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STRONG ERGODIC THEOREMS FOR COMMUTATIVE SEMIGROUPS OF OPERATORS

O. KADA

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Abstract. We prove strong mean convergence theorems and the existence of ergodic projection and retraction for commutative semigroups of operators which is Eberlein-weakly almost periodic.

Introduction

A strong mean convergence theorem for nonexpansive mappings was first established for odd mappings by Baillon [2], and it was generalized to that for asymptotically isometric mappings by Bruck [4]. After works of Miyadera and Kobayasi [21] and Oka [22], we proved the theorem for general commutative semigroups ([16]). On the other hand, Ruess and Summers [26, 28, 29] proved the strong convergence theorem by different methods, such as the following.

A bounded and continuous function f from $\mathbb{R}^+$ to a Banach space E is called Eberlein-weakly almost periodic if its orbit by translation $\{r(s)f; s \in \mathbb{R}^+\}$ is relatively weakly compact in the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to E with supremum norm. Ruess and Summers showed that if f is Eberlein-weakly almost periodic, then $(1/t)\int_0^t f(s+h)ds$ converges strongly as $t \to \infty$ to a point $z \in E$ uniformly in $h \in \mathbb{R}^+$ (they proved for strongly regular kernels). Consequently, for a nonexpansive semigroup $\{T(s); s \in \mathbb{R}^+\}$ on a closed convex subset C of a uniformly convex Banach space E, if $T(\cdot)x$ is asymptotically isometric for $x \in C$, then $(1/t)\int_0^t T(s+h)ds$ converges strongly as $t \to \infty$ to a common fixed point of $T(s)$, $s \in S$, uniformly in $h \in \mathbb{R}^+$.

In section 2, we generalize their results to a commutative semigroup of operators, and give a result of the existence of ergodic projection and retraction. In section 3, we use results of Ruess and Summers ([26]) on Eberlein-almost periodicity for nonexpansive semigroups on $\mathbb{R}^+$ (resp. $\mathbb{Z}^+$), adapted to the d-dimensional case, to apply our results to a pair of commuting nonexpansive maps in uniformly convex Banach spaces.

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1. Preliminaries

Throughout this paper, S denotes a commutative semitopological semigroup with identity, i.e., a commutative semigroup with a Hausdorff topology such that for each $t \in S$, the mapping $s \mapsto s+t$ from S to S is continuous. We assume a Banach space E is real. We also denote by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{R}$ and $\mathbb{R}^+$ the sets of all integers, nonnegative integers, positive integers, real numbers and nonnegative real numbers, respectively. Let $l^\infty(S)$ be the Banach space of all bounded real valued functions on S with the supremum norm, and let $C_b(S)$ be the subspace of $l^\infty(S)$ of all bounded continuous real valued functions on S. Then for each $s \in S$ and $f \in C_b(S)$, we define an element $r(s)f$ in $C_b(S)$ by

$$(r(s)f)(t) = f(t+s) \quad \text{for all } t \in S.$$

An element μ of $C_b(S)^*$, where $C_b(S)^*$ is the dual space of $C_b(S)$, is called a mean on $C_b(S)$ if $\|\mu\| = \mu(1) = 1$. As is known, μ is a mean on $C_b(S)$ if and only if $\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s)$ for each $f \in C_b(S)$. A mean μ on $C_b(S)$ is invariant if $\mu(r(s)f) = \mu(f)$ for all $s \in S$ and $f \in C_b(S)$. It is well known that there exists an invariant mean μ on $l^\infty(S)$; see Day [6]. Since $C_b(S)$ is invariant under $r(s)$, the restriction of μ on $C_b(S)$ is an invariant mean on $C_b(S)$. A finite mean is an element of $\mathrm{co}\{\delta(s); s \in S\}$, where $\delta(s)(f) = f(s)$ for each $f \in l^\infty(S)$ and $\mathrm{co}\, A$ is the convex hull of A. Depending on time and circumstances, the values of mean μ at $f \in C_b(S)$ will also be denoted by $\mu(f)$ or $\mu_s(f(s))$. A commutative semigroup S is a directed system when the binary relation is defined by $s \le t$ if and only if $\{s\} \cup (S+s) \supset \{t\} \cup (S+t)$. We write $x_n \to x$ (resp. $x_n \rightharpoonup x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly (resp. weakly) to x.

