EULER CHARACTERISTIC OF THE MILNOR FIBRE
OF PLANE SINGULARITIES

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(Communicated by Ron Donagi)

Abstract. We give a formula for the Euler characteristic of the Milnor fibre
of any analytic function \( f \) of two variables. This formula depends on the
intersection multiplicities, the Milnor numbers and the powers of the branches
of the germ of the curve defined by \( f \). The goal of the formula is that it use
neither the resolution nor the deformations of \( f \). Moreover, it can be use for
giving an algorithm to compute it.

1. Introduction

In this note we deal with germs of analytic functions \( f \) of two complex variables
with \( f(0) = 0 \) and its factorization \( f = f_1^{q_1} \cdots f_r^{q_r} \) into irreducible factors, such
that \( f_i/f_j, 1 \leq i, j \leq r \), are as power series not units. Let \( (C, 0) \) be the germ of the
plane curve defined by the local equation \( f = 0 \) and let \( (C_i, 0), i = 1, \ldots, r, \) be its
reduced branches defined by \( f_i = 0 \).

The local curve \( C \) defines a link with multiplicities \( L := C \cap S^3_\varepsilon \), in the sphere
of radius \( \varepsilon > 0 \) around \( 0 \in \mathbb{C}^2 \), which does not depend on \( \varepsilon \) provided \( \varepsilon \) is small
enough. The link \( L \) consists of the components \( C_i \cap S^3_\varepsilon \), with multiplicities \( q_i \), and
determines the topological type of the germ \( C \). Moreover, Milnor proved that the
map \( \begin{array}{c}
\pi_1 \rightarrow \mathbb{C}^\infty \\
S^3_\varepsilon \setminus L \rightarrow S^1
\end{array} \) is a \( \mathbb{C}^\infty \)-locally trivial fibration, the Milnor fibration. Any
fibre \( F \) of this fibration is called the Milnor fibre of \( f \) (see [M, Theorem 4.8]).

A’Campo [A] and Eisenbud-Neumann [EN], using different methods, calculated
many topological invariants of the fibration \( \begin{array}{c}
\pi_1 \rightarrow \mathbb{C}^\infty \\
S^3_\varepsilon \setminus L \rightarrow S^1
\end{array} \) from the resolution graph or the
splicing diagrams. We are only interested in the Euler characteristic \( \chi(F) \) of the
surface \( F \). If \( f \) is reduced, i.e. every power \( q_i \) is equal to one, the Euler characteristic
of \( F \) is \( 1 - \mu(C, 0) \), where \( \mu(C, 0) \) is the Milnor number of the isolated singularity
of \( C \). Moreover the Euler characteristic of \( F \) is related to topological and geometric
invariants of its branches by the well-known formula:

\[
\chi(F) = -2 \sum_{1 \leq i < j \leq r} (C_i, C_j)_0 + \sum_{i=1}^{r} (1 - \mu(C_i)),
\]

where \( (C_i, C_j)_0 \) is the intersection multiplicity of \( C_i \) and \( C_j \) at the origin and \( \mu(C_i) \)
is the Milnor number of \( C_i \) at the origin (e.g. see [BK]).

Received by the editors July 24, 1996 and, in revised form, June 27, 1997.

1991 Mathematics Subject Classification. Primary 32S05, 14H20; Secondary 14B05.
Key words and phrases. Euler characteristic, Milnor fibration, Milnor fibre.
This work was done under the partial support of CAYCIT PB94-291.

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On the other hand, when \( f \) is non-reduced Schrauwen [S] expressed the Euler characteristic of \( F \) in terms of special points of suitable deformations of \( f \). For calculating \( \chi(F) \) in this case one can use the methods of A’Campo or Eisenbud-Neumann and construct the resolution graph or the splicing diagram.

The aim of this note is to give a closed formula for the Euler characteristic of \( F \) without the construction of these graphs.

For every \( q \in \mathbb{N}^r \) set
\[
F^q := \{ z \in S_e : \prod_{1 \leq i \leq r, q_i \neq 0} \left( \frac{f_i}{|f_i|} \right)^q_i(z) = 1 \text{ and } f_i(z) \neq 0 \ \forall i = 1, \ldots, r \}.
\]

Note that, for \( \epsilon \) small, the surface \( F^q \) is the Milnor fibre of the local curve \( C^q := \{ f_1^{q_1} \cdots f_r^{q_r} = 0 \} \) if and only if all \( q_i \neq 0 \). If some \( q_i \) are zero, but \( q \neq 0 \), then \( F^q \) is the Milnor fibre of \( \prod_{1 \leq i \leq r, q_i \neq 0} f_i^{q_i} \) with punctures, where the number of punctures equals \( \sum_{1 \leq i \leq r, q_i \neq 0} (C_i)_0(q_i) \). For \( q = 0 \) the space \( F^q \) is just the complement of the link of the curve \( C \).

