ON THE SEMISIMPLICITY OF PURE SHEAVES

LEI FU

Abstract. We obtain a criteria for a pure sheaf to be semisimple. As a corollary, we prove the following: Let $X_0$ and $S_0$ be two schemes over a finite field $\mathbb{F}_q$, and let $f_0 : X_0 \to S_0$ be a proper smooth morphism. Assume $S_0$ is normal and geometrically connected, and assume there exists a closed point $s$ in $S_0$ such that the Frobenius automorphism $F_s$ acts semisimply on $H^i(X_{\bar{s}}, Q_l)$, where $X_{\bar{s}}$ is the geometric fiber of $f_0$ at $s$ (this last assumption is unnecessary if the semisimplicity conjecture is true). Then $R^i f_{0*} Q_l$ is a semisimple sheaf on $S_0$. This verifies a conjecture of Grothendieck and Serre provided the semisimplicity conjecture holds. As an application, we prove that the galois representations of function fields associated to the $l$-adic cohomologies of $K3$ surfaces are semisimple. We also get a theorem of Zarhin about the semisimplicity of the Galois representations of function fields arising from abelian varieties.

The proof relies heavily on Deligne's work on Weil conjectures.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with $q$ elements. Choose an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Throughout this paper, schemes, morphisms and sheaves defined on the base field $\mathbb{F}_q$ are denoted by letters with subscripts 0 and we indicate the base extension from $\mathbb{F}_q$ to $\mathbb{F}$ by dropping the subscripts 0. Schemes and morphisms are separated and of finite type.

Let $X_0$ be a scheme over $\mathbb{F}_q$, let $\mathcal{F}_0$ be a constructible $\overline{Q}_l$-sheaf on $X_0$, and let $\iota : \overline{Q}_l \to \mathbb{C}$ be an isomorphism. Recall that $\mathcal{F}_0$ is called $\iota$-pure with weight $w$ if for every closed point $x$ of $X_0$ and for every eigenvalue $\lambda$ of the (geometric) Frobenius automorphism $F_x$ on $\mathcal{F}_x$, the absolute value of $\iota(\lambda)$ is $N(x)^{w/2}$, where $N(x)$ is the number of elements of the residue field $k(x)$. Also recall that giving a lisse $\overline{Q}_l$-sheaf on a connected scheme is the same as giving a $\overline{Q}_l$-representation of the fundamental group of the scheme. So we can talk about the irreducibility and semisimplicity of a lisse sheaf.

If $X_0$ is a normal geometrically connected scheme over $\mathbb{F}_q$ and if $\mathcal{F}_0$ is a $\iota$-pure lisse $\overline{Q}_l$-sheaf on $X_0$, then by a theorem of Deligne ([D1], 3.4.1 (iii)), $\mathcal{F}$ is a semisimple sheaf on $X$. In this paper, based on Deligne's work, we prove the following:

Theorem. Let $X_0$ be a normal geometrically connected scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}_0$ be a $\iota$-pure lisse $\overline{Q}_l$-sheaf on $X_0$. If there exists a closed point $x$ of...
such that the Frobenius automorphism $F_x$ on $\mathcal{F}_x$ is semisimple, then $\mathcal{F}_0$ is a semisimple sheaf on $X_0$.

Here a linear transformation on a vector space is said to be semisimple if the corresponding matrix is diagonalizable.

**Corollary.** Let $X_0$ and $S_0$ be two schemes over the finite field $\mathbb{F}_q$, and let $f_0 : X_0 \to S_0$ be a proper smooth morphism. Assume $S_0$ is normal and geometrically connected, and assume there exists a closed point $s$ in $S_0$ such that the Frobenius automorphism $F_s$ acts semisimply on $H^i(X_{\overline{s}}, \mathbb{Q}_l)$, where $X_{\overline{s}} = X_0 \otimes_{k(s)} \overline{k(s)}$ is the geometric fiber of $f_0$ at $\overline{s}$. Then $R^if_0_*, \mathbb{Q}_l$ is a semisimple sheaf on $S_0$.

**Proof.** By the proper and smooth base change theorem, we know $R^if_0_*, \mathbb{Q}_l$ is lisse. By Deligne’s theorem (i.e. Weil’s conjecture), $R^if_0_*, \mathbb{Q}_l$ is pure. By assumption, $F_s$ acts semisimply on $(R^if_*, \mathbb{Q}_l)_{\overline{s}} = H^i(X_{\overline{s}}, \mathbb{Q}_l)$. So by the theorem, $R^if_0_*, \mathbb{Q}_l$ is a semisimple sheaf on $S_0$.

The semisimplicity conjecture states that for any proper smooth scheme $X_0$ over $\mathbb{F}_q$, the geometric Frobenius correspondence $F$ acts semisimply on $H^i(X, \mathbb{Q}_l)$. If this conjecture is true, then in the above corollary, we don’t need the assumption that $F_s$ acts semisimply on $H^i(X_{\overline{s}}, \mathbb{Q}_l)$. So provided the semisimplicity conjecture holds, the above corollary verifies for schemes over finite field a conjecture of Serre and Grothendieck stated in [T].