Let C be a closed subset of a Banach space E. A mapping V of C into itself is called *Lipschitzian* if there exists $K \ge 0$ such that $\|Vx - Vy\| \le K\|x-y\|$ for every $x, y \in C$. We define the Lipschitz norm of V by $\|V\| := \sup\{\frac{\|Vx-Vy\|}{\|x-y\|}; x, y \in C, x \ne y\}$. If $\|V\| \le 1$, then V is called *nonexpansive*. We denote by $\mathrm{Lip}(C), \mathrm{Cont}(C)$ and $L(E)$ the semitopological semigroup of all Lipschitzian self-mappings of C, the semitopological semigroup of all nonexpansive self-mappings of C and the semitopological semigroup of all bounded linear mappings of E, under composition and pointwise convergence topology, respectively. Let $T\colon S \to \mathrm{Lip}(C)$ or $\mathrm{Cont}(C)$ (resp. $L(E)$) be a representation, i.e., $T(s+t) = T(s)T(t)$ for every $s, t \in S$, and $T(\cdot)x$ is continuous for every $x \in C$ (resp. $x \in E$). We denote by $\mathrm{Fix}(T)$ the set of common fixed points of $T(s)$, $s \in S$. We also denote by $B(E) = \{x \in E; \|x\| \le 1\}$ the unit ball of E. For two Banach spaces E and F, $L(E,F)$ is the set of all bounded linear mappings from E to F.

Let S be a topological space and let E be a Banach space. Then we denote by $C_b(S,E)$ the Banach space of all bounded continuous mappings from S to E with supremum norm, and by $C_C(S,E)$ the set of all elements $f \in C_b(S,E)$ such that $f(S) := \{f(s); s \in S\}$ is relatively weakly compact. It is obvious that $C_C(S,E)$ is a linear subspace of $C_b(S,E)$. Let $\kappa\colon E \to C_b(S,E)$ be a mapping such that for $x \in E$ and $s \in S$, $\kappa(x)(s) = x$. Since κ is a norm preserving isomorphism of E into $C_b(S,E)$, we consider E as a subspace of $C_b(S,E)$.

For any mean μ on $C_b(S)$, we define a "vector valued mean" $\tau(\mu)$ homomorphically, i.e., $\tau(\mu)$ is an element of $L(C_C(S,E),E)$ such that $\tau(\mu)x = x$ for each $x \in E$, and $\|\tau(\mu)\| = 1$. More generally, we give a definition as follows; see [16].

Definition 1.1. Let E be a Banach space. For $\mu \in C_b(S)^*$ and $f \in C_C(S,E)$, let $x^{**}_{\mu,f}\colon x^* \mapsto \mu_s\langle f(s), x^*\rangle$ be an element of E^{**}. Then $x^{**}_{\mu,f} \in E$. We define an element $\tau = \tau^E \in L(C_b(S)^*, L(C_C(S,E),E))$ by $\tau(\mu)f = x^{**}_{\mu,f}$.

Remark 1.2. The following holds; see [16]. (i) τ is injective and $\|\tau\| \le 1$; (ii) $\tau(\mu)x = \mu(1)x$ for all $x \in E$, and if μ is a mean on $C_b(S)$, then $\|\tau(\mu)\| = 1$; (iii) τ maps the point evaluation $\delta(s)$ to the point evaluation $\varepsilon(s)$, $s \in S$, where $\varepsilon(s)f = f(s)$ for $f \in C_C(S,E)$; (iv) $\tau(r(s)^*\mu) = r(s)^*\tau(\mu)$.

Let $T\colon S \to \mathrm{Lip}(C)$ be a representation such that $T(\cdot)x \in C_C(S,E)$ for some $x \in C$. Then we shall denote $\tau(\mu)(T(\cdot)x)$ by $T(\mu)x$, which is an element of E.

2. Ergodic theorems for weakly almost periodic functions

In this section we prove strong mean convergence theorems and a result about the existence of ergodic projection and retraction of Eberlein-weakly almost periodic functions for commutative semigroups.

