional family of such n-gons obtained by moving any of the vertices p_j over $\mathcal{E}_{\lambda_j}$.

2.2 The curve. As with $g = 1$, the hyperelliptic curve X can be defined using different projective constructions:

(a) $\mu^2 = \prod_{i=1}^{g+1}(\lambda - a_i)\prod_{j=1}^{g}(\lambda - b_j) = f(\lambda)$ gives the hyperelliptic integrals which, in analogy to (1.4.1) and (1.4.2), linearize the geodesic equations ([M], [CCS2], [CF]).

(b) For a generic point of $\mathcal{E}_k$, a line $l \in T$ meets $\mathcal{E}_\lambda$ in two points; the curve is the double cover of $\lambda \in \mathbf{P}^1$ given by those two points. More precisely, it is enough to choose a point of $\mathcal{E}_k$ whose dual in $(\mathbf{P}^{g+1})^*$ does not belong to the $(g-1)$-dimensional base locus of the pencil $\mathcal{E}^*_\lambda$.

(c) In Reid's model, we consider the pencil of quadrics in $\mathbf{P}^{2g+1}$ spanned by $Q_1 : \sum_1^{g+1} a_i x_i^2 - \sum_1^g b_i y_i^2 = x_0^2$, $Q_2 : \sum_1^{g+1} x_i^2 - \sum_i^g y_i^2 = 0$ [K1]; the curve is the 2:1 cover of the pencil which to each point associates the 2 families of $\mathbf{P}^g$'s contained in the corresponding quadric. It can also be described as the closure of the set of $\mathbf{P}^{g-1}$'s l in the base locus $M = Q_1 \cap Q_2$ that meet a fixed $\mathbf{P}^{g-1} = l_0 \subset M$ in a subspace of codimension one. The double cover sends these points to the unique quadric of the pencil that contains the span $\langle l_0, l\rangle$. The latter description includes a choice of origin to represent the curve inside its Jacobian.

2.3 The point. Keeping track of the passage from one side of the polygon to the next by means of the abelian sum is somewhat more delicate than in the case $g = 1$, for which the curve is isomorphic to its Jacobian. In higher genus, first we associate a point of Jac X to a point of the phase space $(x_1, \dots, x_{g+1}; \dot{x}_1, \dots, \dot{x}_{g+1})$ with $\sum_{i=1}^{g+1} \frac{x_i^2}{a_i} = 1$, $\sum_{i=1}^{g+1} \frac{x_i \dot{x}_i}{a_i} = 0$ or, in ellipsoidal coordinates,

$$(\lambda_1, \dots, \lambda_{g+1}, \sqrt{f(\lambda_1)}, \dots, \sqrt{f(\lambda_{g+1})})$$

with $\sum_{i=1}^{g+1} \frac{x_i^2}{a_i - \lambda_j} = 0$, $j = 1, \dots, g+1$. Then, to a point of the phase space, which we can view as a point on a billiard trajectory or a geodesic, we associate the trajectory obtained by bouncing on a confocal quadric $\mathcal{E}_\lambda$, or replacing the geodesic γ by the geodesic γ' traced by the reflected ray ([K1], 4.9). This is interpreted as adding the point $P_\lambda - \iota P_\lambda$ of Jac X, as in $g = 1$.

(a) In abelian coordinates, after normalizing the branch points of X by the linear fractional transformation $\lambda \mapsto z = \frac{a_1 - \lambda}{a_2 - a_1}$ and renaming them: $\chi_1 = 0, \chi_2 = 1$; $\chi_3, \chi_4; \dots\ \chi_{2g+1}, \infty$, the billiard flow is given by equations:

$$\sum_{i=0}^{g-1} \int_{\chi_{2i+1}}^{z_{i+1}} \frac{z^j dz}{\sqrt{f(z)}} = (\text{const.})_j, \quad j = 0, \dots, g-1, \tag{2.3.1}$$

$$\sum_{i=0}^{g-1} \int_{\chi_{2i+1}}^{z_{i+1}} \frac{z^g dz}{\sqrt{f(t)}} = t \tag{2.3.2}$$

and bouncing on $\mathcal{E}_\lambda$ is given by the addition of $2\int_{\chi_{2g+1}}^{\lambda} \frac{z^j dz}{\sqrt{f(z)}}$ to the corresponding coordinate.

(b) If the embedding of the curve viewed as the double cover of the confocal family as in 2.2(b) in its Jacobian is effected by sending a branch point to the origin, the operation in (a) corresponds to adding $\pm(P_\lambda - \iota P_\lambda)$ to the given point, up to translation by a branch point.

