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A GEOMETRIC PROOF OF A THEOREM ABOUT NON-DUAL RENORMINGS

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ABSTRACT. We give a simple geometric proof of a result by Davis and Johnson that every nonreflexive Banach space X admits an equivalent norm in which X is not isometric to a dual space. Moreover, our renorming keeps unchanged the original norm on a given finite-codimensional subspace and makes this subspace norm-one complemented.

Let X be a real Banach space. We shall say that a norm $\|\cdot\|$ on X (or the space $(X,\|\cdot\|)$) is *non-dual* if $(X,\|\cdot\|)$ is not isometric to a dual space.

W. J. Davis and W. B. Johnson proved in [DJ] that every nonreflexive Banach space admits a non-dual renorming. This result was strengthened by D. van Dulst and I. Singer ([vDS]), who produced a renorming of any nonreflexive space such that the renormed space is not norm-one complemented in its bidual. Finally, S. V. Konyagin gave a quite simple proof of the following yet stronger result [Ko]: *every nonreflexive Banach space has an equivalent norm in which a three-point set fails to have Chebyshev centers.* (See, e.g., [Ho] for the definition of Chebyshev centers and for the fact that each bounded subset of X admits a Chebyshev center whenever X is norm-one complemented in its bidual.)

We present here a simple short geometric proof of a stronger form of the result by Davis and Johnson (Corollary 2). Here "simple" means that it uses only classical tools (James' theorem, Krein-Šmulyan theorem) and not results from renorming theory.

For a Banach space X, we denote by B_X its closed unit ball and by X^* its dual space.

Theorem. *Let $(X,\|\cdot\|)$ be a nonreflexive Banach space. Then there exists a norm $|\!|\!|\cdot|\!|\!|$ on $X\oplus\mathbb{R}$ such that*

(a) $|\!|\!|(x,0)|\!|\!| = \|x\|$*;*

(b) *the natural projection $P\colon X\oplus\mathbb{R}\to X$, $P(x,t)=x$, has norm one;*

(c) $\big(X\oplus\mathbb{R}, |\!|\!|\cdot|\!|\!|\big)$ *is non-dual.*

Proof. Let $f\in X^*$ be any functional that does not attain its norm (cf. [Ja]). Let us denote

$$L = B_X\cap f^{-1}(0), \qquad C = \overline{\operatorname{conv}}\Big[\big(B_X\times\{0\}\big)\cup\big(L\times\{1\}\big)\cup\big(L\times\{-1\}\big)\Big].$$

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(The reader is invited to use an easily made diagram.) Then $B_{X\oplus_1\mathbb{R}} \subset C \subset B_{X\oplus_\infty\mathbb{R}}$, and C is symmetric. (The symbols $\oplus_1$ and $\oplus_\infty$ denote respectively the ℓ_1-sum and the ℓ_∞-sum.) Thus C is the unit ball of an equivalent norm $|\!|\!|\cdot|\!|\!|$ on $X \oplus \mathbb{R}$, and moreover,

$$B_X = \{x \in X : (x, 0) \in C\} \subset P(C) \subset P(B_{X\oplus_\infty\mathbb{R}}) = B_X.$$

Thus (a), (b) hold.

To prove (c), suppose that $\big(X \oplus \mathbb{R}, |\!|\!|\cdot|\!|\!|\big)$ is isometric to Z^*. If we denote by w^* the weak-star topology $\sigma\big((X \oplus \mathbb{R}, |\!|\!|\cdot|\!|\!|), Z\big)$, then C is w^*-compact, and hence also the set

$$L_0 := L \times \{0\} = \big((0,1) + C\big) \cap \big((0,-1) + C\big)$$

is w^*-compact. This implies that also

$$\big(f^{-1}(0) \times \mathbb{R}\big) \cap C = L \times [-1,1] = \operatorname{conv}\Big[\big((0,1) + L_0\big) \cup \big((0,-1) + L_0\big)\Big]$$

is w^*-compact. By the Krein-Šmulyan theorem (cf. [D-S] or [Sch]), $f^{-1}(0) \times \mathbb{R}$ is w^*-closed since it is bw^*-closed. But $f^{-1}(0) \times \mathbb{R}$ is the kernel of the functional $F = (f, 0) \in X^* \oplus \mathbb{R} = (X \oplus \mathbb{R})^*$. Hence F can be identified with an element of Z, and F attains its norm on C. In other words, $F^{-1}\big(|\!|\!|F|\!|\!|\big)$ intersects C. But this is in contradiction with the fact that f does not attain its norm on B_X. Indeed, $f^{-1}(\|f\|)$ does not intersect B_X, and (by (a), (b)) $\|f\| = |\!|\!|F|\!|\!|$; thus $F^{-1}\big(|\!|\!|F|\!|\!|\big) = f^{-1}(\|f\|) \times \mathbb{R}$ does not intersect $B_X \times \mathbb{R}$ and the latter set contains C. □

Corollary 1. *For every positive integer n, each nonreflexive Banach space is isometric to an n-codimensional norm-one complemented subspace of a non-dual Banach space.*

Proof. Apply the Theorem n times. □

From Corollary 1, we obtain the following strengthening of the theorem by Davis and Johnson.

Corollary 2. *Let E be a nonreflexive Banach space and let $X \subset E$ be a proper closed subspace of finite codimension. Then E has an equivalent non-dual norm which coincides with the original norm on X and makes X norm-one complemented.*

Proof. Observe that E is isomorphic with $X \oplus \mathbb{R}^n$ for $n = \operatorname{codim} X$ (where the norm on X is the one inherited from E), and X is nonreflexive. Apply Corollary 1. □

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