Definition 2.1. Let E be a Banach space. A function $f \in C_b(S,E)$ is called *Eberlein-weakly almost periodic* if $\{r(s)f; s \in S\}$ is a relatively weakly compact subset of $C_b(S,E)$. We denote by $W(S,E)$ the set of all of Eberlein-weakly almost periodic functions. Let $UC_b(S,E)$ be the set of all *bounded uniformly continuous functions* from S to E, i.e., the set of all $f \in C_b(S,E)$ such that the map $s \mapsto r(s)f$ from S to $C_b(S,E)$ is continuous. Let C be a closed subset of E. A representation $T\colon S \to \mathrm{Lip}(C)$ is called *Eberlein-weakly almost periodic* on a subset D of C if for any $x \in D$, $T(\cdot)x \in W(S,E)$.

Remark 2.2. (a) $W(S,E)$ and $UC_b(S,E)$ are closed translation invariant linear subspaces of $C_b(S,E)$.

(b) We do not know whether $W(S,E) \subset UC_b(S,E)$. This holds when $S = \mathbb{R}^+$; see [29, Proposition 2.1].

Definition 2.3. Let $\{\mu_\alpha; \alpha \in A\}$ be a net of means on $C_b(S)$. Then we call $\{\mu_\alpha\}$ *strongly asymptotically invariant* (cf. [30], [24]) if $\lim_\alpha \|\mu_\alpha - r(s)^*\mu_\alpha\| = 0$ for every $s \in S$.

Remark 2.4. (a) Since S is commutative, a strongly asymptotically invariant net of finite means on $C_b(S)$ always exists; see Day [6].

(b) The following are examples of strongly asymptotically invariant net of means.

(i) Let $S = \mathbb{Z}^+$. Then putting $\mu_n(f) = (1/n)\sum_{k=0}^{n-1} f(k)$ for $f \in C_b(\mathbb{Z}^+) = l^\infty(\mathbb{Z}^+)$, $\{\mu_n; n \in \mathbb{N}\}$ is an asymptotically invariant net of means.

(ii) Let $S = \mathbb{Z}^+$. Then putting $\mu_s(f) = (1-s)\sum_{k=0}^{\infty} s^k f(k)$ for $f \in C_b(\mathbb{Z}^+) = l^\infty(\mathbb{Z}^+)$, $\{\mu_s; s \in (0,1)\}$ is an asymptotically invariant net of means.

(iii) Let $S = (\mathbb{Z}^+)^2$. Then putting $\mu_n(f) = (1/n^2)\sum_{i,j=0}^{n-1} f(i,j)$ for $f \in C_b((\mathbb{Z}^+)^2) = l^\infty((\mathbb{Z}^+)^2)$, $\{\mu_n; n \in \mathbb{N}\}$ is an asymptotically invariant net of means.

(iv) Let $S = \mathbb{R}^+$. Then putting $\mu_s(f) = (1/s)\int_0^s f(t)dt$ for $f \in C_b(\mathbb{R}^+)$, $\{\mu_s; s \in \mathbb{R}^+\backslash\{0\}\}$ is an asymptotically invariant net of means.

(v) Let $S = \mathbb{R}^+$. Then putting $\mu_s(f) = s\int_0^\infty e^{-st} f(t)dt$ for $f \in C_b(\mathbb{R}^+)$, $\{\mu_s; s \in \mathbb{R}^+\}$ is an asymptotically invariant net of means.

Definition 2.5 (Eberlein [9])**.** Let E be a Banach space. $\{T_\alpha; \alpha \in A\}$ is called a *system of almost invariant integrals* of a representation $U\colon S \to L(E)$ on a subset

D of E if the following hold:

I. $\{T_\alpha\} \subset L(E)$, $\sup_{\alpha\in A} \|T_\alpha\| < \infty$;
II. for any $z \in E$ and $\alpha \in A$, $T_\alpha z \in \mathrm{clco}\{U(s)z; s \in S\}$;
III. for any $s \in S$ and $x \in D$,
  (a) $\lim_\alpha \|U(s)T_\alpha x - T_\alpha x\| = 0$,
  (b) $\lim_\alpha \|T_\alpha U(s)x - T_\alpha x\| = 0$.

