A GEOMETRIC PROOF OF A THEOREM ABOUT NON-DUAL RENORMINGS

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ABSTRACT. We give a simple geometric proof of a result by Davis and Johnson that every nonreflexive Banach space $X$ admits an equivalent norm in which $X$ is not isometric to a dual space. Moreover, our renorming keeps unchanged the original norm on a given finite-codimensional subspace and makes this subspace norm-one complemented.

Let $X$ be a real Banach space. We shall say that a norm $\| \cdot \|$ on $X$ (or the space $(X, \| \cdot \|)$) is non-dual if $(X, \| \cdot \|)$ is not isometric to a dual space.

W. J. Davis and W. B. Johnson proved in [DJ] that every nonreflexive Banach space admits a non-dual renorming. This result was strengthened by D. van Dulst and I. Singer ([vDS]), who produced a renorming of any nonreflexive space such that the renormed space is not norm-one complemented in its bidual. Finally, S. V. Konyagin gave a quite simple proof of the following yet stronger result [Ko]:

every nonreflexive Banach space has an equivalent norm in which a three-point set fails to have Chebyshev centers. (See, e.g., [Ho] for the definition of Chebyshev centers and for the fact that each bounded subset of $X$ admits a Chebyshev center whenever $X$ is norm-one complemented in its bidual.)

We present here a simple short geometric proof of a stronger form of the result by Davis and Johnson (Corollary 2). Here “simple” means that it uses only classical tools (James’ theorem, Krein-Šmulian theorem) and not results from renorming theory.

For a Banach space $X$, we denote by $B_X$ its closed unit ball and by $X^*$ its dual space.

**Theorem.** Let $(X, \| \cdot \|)$ be a nonreflexive Banach space. Then there exists a norm $\| \cdot \|$ on $X \oplus \mathbb{R}$ such that

(a) $\|(x, 0)\| = \|x\|$;
(b) the natural projection $P: X \oplus \mathbb{R} \to X$, $P(x, t) = x$, has norm one;
(c) $(X \oplus \mathbb{R}, \| \cdot \|)$ is non-dual.

**Proof.** Let $f \in X^*$ be any functional that does not attain its norm (cf. [Ja]). Let us denote

$$L = B_X \cap f^{-1}(0), \quad C = \text{conv} \left[ (B_X \times \{0\}) \cup (L \times \{1\}) \cup (L \times \{-1\}) \right].$$
(The reader is invited to use an easily made diagram.) Then \( B_{X_{\oplus 1}} \subset C \subset B_{X_{\oplus \infty}} \), and \( C \) is symmetric. (The symbols \( \oplus_1 \) and \( \oplus_\infty \) denote respectively the \( \ell_1 \)-sum and the \( \ell_\infty \)-sum.) Thus \( C \) is the unit ball of an equivalent norm \( \| \cdot \| \) on \( X_{\oplus \mathbb{R}} \), and moreover,
\[
B_X = \{ x \in X : (x, 0) \in C \} \subset P(C) \subset P(B_{X_{\oplus \infty}}) = B_X.
\]
Thus (a), (b) hold.

To prove (c), suppose that \( (X_{\oplus \mathbb{R}}, \| \cdot \|) \) is isometric to \( Z^* \). If we denote by \( w^* \) the weak-star topology \( \sigma((X_{\oplus \mathbb{R}}, \| \cdot \|), Z) \), then \( C \) is \( w^* \)-compact, and hence also the set
\[
L_0 := L \times \{ 0 \} = \{(0, 1) + C \} \cap \{(0, -1) + C \}
\]
is \( w^* \)-compact. This implies that also
\[
(f^{-1}(0) \times \mathbb{R}) \cap C = L \times [-1, 1] = \text{conv}\left[ \{(0, 1) + L_0 \} \cup \{(0, -1) + L_0 \} \right]
\]
is \( w^* \)-compact. By the Krein-Šmulian theorem (cf. [D-S] or [Sch]), \( f^{-1}(0) \times \mathbb{R} \) is \( w^* \)-closed since it is \( bw^* \)-closed. But \( f^{-1}(0) \times \mathbb{R} \) is the kernel of the functional \( F = (f, 0) \in X^* \oplus \mathbb{R} = (X_{\oplus \mathbb{R}})^* \). Hence \( F \) can be identified with an element of \( Z \), and \( F \) attains its norm on \( C \). In other words, \( F^{-1}(\| F \|) \) intersects \( C \). But this is in contradiction with the fact that \( f \) does not attain its norm on \( B_X \). Indeed, \( f^{-1}(\| f \|) \) does not intersect \( B_X \), and (by (a), (b)) \( \| f \| = \| F \| \); thus \( F^{-1}(\| F \|) = f^{-1}(\| f \|) \times \mathbb{R} \) does not intersect \( B_X \times \mathbb{R} \) and the latter set contains \( C \).

**Corollary 1.** For every positive integer \( n \), each nonreflexive Banach space is isometric to an \( n \)-codimensional norm-one complemented subspace of a non-dual Banach space.

**Proof.** Apply the Theorem \( n \) times.

From Corollary 1, we obtain the following strengthening of the theorem by Davis and Johnson.

**Corollary 2.** Let \( E \) be a nonreflexive Banach space and let \( X \subset E \) be a proper closed subspace of finite codimension. Then \( E \) has an equivalent non-dual norm which coincides with the original norm on \( X \) and makes \( X \) norm-one complemented.

**Proof.** Observe that \( E \) is isomorphic with \( X_{\oplus \mathbb{R}^n} \) for \( n = \text{codim} \ X \) (where the norm on \( X \) is the one inherited from \( E \)), and \( X \) is nonreflexive. Apply Corollary 1.

**References**


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