(c) In Reid's model, with suitable choices as in (b) above and [D], [K1], the operation $\pm(P_\lambda - \iota P_\lambda)$ corresponds to composing the two involutions associated to

P_λ and ιP_λ; each involution takes a $\mathbf{P}^{g-1} \subset M$ to the other $\mathbf{P}^{g-1} \subset M$ contained in one of the two $\mathbf{P}^g$'s $\subset Q_\lambda$ that go through the original $\mathbf{P}^{g-1}$.

Proof. (a) is given in [CCS2] and [M], (b) and (c) in [D] and [K1]. □

2.4 Theorem. *Versions I and II of Poncelet's theorem in space hold; the closure is equivalent to the point* $n(P_\lambda - \iota P_\lambda)$, *and* $(P_1 - \iota P_1) + (P_2 - \iota P_2) + \ldots + (P_n - \iota P_n)$ *resp., being zero.*

2.5 Corollary (Darboux's theorem in space)**.** *If a Poncelet n-gon occurs as in 2.1, then in the dual* $(\mathbf{P}^{g+1})^*$ *any $(g+1)$-tuple of hyperplanes tangent to* $\mathcal{E}^*_\lambda$ *at the point of contact with a side of the polygon meet at a point of a surface of degree $n-1$ which splits into* $[\frac{n-1}{2}]$ *quadrics, and a hyperplane if n is even (cf.* [T]*, 7.3, for a result of similar nature).*

§3. Problems

Just investigating this simple description of the hyperelliptic sum can lead into three deep directions.

Direction I. Cayley [GH2] gave an algebraic condition for given conics C, D to fit in the n-gon configuration. By using $tQ_1 + Q_2$ in $\mathbf{P}^{g+1}$ it should be possible to imitate his coordinatization and give a determinantal condition for the closure to hold. In higher genus this will have a finite number of solutions [Ra] as was conjectured by S. Lang, as it corresponds to points $P_\lambda - \iota P_\lambda$ of finite order. Note here again as in 2.3 the fundamental difference between genus 1 or higher. In genus 1, the number of conics in a given pencil n-circumscribed to a fixed one is related to the number of points of order n of a given elliptic curve, as well as to the number of corresponding sections of a suitable modular curve [BM]. This is because a conic in the pencil can be viewed both as a variable point on a fixed curve, and as a varying curve. For higher genus, in the analog of the Cayley equations the curve varies. A question inspired by the approach of [H] to enumerative problems, then, is whether the Galois group of the solutions is solvable, which amounts to calculating a monodromy group. Moreover, recent evidence ([H],[S]) points to the fact that there is a relationship between the structure of the Galois group and the existence of real configurations. As regards reality conditions, it is worth pointing out that curves with a real point of finite order $P - \iota P$ are "Toda curves" and were shown to be dense in hyperelliptic moduli space [McKvM].

Direction II. The moduli space of rank r bundles over a hyperelliptic curve has also been related to certain projective configurations (cf. e.g. [DR], [N], [R]) connected with the intersection of two quadrics. One could ask if tensoring by a line bundle of finite order over X, or over a "spectral curve" $S \to X$ whose line bundles push forward to given vector bundles, produces Poncelet configurations; and then if this is a "stroboscope" (discretized motion) of the Hitchin system (cf. [vGP]).

Direction III. Degenerate configurations correspond to generalized Jacobians and singular pencils of quadrics. As noted in 1.3, different limits already abound in $g = 1$. In higher genus, the moduli spaces are still open to investigation. Even in genus 1, an explicit deformation that does not appear to have been studied is the agM transformation [C], by which the elliptic curve approaches a rational limit.

The agM can be interpolated by a continuous flow in moduli space (P. Deift and C. Tomei, private communication); do the intermediate polygons stay closed?

Appendix. Discrete integrable systems: Two views of the billiard

It often happens that an algebraically completely integrable system admits more than one "spectral curve" whose abelian integrals provide linear coordinates for the flow. In the case of billiards with g-dimensional boundary, it is known that the flow can be viewed as a degenerate geodesic flow for a $(g+1)$-dimensional ellipsoid when one of the axes approaches zero [KT]. However, we saw in 2.2 that the flow can be linearized on the Jacobian of a genus g hyperelliptic curve, as can the geodesic flow on a g-dimensional quadric, confocal with the boundary of the billiard. This interesting phenomenon is an example of "stroboscope" (cf. [DLT]) and was noted in [V], where the coordinates for the successive vertices are given in terms of theta functions; as in [K1], these involve theta characteristics associated with the branchpoints $a_1, \ldots, a_{g+1}$, and successive additions of a vector $\vec{U}$ which corresponds (in Abel coordinates) to $P_\lambda - \iota P_\lambda$.