We give an example of a system of almost invariant integrals:

Proposition 2.6. *Let E be a Banach space and let $U(s) = r(s)$, $s \in S$, be the translation operator on $F := W(S,E) \cap UC_b(S,E)$. Let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on $C_b(S)$, and define $\{T_\alpha\} \subset L(F)$ by $(T_\alpha f)(t) = \tau(r(t)^*\mu_\alpha)f$ for $f \in F$ and $t \in S$ (see Definition 1.1 for τ). Then $\{T_\alpha\}$ is a system of asymptotically invariant integrals of a representation $U\colon S \to L(F)$ on F.*

We note that since $W(S,E) \subset C_C(S,E)$, T_α is well defined. For this proof, we give a lemma and a proposition.

Lemma 2.7 (Ruess and Summers [27, proof of Theorem 2.1])**.** *Let E be a Banach space. Let us equip the closed unit ball $B(E^*)$ of E^*, and the set of point evaluations $\delta(S) := \{\delta(s); s \in S\} \subset C_b(S)^*$ on $C_b(S)$, with the topologies induced by the w^*-topology and the strong topology, respectively. Then putting $\Phi(f)((\delta(s), x^*)) = \langle f(s), x^*\rangle$ for $f \in C_b(S,E)$ and $(\delta(s), x^*) \in \delta(S) \times B(E^*)$, Φ is a norm preserving isomorphism from $C_b(S,E)$ into $C_b(\delta(S) \times B(E^*))$.*

Proposition 2.8. *Let E be a Banach space and let μ be a mean on $C_b(S)$. Then for any $f \in F := W(S,E) \cap UC_b(S,E)$, $\tau(r(\cdot)^*\mu)f \in \mathrm{clco}\{r(s)f; s \in S\} \subset F$.*

Proof. First we prove $\tau(r(\cdot)^*\mu)f \in C_b(S,E)$. For any $t, s \in S$,

$$\begin{aligned}\|\tau(r(t)^*\mu)f - \tau(r(s)^*\mu)f\| &= \|\tau(\mu)(r(t)f) - \tau(\mu)(r(s)f)\| \\ &\le \|r(t)f - r(s)f\|,\end{aligned}$$

and since $f \in UC_b(S,E)$, the assertion follows. Assume $\tau(r(\cdot)^*\mu)f \notin A_f := \mathrm{clco}\{r(s)f; s \in S\}$. We consider $C_b(S,E)$ as a subspace of $C_b(\delta(S) \times B(E^*))$ by Lemma 2.7. Since A_f is weakly compact, A_f is closed in the pointwise convergence topology. So, by the separation theorem, there exists a functional T in the dual of $C_b(S,E)$ with respect to the pointwise topology on $\delta(S) \times B(E)^*$ such that $T(\tau(r(\cdot)^*\mu)f) < \inf\{T(g); g \in A_f\}$. Put $T = \sum_{i=1}^n \delta(s_i) \otimes x_i^*$, $n \in \mathbb{N}$, $s_i \in S$, $x_i^* \in E^*$. Then we have

$$\begin{aligned}\sum_{i=1}^n \langle \tau(r(s_i)^*\mu)f, x_i^*\rangle &< \inf\left\{\sum_{i=1}^n \langle g(s_i), x_i^*\rangle; g \in A_f\right\} \le \inf_{s\in S}\left\{\sum_{i=1}^n \langle (r(s)f)(s_i), x_i^*\rangle\right\} \\ &\le \mu_s\left(\sum_{i=1}^n \langle (r(s)f)(s_i), x_i^*\rangle\right) = \sum_{i=1}^n \mu_s\langle (r(s)f)(s_i), x_i^*\rangle \\ &= \sum_{i=1}^n (r(s_i)^*\mu)\langle f, x_i^*\rangle = \sum_{i=1}^n \langle \tau(r(s_i)^*\mu)f, x_i^*\rangle,\end{aligned}$$

which is a contradiction. Hence we have $\tau(r(t)^*\mu)f \in A_f$. By Remark 2.2 (a), we have $A_f \subset F$. □

Proof of Proposition 2.6. By Proposition 2.8, Definition 2.5 II is satisfied, and $T_\alpha F \subset F$ for any $\alpha \in A$. Clearly T_α is linear, and for any $f \in F$,

$$\begin{aligned}\|T_\alpha f\| &= \sup_t |T_\alpha f(t)| = \sup_t |\tau(r(t)^*\mu_\alpha)f| \\ &\le \sup_t \|\tau(r(t)^*\mu_\alpha)\|\,\|f\| \le \|f\|,\end{aligned}$$

implying $\|T_\alpha\| \le 1$. Hence we have $T_\alpha \in L(F)$. So, Definition 2.5 I is satisfied.

