A SEPARABLE SPACE
WITH NO SCHAUDER DECOMPOSITION

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Abstract. We combine some known results to remark that there exists a
separable Banach space which fails to have a Schauder decomposition. It can
be chosen as a subspace of Gowers-Maurey space without any unconditional
basic sequence.

The following problem was raised in [Si] (Problem 15.1, p. 494): Does every
separable Banach space have a Schauder decomposition? This question goes back
to J. R. Retherford [R].

Recall that a sequence \( \{X_n\}_{n=1}^\infty \) of closed subspaces of a Banach space \( X \) is said
to be a Schauder decomposition of \( X \) if every \( x \in X \) has a unique representation
of the form \( x = \sum_{n=1}^\infty x_n \), with \( x_n \in X_n \) for every \( n \).

Let \( GM \) be Gowers-Maurey space which does not contain any unconditional
basic sequence [GM]. As was observed by W. B. Johnson, \( GM \) has in fact a
stronger property, namely it is hereditarily indecomposable (H.I.); i.e., no infinite-
dimensional closed subspace can be written as a direct sum \( Y \oplus Z \), where \( Y \) and
\( Z \) are infinite-dimensional closed subspaces. It is known that every block subspace
of \( GM \) contains uniform copies of \( \ell_1^n \). This follows from the lower \( f \)-estimate and
Krivine’s theorem as in [S]. Then, by Szankowski’s refinement of Enflo’s criterion
(see [LT2, p. 111, Remark 1]), we immediately obtain the following.

Proposition. There exists a subspace \( X \) of \( GM \) which does not have the compact
approximation property (C.A.P.).

Remark 1. For the same purpose we can as well use other H.I. spaces constructed
after the breakthrough of W. T. Gowers and B. Maurey. For example, there are
subspaces without the C.A.P. of the super-reflexive H.I. spaces in [F] in the case
when they contain uniform copies of \( \ell_p^n \) for \( p \neq 2 \). One can also use the asymptotic
\( \ell_1 \) hereditarily indecomposable spaces constructed in [AD] and [ADKM]. The existence
of uniform copies of \( \ell_1^n \) in these spaces follows directly from the definition and
one does not need to apply Krivine’s theorem. Therefore, they also have subspaces
without the C.A.P.

Corollary. The space \( X \) is an example of a separable Banach space with no
Schauder decomposition.

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Proof. Assume the contrary, i.e. X has a Schauder decomposition \( \{ X_n \}_{n=1}^{\infty} \).

Case 1. \( \{ X_n \}_{n=1}^{\infty} \) is a finite-dimensional decomposition. This is impossible since the existence of an F.D.D. implies B.A.P. which in turn implies C.A.P. (see [LT1]) and this contradicts the above Proposition.

Case 2. There exists \( m \) such that \( X_m \) is infinite-dimensional. Denote \( Y = [X_n : n \neq m] \). Then \( X = X_m \oplus Y \) which is also impossible because \( X_m \) and \( Y \) are closed infinite-dimensional subspaces of \( X \), \( X \) is a closed subspace of \( GM \), and \( GM \) is H.I.

Remark 2. Clearly, the result is true hereditarily in all the above mentioned H.I. spaces, e.g. we have that every subspace of \( GM \) has a further subspace which has no Schauder decomposition.

References


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