The semisimplicity conjecture is proved to be true for the following varieties over finite field:

(a) smooth projective curves ([W]),
(b) abelian varieties ([W]),
(c) $K3$ surfaces ([D2], [PS]).

Denote the function field of $S_0$ by $K$. Let $Y$ be a smooth projective variety defined over $K$. Assume $Y$ is in one of the following families:

(a) smooth projective curves,
(b) abelian varieties,
(c) $K3$ surfaces.

Then by the above corollary the Galois representation

$$\text{Gal}(\overline{K}/K) \to H^i(Y \otimes_K \overline{K}, \mathbb{Q}_l)$$

is semisimple. The semisimplicity of the Galois representations of function fields arising from abelian varieties was first proved by Zarhin ([Z]).

**2. Proof of the Theorem**

We first prove some lemmas.

**Lemma 1.** Let $G^0$ be a subgroup of $G$ with finite index. Let $G \to GL(V)$ be a finite dimensional representation of $G$. If the restriction $G^0 \to GL(V)$ is semisimple, then $G \to GL(V)$ is also semisimple.

**Proof.** Let $U$ be a $G$-stable subspace of $V$. We need to show $U$ has a complement which is also $G$-stable. It is enough to show that there exists a homomorphism $P : V \to U$ such that $P|_U$ is identity and $P$ is invariant under the action of $G$. Since $V$ is a semisimple representation of $G^0$, we can find a homomorphism $P_0$
such that \( P_{0|U} \) is identity and that \( P_0 \) is \( G^0 \)-invariant. Let \( G^0 g_1, \ldots, G^0 g_n \) be representatives of the right cosets of \( G^0 \), where \( n = [G : G^0] \). Define

\[
P = \frac{1}{n} \sum_{i=1}^{n} g_i^{-1} P_0 g_i.
\]

One can check \( P_{0|U} \) is identity. To see \( P \) is \( G \)-invariant, taking \( g \in G \), then \( G^0 g_1 g, \ldots, G^0 g_n g \) are also representatives of the right cosets. So there is a permutation \( \sigma \) of \( \{1, \ldots, n\} \) such that \( G^0 g_i g = G^0 g_{\sigma(i)} \). Hence there exists \( g_i^0 \in G^0 \) such that \( g_i g = g_i^0 g_{\sigma(i)} \). We have

\[
P g = \frac{1}{n} \sum g_i^{-1} P_0 g_i g = \frac{1}{n} \sum g_i^{-1} P_0 g_i^0 g_{\sigma(i)}
\]

\[
= \frac{1}{n} \sum g_i^{-1} g_i^0 P_0 g_{\sigma(i)} = \frac{1}{n} \sum g g_{\sigma(i)}^{-1} P_0 g_{\sigma(i)} = g P.
\]

So \( P \) is \( G \)-invariant.

**Lemma 2.** Let \( G^0 \) be a normal subgroup of \( G \). Assume there exists an exact sequence

\[
0 \to G^0 \to G \xrightarrow{deg} \mathbb{Z} \to 0.
\]

We call the homomorphism \( G \xrightarrow{deg} \mathbb{Z} \) the degree homomorphism. Assume there exists an element in the center of \( G \) with nonzero degree. Let \( G \to GL(V) \) be a finite dimensional representation of \( G \). If the restriction \( G^0 \to GL(V) \) is semisimple, and if there exists an element \( g \in G \) with nonzero degree such that matrix corresponding to \( g \) is diagonalizable, then \( G \to GL(V) \) is semisimple.

**Proof.** Let \( G_d = \{ x \in G : d| \text{deg}(x) \} \), where \( d \) is the degree of an element in the center of \( G \) with nonzero degree. Then \( G_d \) is a subgroup of \( G \) and \( G/G_d \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \). By Lemma 1, to prove \( G \to GL(V) \) is semisimple, it is enough to prove the representation \( G_d \to GL(V) \) is semisimple. Replacing \( G \) by \( G_d \) and \( deg \) by \( deg/d \), we may thus assume that there exists an element in the center of \( G \) with degree 1; that is, we may assume \( G = G^0 \times \mathbb{Z} \).