For $f \in F$,

$$\begin{aligned}\|U(s)T_\alpha f - T_\alpha f\| &= \|r(s)\tau(r(\cdot)^*\mu_\alpha)f - \tau(r(\cdot)^*\mu_\alpha)f\| \\ &= \sup_t \|\tau(r(t+s)^*\mu_\alpha)f - \tau(r(t)^*\mu_\alpha)f\| \\ &= \sup_t \|\tau(r(s)^*\mu_\alpha - \mu_\alpha)(r(t)f)\| \\ &\le \|r(s)^*\mu_\alpha - \mu_\alpha\|\,\|f\|.\end{aligned}$$

Hence, III (a) is satisfied. Since $T_\alpha U(s)f = \tau(r(\cdot)^*\mu_\alpha)(r(s)f) = \tau(r(s)^*r(\cdot)^*\mu_\alpha)f = \tau(r(\cdot+s)^*\mu_\alpha)f = r(s)\tau(r(\cdot)^*\mu_\alpha)f = U(s)T_\alpha f$, III (b) is satisfied. This completes the proof. □

From Eberlein [9] and Kido and Takahashi [18], we have the following theorem.

Theorem 2.9. *Let E be a Banach space and let $\{T_\alpha\}$ be a system of almost invariant integrals of representation $U\colon S \to L(E)$ on $\{x\}$ for some point $x \in E$. Assume $\sup_{s\in S}\|U(s)\| < \infty$ and a weak cluster point y of $\{T_\alpha x\}$ exists. Then $T_\alpha x$ converges strongly to y. In this case, if $\{U(s)x; s \in S\}$ is relatively weakly compact, then $y = U(\mu)x$ for every invariant mean on $C_b(S)$, and $\{U(\mu)x\} = \mathrm{clco}\{U(s)x; s \in S\} \cap \mathrm{Fix}(U)$.*

Definition 2.10. Let C be a closed subset of a Banach space E, and let D be a subset of C. A mapping $P\colon C \to D$ is called a *retraction* if $P^2 = P$. Furthermore, assume C and D are linear subspaces of E, and P is linear. Then P is called a *projection*.

Put $G = W(S,E)$ and $\tau' = \tau^G \in L(C_b(S)^*, L(C_C(S,G),G))$, where τ^G is as in Definition 1.1. The following theorem is a strong ergodic theorem for Eberlein-weakly almost periodic representations.

Theorem 2.11. *Let E be a Banach space and let $U\colon S \to L(W(S,E))$ be the translating representation. Let μ be an invariant mean on $C_b(S)$. Then the following hold:*

(a) *For each invariant mean μ on $C_b(S)$, $U(\mu)$ is a nonexpansive projection from $W(S,E)$ onto $\mathrm{Fix}(U) = E$ such that $U(\mu)U(s) = U(s)U(\mu) = U(\mu)$ for every $s \in S$ and $\{U(\mu)f\} = E \cap \mathrm{clco}\{U(s)f; s \in S\}$ for every $f \in W(S,E)$;*
(b) *$U(\mu)f = \tau'(\mu)(U(\cdot)f) = \tau(\mu)f$ for every $f \in W(S,E)$;*
(c) *for any strongly asymptotically invariant net of means $\{\mu_\alpha; \alpha \in A\}$ on $C_b(S)$ and $f \in F := W(S,E) \cap UC_b(S,E)$, $\tau(r(h)^*\mu_\alpha)f$ converges strongly to $y \in E$ uniformly in $h \in S$. Here $y = U(\mu)f$ for every invariant mean μ on $C_b(S)$ and is the only point in $E \cap \mathrm{clco}\{r(s)f; s \in S\}$.*

Proof. (a) We note that $U(\cdot)f \in C_C(S, W(S,E))$ for $f \in W(S,E)$. So, $U(\mu)f = \tau'(\mu)(U(\cdot)f) \in W(S,E)$ is well defined. From [18, Theorem 2, Lemma 4], the assertion follows.