References

[BB] W. Barth and Th. Bauer, Poncelet theorems, *Expositiones Math.* **14** (1996), 125–144. MR **97f:**14051

[BM] W. Barth and J. Michel, Modular curves and Poncelet polygons, *Math. Ann.* **295** (1993), 25-49. MR **94c:**14045

[BKOR] H.J.M. Bos, C. Kers, F. Oort and D.W. Raven, Poncelet's closure theorem, *Expositiones Math.* **5** (1987), 238-364. MR **88m:**14041

[CCS1] S.-J. Chang, B. Crespi and K.-J. Shi, Elliptical billiard systems and the full Poncelet's theorem in n dimensions, *J. Math. Phys.* **34** (1993), 2242-2256. MR **94g:**58092

[CCS2] B. Crespi, S.-J. Chang and K.-J. Shi, Elliptical billiards and hyperelliptic functions, *J. Math. Phys.* **34** (1993), 2257-2289. MR **94g:**58093

[Co] D.A. Cox, The arithmetic-geometric mean of Gauss, *Enseign. Math.* **30** (1984), 275-330. MR **86a:**01027

[CF] S.-J. Chang and R. Friedberg, Elliptical billiards and Poncelet's theorem, *J. Math. Phys.* **29** (1988), 1537-1550. MR **89j:**58043

[DLT] P. Deift, L.-C. Li and C. Tomei, Loop Groups, Discrete Versions of Some Classical Integrable Systems, and Rank 2 Extensions, *Memoirs of the Amer. Math. Soc.* **479** (1992). MR **93d:**58065

[DR] U. Desale and S. Ramanan, Classification of vector bundles of rank 2 over hyperelliptic curves, *Invent. Math.* **38** (1977), 161-186. MR **55:**2906

[D] R. Donagi, The group law on the intersection of two quadrics, *Ann. Scuola Norm. Sup. Pisa* **7** (1980), 217-240. MR **82b:**14025

[vGP] B. van Geemen and E. Previato, On the Hitchin system, *Duke Math. J.* **85** (1996), 659-683. MR **97k:**14010

[GH1] P. Griffiths and J. Harris, A Poncelet theorem in space, *Comment. Math. Helvetici* **52** (1977), 145-160. MR **58:**16695

[GH2] P. Griffiths and J. Harris, On Cayley's explicit solution to Poncelet's porism, *Enseign. Math.* **24** (1978), 31-40. MR **80g:**51017

[H] J. Harris, Galois groups of enumerative problems, *Duke Math. J.* **46** (1979), 685-724. MR **80m:**14038

[J] C.M. Jessop, *A treatise on the line complex*, Cambridge University Press, 1903.

[K1] H. Knörrer, Geodesics on the ellipsoid, *Invent. Math.* **59** (1980), 119-143. MR **81h:**58050

[K2] H. Knörrer, Geodesics on quadrics and a mechanical problem of C. Neumann, *J. Reine Angew. Math.* **334** (1982), 69-78. MR **84b:**58089

[KT] V.V. Kozlov and D.V. Treshchëv, *Billiards. A genetic introduction to the dynamics of systems with impacts*, AMS Translations Math. Monographs **89** (1991). MR **93k:**58094a

[McKvM] H.P. McKean and P. van Moerbeke, Hill and Toda curves, *Comm. Pure Appl. Math.* **33** (1980), 23-42. MR **81b:**14016

[M] J. Moser, Geometry of quadrics and spectral theory, *Chern Sympos.*, Springer-Verlag 1980, pp. 147-188. MR **82j:**58064

[N] P. E. Newstead, Stable bundles of rank 2 and odd degree over a curve of genus 2, *Topology* **7** (1968), 205-215. MR **38:**5782

[R] S. Ramanan, Orthogonal and spin bundles over hyperelliptic curves, in *Geometry and Analysis*, Papers Dedicated to V.K. Patodi, Springer 1981, pp. 151-166. MR **83f:**14014

[Ra] M. Raynaud, Courbes sur une variété abélienne et points de torsion, *Invent. Math.* **71** (1983), 207-233. MR **84c:**14021

[Re] M. Reid, The complete intersection of two or more quadrics, Thesis, Cambridge Univ. 1972.

[S] F. Sottile, Enumerative geometry for the real Grassmannian of lines in projective space, *Duke Math. J.* **87** (1997), 59–85. MR **99a:**14079

[T] G. Trautmann, Poncelet curves and associated theta characteristics, *Expositiones Math.* **6** (1988), 29-64. MR **89c:**14047

[V] A.P. Veselov, Integrable discrete-time systems and difference operators, *Functional Anal. Appl.* **22** (1988), 83-93. MR **90a:**58081

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