AMPLE DIVISORS ON THE BLOW UP OF $\mathbb{P}^n$ AT POINTS

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Abstract. Fix integers $n, k, d$ with $n \geq 2, d \geq 2$ and $k > 0$; if $n = 2$ assume $d \geq 3$. Let $P_1, \ldots, P_k$ be general points of the complex projective space $\mathbb{P}^n$ and let $\pi : X \to \mathbb{P}^n$ be the blow up of $\mathbb{P}^n$ at $P_1, \ldots, P_k$ with exceptional divisors $E_i := \pi^{-1}(P_i), 1 \leq i \leq k$. Set $H := \pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. We here prove that the divisor $L := dH - \sum_{1 \leq i \leq k} E_i$ is ample if and only if $L^n > 0$, i.e. if and only if $d^n > k$.

Theorem 0.1. Fix integers $n, k, d$ with $n \geq 2, d \geq 2$ and $k > 0$; if $n = 2$ assume $d \geq 3$. Let $P_1, \ldots, P_k$ be general points of the complex projective space $\mathbb{P}^n$ and let $\pi : X \to \mathbb{P}^n$ be the blow up of $\mathbb{P}^n$ at $P_1, \ldots, P_k$ with exceptional divisors $E_i := \pi^{-1}(P_i), 1 \leq i \leq k$. Set $H := \pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Then the divisor $L := dH - \sum_{1 \leq i \leq k} E_i$ is ample if and only if $L^n > 0$, i.e. if and only if $d^n > k$.

Here “general points” means “outside countably many proper subvarieties of the symmetric product $S^k(\mathbb{P}^n)$ of $\mathbb{P}^n$”. For $n = 2$ and $3$ Theorem 0.1 was proved in [1] (at least for $d \geq 5$). In [1], §3, the proof of Theorem 0.1 was reduced to the proof of a general lemma ([1], Lemma 2.1) which was proved there only for $n = 2$ and 3 (see in particular page 45, lines 4 and 5 from the bottom). We will prove that lemma for every $n$ and hence obtain for free a proof of Theorem 0.1. Indeed we will prove the following more general result which is very classical (for $n = 2$ being due to Kronucker and Castelnuovo). 

Lemma 0.2. Let $X$ be a smooth projective variety of dimension $n \geq 2$. Take $H \in \text{Pic}(X)$, $H$ very ample. Assume that $(X, H)$ is not a scroll over a smooth curve, i.e. assume that $X$ is not a $\mathbb{P}^{n-1}$-bundle $\pi : X \to C$ over a smooth curve with $H$ degree 1 line bundle on each fiber. If $n = 2$ and $X = \mathbb{P}^2$ assume $\deg(H) \neq 2$.

Let $V$ be a linear subspace of $H^0(X, H)$ which induces an embedding of $X$. Let $W$ be a general linear subspace of $V$ with $\dim(W) = n$. Then for every hyperplane $M$ of $W$ the base locus of $M$ is a reduced and irreducible curve.

Proof. First assume $n = 2$. Consider the embedding $X \subset \mathbb{P}(V)$ associated to $V$. By Lefschetz’ theorem for a general pencil $\{H(\lambda)\}_{\lambda \in \mathbb{P}^1}$ of hyperplanes of $X$ each $H(\lambda) \cap X$ is either a smooth curve or it has a single ordinary double point. We claim that, except for the list given, this implies that every $H(\lambda) \cap X$ is reduced and irreducible. Since $\text{Sing}(H(\lambda) \cap X)$ is finite, $H(\lambda) \cap X$ is reduced. In order to obtain a contradiction we assume that $H(\lambda) \cap X$ is reducible, say $H(\lambda) \cap X = A \cup B$ with
A irreducible and \( B \neq \emptyset \). Since \( H \) is very ample, \( H(\lambda) \cap X \) is 1-connected (see e.g. [3], Remark after the statement of Th. I). Furthermore, outside the list given \( H(\lambda) \cap X \) is 2-connected ([3], Th. I). This implies that either \( \text{card}(\text{Sing}(H(\lambda) \cap X)) \geq \text{card}(A \cap B) \geq 2 \) or there is \( P \in A \cap B \) with \( A \cup B \) not an ordinary nodal curve at \( P \), a contradiction. Now assume \( n \geq 3 \). Taking the 0-locus \( F_1, \ldots, F_{n-2} \) of \( n-2 \) general elements of \( V \) we obtain a smooth surface \( X' := F_1 \cap \cdots \cap F_{n-2} \). Apply the first part to the case \((X', H|_{X'}, V|_{X'})\). We may apply the case \( n = 2 \) because we did not require that \( V = H^0(X, H) \), but only that \( V \) embeds \( X \). Assume that \( X' \cong \mathbb{P}^2 \) and \( \deg(H|_{X'}) = 2 \) or that \((X', H|_{X'})\) is a scroll over a smooth curve. By the adjunction formula we have \( K_{X'} \cong K_X \otimes H^{\otimes n-2}|_{X'} \). Hence in both cases \( K_X \otimes H^{\otimes n-1} \) is not spanned by global sections. By [2], Th. 1, and the assumption \( n \geq 3 \) the pair \((X, H)\) is a scroll over a smooth curve.

References
1. F. Angelini, Ample divisors on the blow up of \( \mathbb{P}^3 \) at points, Manuscripta Math. 93 (1997), 39–48. MR 98d:14005

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