(b) Let $f \in W(S,E)$. By Lemma 2.7, we consider $C_b(S,E)$ as a subspace of $C_b(\delta(S) \times B(E^*))$. Hence, by (a), $U(\mu)f \in \mathrm{Fix}(U) = E \subset W(S,E) \subset C_b(S,E) \subset C_b(\delta(S) \times B(E^*))$. For any $x^* \in B(E^*)$ and $t \in S$,

$$\begin{aligned}
\langle U(\mu)f, x^*\rangle &= \langle (U(\mu)f)(t), x^*\rangle \text{ (considering } U(\mu)f \in C_b(S,E)) \\
&= (U(\mu)f)((\delta(t), x^*)) \text{ (considering } U(\mu)f \in C_b(\delta(S) \times B(E^*))) \\
&= \langle U(\mu)f, (\delta(t), x^*)\rangle \text{ (considering } U(\mu)f \in W(S,E) \text{ and since} \\
&\qquad\qquad (\delta(t), x^*)\colon g \mapsto g((\delta(t), x^*)) = \langle g(t), x^*\rangle, \\
&\qquad\qquad\qquad g \in W(S,E), \text{ is in } W(S,E)^*) \\
&= \mu_s \langle U(s)f, (\delta(t), x^*)\rangle \\
&= \mu_s \langle (U(s)f)(t), x^*\rangle = \mu_s \langle f(t+s), x^*\rangle \\
&= \mu_s \langle f(s), x^*\rangle = \langle \tau(\mu)f, x^*\rangle.
\end{aligned}$$

Since $x^* \in B(E^*)$ is arbitrary, we have $\tau'(\mu)(U(\cdot)f) = U(\mu)f = \tau(\mu)f$.

(c) From Proposition 2.8 we see that there exists a weak cluster point g of $\{\tau(r(\cdot)^*\mu_\alpha)f; \alpha \in A\}$. From Proposition 2.6 and Theorem 2.9, $\tau(r(\cdot)^*\mu_\alpha)f$ converges strongly to $g = U(\mu)f \in \mathrm{Fix}(U) = E$ in the supremum norm and $\{U(\mu)f\} = E \cap \mathrm{clco}\{r(s)f; s \in S\}$. So, the assertion follows. □

Corollary 2.12. *Let C be a closed subset of a Banach space E and let $T\colon S \to \mathrm{Lip}(C)$ be an Eberlein-weakly almost periodic representation on a subset D of C such that $\sup_{s\in S} \|T(s)\| < \infty$. We assume $\mathrm{Fix}(T) \subset D$. Let $U(s)$, $s \in S$, be the translation operator on $W(S,E)$ and let μ be an invariant mean on $C_b(S)$. Then the following hold:*

(a) *For any strongly asymptotically invariant net of means $\{\mu_\alpha\}$ on $C_b(S)$ and $x \in D$, $T(r(h)^*\mu_\alpha)x$ converges strongly to $y \in E$ uniformly in $h \in S$. Here $y = U(\mu)(T(\cdot)x)$ and is the only point in $E \cap \mathrm{clco}\{r(s)f; s \in S\}$;*

(b) *if C is a closed convex subset of a uniformly convex Banach space E and $T\colon S \to \mathrm{Cont}(C)$, putting $Px = U(\mu)(T(\cdot)x)$ for $x \in D$, P is a nonexpansive retraction from D onto $\mathrm{Fix}(T)$ such that $PT(s) = T(s)P = P$ for every $s \in S$ and $Px \in \mathrm{clco}\{T(s)x; s \in S\}$ for every $x \in D$.*

Proof. Putting $f = T(\cdot)x$, $f \in W(S,E) \cap UC_b(S,E)$. So, (a) follows from Theorem 2.11 (c), and (b) follows from Theorem 2.11 (b) and [12, Theorem 3]. □

Remark 2.13. By a result of [14], we see that Theorem 2.11 (c) and Corollary 2.12 (a) hold for strongly regular net $\{\mu_\alpha\}$ which is a generalization of strongly regular kernel (see [23], [26]). Here a net $\{\mu_\alpha\} \subset C_b(S)^*$ is called *strongly regular* ([12]) if it satisfies the following conditions: (a) $\sup_\alpha \|\mu_\alpha\| < \infty$; (b) $\lim_\alpha \mu_\alpha(1) = 1$; (c) $\lim_\alpha \|\mu_\alpha - r(s)^*\mu_\alpha\| = 0$ for every $s \in S$.