Our generalized and closed formula is:
\[
\chi(F^q) = - \sum_{1 \leq i < j \leq r} (C_i, C_j)_0(q_i + q_j) + \sum_{i = 1}^r q_i (1 - \mu(C_i)).
\]

I am indebted to the referee for suggesting how to improve the presentation of the proof of the formula.

2. Proof of the formula

The formula follows from the two following lemmas.

**Lemma 1.** The function \( q \in \mathbb{N}^r \to \chi(F^q) \) is additive.

**Proof.** Let \( \pi : X \to \mathbb{C}^2 \) be a proper modification of \( \mathbb{C}^2 \) above the origin such that, for every point on the divisor \( E := \pi^{-1}(0) \), the total transform of the \( \bigcup_{1 \leq i \leq r} \tilde{C}_i \) has normal crossing singularities. Let \( \tilde{C}_i \) be the strict transform of \( C_i \) by \( \pi \) and \( E_\alpha, \alpha \in A \), the components of \( E \).

First assume \( q \neq 0 \). Put \( f^q = \prod_{1 \leq i \leq r, q_i \neq 0} f_i^{q_i} \). Observe that \( F^q \) retracts on \( E \setminus \left( \bigcup_{1 \leq i \leq r, q_i = 0} \tilde{C}_i \right) \). With the formula of A’Campo we get:
\[
\chi(F^q) = \sum_{\alpha \in A} m(f^q, E_\alpha) \chi(\tilde{E}_\alpha),
\]
where \( \tilde{E}_\alpha := E_\alpha \setminus \left( \bigcup_{\beta \neq \alpha} E_\beta \cup \bigcup_{1 \leq i \leq r} \tilde{C}_i \right) \). Then
\[
\chi(F^q) = \sum_{\alpha \in A} \sum_{i = 1}^r q_i m(f_i, E_\alpha) \chi(\tilde{E}_\alpha),
\]
since \( m(f^q, E_\alpha) = \sum_{1 \leq i \leq r} q_i m(f_i, E_\alpha) \).

To prove the additivity it remains to observe that \( \chi(F^0) = 0 \).

Put \( \epsilon_i = (0, \ldots, 1, \ldots, 0) \). From the additivity we get:
\[
\chi(F^q) = \sum_{i = 1}^r q_i \chi(F^{\epsilon_i}).
\]
Lemma 2.
\[ \chi(F^e_i) = - \sum_{j=1 \ldots r, i \neq j} (C_i, C_j)_0 + (1 - \mu(C_i)). \]

Proof. Remember that $F^e_i$ is the Milnor fibre $F_i$ with $\sum_{1 \leq j \leq r, j \neq i} (C_i, C_j)_0$ punctures.

Remark that Lemma 1 holds for the case where the germs of the curves $C_i$ are reduced and have no branch in common. Thus, if we assume

1. each $f_i$ has no multiple components (i.e. $f_i$ is squarefree) and
2. for $i, j \in \{1, \ldots, r\}$, $i \neq j$, the germ $f_i f_j$ has no multiple components,

then we finally get for the Euler characteristic of the Milnor fibre of $F$ of $f = f_1^{q_1} \cdots f_s^{q_s}$, $q_i > 0$, the formula:

\[ \chi(F) = - \sum_{1 \leq i < j \leq s} (C_i, C_j)_0 (q_i + q_j) + \sum_{i=1}^{s} q_i (1 - \mu(C_i, 0)). \]

To have this formula for squarefree factorization is particularly useful for inductive calculations. If $R$ is a computable ring with $\text{char}(R) = 0$ and $f$ is a polynomial in $R[x, y]$, then there exists an algorithm that computes a squarefree decomposition of $f$ in $R[x, y]$ (see [BWK, Proposition 2.86, Corollary 2.92]). This is also a squarefree decomposition in $R\{x, y\}$ and one may then compute the intersection multiplicities and the Milnor numbers. I would like to thank Bernd Martin for showing me the implementation of this algorithm using the computer algebra system SINGULAR, [GPS].

References

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