By assumption, the restriction of \( G^0 \times \mathbb{Z} \to GL(V) \) to \( G^0 \) is semisimple. So we have an isomorphism of \( G^0 \)-representations:

\[
\phi : \bigoplus_j (W_j \otimes \text{Hom}_{G^0}(W_j, V)) \to V,
\]

\[(w, f) \mapsto f(w),\]

where the direct sum on the left-hand side sums over all the irreducible representations \( W_j \) of \( G^0 \). If we let \( n \in \mathbb{Z} \) act on \( f \in \text{Hom}_{G^0}(W_j, V) \) by \( (nf)(w) = nf(w) \), where \( nf(w) \) is the action of \( n \in \mathbb{Z} \subset G^0 \times \mathbb{Z} \) on \( f(w) \in V \), then \( \phi \) is also an isomorphism of \((G^0 \times \mathbb{Z})\)-representations. So we may assume \( V = \bigoplus_j (W_j \otimes U_j) \) as \((G^0 \times \mathbb{Z})\)-representations, where \( W_j \) are irreducible representations of \( G^0 \) and \( U_j \) are some representations of \( \mathbb{Z} \). By assumption, there exists an element \( (g, n) \) with nonzero degree \( n \) such that it corresponds to a diagonalizable matrix in \( GL(V) \). So \( (g, n) \) corresponds to a diagonalizable matrix in \( GL(W_j \otimes U_j) \) for each \( j \). Then \( n \) corresponds to a diagonalizable matrix in \( GL(U_j) \) because of the fact that if the tensor product of two matrices is diagonalizable, then each is. Hence \( U_j \) is a semisimple representation of \( \mathbb{Z} \). We can write \( U_j = \bigoplus_k U_{jk} \), where each \( U_{jk} \) is a
one-dimensional representation of \( Z \). We then have \( V = \bigoplus_{jk} (W_j \otimes U_{jk}) \). Obviously each \( W_j \otimes U_{jk} \) is an irreducible representation of \( G^0 \times Z \). So \( V \) is a semisimple representation of \( G^0 \times Z \).

Now let’s prove the theorem. I learned the following proof (and the above Lemma 2) from P. Deligne.

We use the same notation as in [D1]. The lisse \( \mathbb{Q}_l \)-sheaf \( F_0 \) gives rise to a \( \mathbb{Q}_l \)-representation \( \pi_1(X_0, \bar{x}) \rightarrow GL(F_{\bar{x}}) \). Applying the construction [D1] 1.3.7 to \( F_0 \), we get

\[
\begin{array}{cccc}
0 & \rightarrow & \pi_1(X, \bar{x}) & \rightarrow & W(X_0, \bar{x}) & \rightarrow & Z & \rightarrow & 0 \\
\downarrow & & \downarrow & & \deg & \rightarrow & \| & & \\
0 & \rightarrow & G^0 & \rightarrow & G & \rightarrow & Z & \rightarrow & 0 \\
& & & & & \rightarrow & GL(F_{\bar{x}})
\end{array}
\]

where \( G^0 \) is the Zariski closure of the image of \( \pi_1(X, \bar{x}) \rightarrow GL(F_{\bar{x}}) \), the group \( W(X_0, \bar{x}) \) is the Weil group of \( X_0 \), and the second short exact sequence is obtained by pushing forward the first one using \( \pi_1(X, \bar{x}) \rightarrow G^0 \). By [D1] 3.4.1 (iii), \( F \) is a semisimple sheaf on \( X \), that is, the representation \( \pi_1(X, \bar{x}) \rightarrow GL(F_{\bar{x}}) \) is semisimple. By [D1] 1.3.9, the algebraic group \( G^0 \) is an extension of a finite group by a semi-simple group, and by [D1] 1.3.11, if \( Z \) is the center of \( G \), then the restriction of the degree map on \( Z \) has finite kernel and cokernel. ([D1] 1.3.9 and 1.3.11 hold if one just assumes \( F \) is semisimple, as Deligne mentions in the proof of 1.3.9.) In particular, there exists an element in the center of \( G \) with nonzero degree.

Now assume the Frobenius automorphism \( F_x \) acts semisimply on \( F_{\bar{x}} \), and let’s prove the representation \( \pi_1(X_0, \bar{x}) \rightarrow GL(F_{\bar{x}}) \) is semisimple. It is not hard to see the following statements are equivalent:

1. \( \pi_1(X_0, \bar{x}) \rightarrow GL(F_{\bar{x}}) \) is semisimple.
2. \( W(X_0, \bar{x}) \rightarrow GL(F_{\bar{x}}) \) is semisimple.
3. \( G \rightarrow GL(F_{\bar{x}}) \) is semisimple.

We will prove (3) is true. By [D1] 3.4.1 (iii), we know \( \pi_1(X, \bar{x}) \rightarrow GL(F_{\bar{x}}) \) is semisimple. It is not hard to see this implies that \( G^0 \rightarrow GL(F_{\bar{x}}) \) is semisimple. By assumption, \( F_x \) acts semisimply on \( F_{\bar{x}} \). Moreover there exists an element in the center of \( G \) with nonzero degree, and the degree of \( F_x \) is nonzero. So (3) holds by Lemma 2. This proves the theorem.

ACKNOWLEDGEMENTS

I thank Professor Deligne for suggesting Lemma 2 which greatly simplifies the proof of the theorem. This paper is written during my visit at Indiana University. I thank Professor Dadok for inviting me and for many interesting discussions on group theory.

REFERENCES


Institute of Mathematics, Nankai University, Tianjin, People’s Republic of China

E-mail address: leifu@sun.nankai.edu.cn