Definition 2.14. Let C be a closed subset of a Banach space E. A representation $T\colon S \to \mathrm{Lip}(C)$ is called *strongly asymptotic regular* on a subset D of C if for any $x \in D$, $\lim_s \|T(s+t)x - T(s)x\| = 0$ for every $t \in S$.

Corollary 2.15. *Let C be a closed subset of a Banach space E and let $U(s)$, $s \in S$, be the translation operator on $W(S,E)$. Assume a representation $T\colon S \to \mathrm{Lip}(C)$ is Eberlein-weakly almost periodic and strongly asymptotic regular on a subset D of C such that $K := \sup_{s\in S} \|T(s)\| < \infty$. Let μ be an invariant mean on $C_b(S)$.*

Then the following hold:

(a) $T(\mu)$ *is a Lipschitzian retraction from* D *onto* $\mathrm{Fix}(T)$ *with* $\|T(\mu)\| \leq K$ *such that* $T(\mu)T(s) = T(s)T(\mu) = T(\mu)$ *for every* $s \in S$ *and* $T(\mu)x \in \mathrm{clco}\{T(s)x; s \in S\}$ *for every* $x \in D$*;*

(b) *for any* $x \in D$, $T(t+s)x$ *converges strongly to* $y \in \mathrm{Fix}(T)$ *uniformly in* $t \in S$*. Here* $y = U(\mu)(T(\cdot)x) = T(\mu)x$*.*

Proof. It is easy to see that $\{U(t); t \in S\}$ is a system of almost invariant integrals of a representation $U\colon S \to L(W(S,E))$ on $\{T(\cdot)x; x \in D\}$. So, from Theorem 2.9 and Theorem 2.11 (b), for any $x \in D$, $T(t+s)x \to U(\mu)(T(\cdot)x) = \tau(\mu)(T(\cdot)x) = T(\mu)x$ uniformly in $t \in S$. Then for any $q \in S$, $T(q)T(t+s)x \to T(q)T(\mu)x$, implying $T(\mu)x = T(q)T(\mu)x$. So, $T(\mu)x \in \mathrm{Fix}(T)$. The other assertions are easy. □

3. A weakly almost periodic representation

In the previous section, we proved the strong ergodic theorem for Eberlein-weakly almost periodic representations. Ruess and Summers [26] proved that if a representation $T\colon S \to \mathrm{Cont}(C)$ is asymptotically isometric, then T is Eberlein-weakly almost periodic when $S = \mathbb{R}^+$ or $\mathbb{Z}^+$, and E is uniformly convex. Their method of proof works as well for the d-dimensional case.

First, we give a definition.

Definition 3.1. A representation $T\colon S \to \mathrm{Cont}(C)$ is called *asymptotically isometric* on D if for any $x, y \in D$, $\lim_s \|T(s+h)x - T(s+k)y\|$ exists uniformly over $h, k \in S$.

Let S be $(\mathbb{Z}^+)^d$ or $(\mathbb{R}^+)^d$, $d \in \mathbb{N}$, and let $\|\cdot\|$ be the restriction to S of any norm on $(\mathbb{R})^d$. We call a representation $T\colon S \to \mathrm{Cont}(C)$ *strongly asymptotically isometric* on a subset D of C if for any $x, y \in D$,

$$\lim_{\|s\|\to\infty} \|T(s+h)x - T(s)y\| \quad \text{exists uniformly over } h \in S.$$

Remark 3.2. The following hold; see [16].

(1) Assume S is totally ordered. Then T is asymptotically isometric on D if and only if for any $x, y \in D$, $\lim_s \|T(s+h)x - T(s)y\|$ exists uniformly over $h \in S$; see Bruck [4] and Oka [22].

(2) Let C be a closed convex subset of a Hilbert space E. Then T is asymptotically isometric on a subset D of C and $0 \in \mathrm{Fix}(T)$ if and only if there exists a function $\varepsilon(\cdot,\cdot)\colon S \times S \to \mathbb{R}$ such that $\lim_{s,t\to\infty} \varepsilon(s,t) = 0$ and for any $x, y \in D$, $q \in S$, $\|T(s+q)x + T(t+q)y\|^2 \leq \|T(s)x + T(t)y\|^2 + \varepsilon(s,t)$.

(3) Assume there exist a subnet $\{s_\alpha\}$ of S and $x, y \in C$ such that $T(s_\alpha)x$ converges strongly to y. Then T is asymptotically isometric on $\{T(s)x; s \in S\} \cup \mathrm{Fix}(T)$.

(4) Let C be a closed convex subset of a Hilbert space E and each $T(s)$, $s \in S$, is affine. Then T is asymptotically isometric on C.

Theorem 3.3 (Ruess and Summers [27])**.** *Let* S *be* $(\mathbb{Z}^+)^d$ *or* $(\mathbb{R}^+)^d$, $d \in \mathbb{Z}^+$*. Let* C *be a closed convex subset of a uniformly convex Banach space* E *and let* $x \in C$*. Let* $T\colon S \to \mathrm{Cont}(C)$ *be a strongly asymptotically isometric representation on* $\{x\}$*. Then the following are equivalent:*

(a) $T(\cdot)x$ *is Eberlein-weakly almost periodic;*

(b) $\{T(s)x; s \in S\}$ *is relatively weakly compact.*

Out of the corollaries which we can get by applying Remark 2.13 and Remark 2.4 (b), we give only the following:

Corollary 3.4. *Let C be a closed convex subset of a uniformly convex Banach space E, and let $V, W \in \mathrm{Cont}(C)$ be such that $VW = WV$ and $\mathrm{Fix}\, V \cap \mathrm{Fix}\, W \neq \varnothing$. Let $x \in C$ and assume $\lim_{\|(s_1,s_2)\|\to\infty} \|V^{s_1+h_1}W^{s_2+h_2}x - V^{s_1}W^{s_2}x\|$ exists uniformly over $h_1, h_2 \in \mathbb{Z}^+$. Then putting $f(s) = V^{s_1}W^{s_2}x$ for $s = (s_1, s_2) \in (\mathbb{Z}^+)^2$, f is an Eberlein-weakly almost periodic function from $(\mathbb{Z}^+)^2$ to E.*

And $(1/n^2)\sum_{i,j=1}^{n-1} V^{i+h_1}W^{j+h_2}x$ converges strongly as $n \to \infty$ to a common fixed point of V and W uniformly over $h_1, h_2 \in \mathbb{Z}^+$.

Furthermore, assume $\lim_{s_1,s_2\to\infty} \|V^{s_1+t_1}W^{s_2+t_2}x - V^{s_1}W^{s_2}x\| = 0$ for every $t_1, t_2 \in \mathbb{Z}^+$. Then $V^{s_1+t_1}W^{s_2+t_2}x$ converges strongly as $s_1, s_2 \to \infty$ to a common fixed point of V and W uniformly in $t_1, t_2 \in \mathbb{Z}^+$.

Proof. Define $T\colon (\mathbb{Z}^+)^2 \to \mathrm{Cont}(C)$ by $T((s_1, s_2))x = V^{s_1}W^{s_2}x$ for $(s_1, s_2) \in (\mathbb{Z}^+)^2$, $x \in C$. Let μ_n be that of Remark 2.4 (b)(iii). Then $T(r(h)^*\mu_n)x = (1/n^2)\sum_{i,j=1}^{n-1} V^{i+h_1}W^{j+h_2}x$ for $h = (h_1, h_2) \in (\mathbb{Z}^+)^2$. So, the assertions follow from Remark 2.4 (b)(iii), Theorem 3.3, Corollary 2.12 and Corollary 2.15. □

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Department of Information Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan

E-mail address: kada@math.sci.ynu.ac.jp