STRONG ERGODIC THEOREMS FOR COMMUTATIVE SEMIGROUPS OF OPERATORS

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Abstract. We prove strong mean convergence theorems and the existence of ergodic projection and retraction for commutative semigroups of operators which is Eberlein-weakly almost periodic.

Introduction

A strong mean convergence theorem for nonexpansive mappings was first established for odd mappings by Baillon [2], and it was generalized to that for asymptotically isometric mappings by Bruck [4]. After works of Miyadera and Kobayasi [21] and Oka [22], we proved the theorem for general commutative semigroups ([16]). On the other hand, Ruess and Summers [26, 28, 29] proved the strong convergence theorem by different methods, such as the following.

A bounded and continuous function \( f \) from \( \mathbb{R}^+ \) to a Banach space \( E \) is called Eberlein-weakly almost periodic if its orbit by translation \( \{ r(s)f; s \in \mathbb{R}^+ \} \) is relatively weakly compact in the Banach space of all bounded and continuous functions from \( \mathbb{R}^+ \) to \( E \) with supremum norm. Ruess and Summers showed that if \( f \) is Eberlein-weakly almost periodic, then \( \frac{1}{t} \int_0^t f(s+h)ds \) converges strongly as \( t \to \infty \) to a point \( z \in E \) uniformly in \( h \in \mathbb{R}^+ \) (they proved for strongly regular kernels). Consequently, for a nonexpansive semigroup \( \{ T(s); s \in \mathbb{R}^+ \} \) on a closed convex subset \( C \) of a uniformly convex Banach space \( E \), if \( T(\cdot)x \) is asymptotically isometric for \( x \in C \), then \( \frac{1}{t} \int_0^t T(s+h)ds \) converges strongly as \( t \to \infty \) to a common fixed point of \( T(s) \), \( s \in S \), uniformly in \( h \in \mathbb{R}^+ \).

In section 2, we generalize their results to a commutative semigroup of operators, and give a result of the existence of ergodic projection and retraction. In section 3, we use results of Ruess and Summers ([26]) on Eberlein-almost periodicity for nonexpansive semigroups on \( \mathbb{R}^+ \) (resp. \( \mathbb{Z}^+ \)), adapted to the \( d \)-dimensional case, to apply our results to a pair of commuting nonexpansive maps in uniformly convex Banach spaces.
1. Preliminaries

Throughout this paper, \( S \) denotes a commutative semitopological semigroup with identity, i.e., a commutative semigroup with a Hausdorff topology such that for each \( t \in S \), the mapping \( s \mapsto s + t \) from \( S \) to \( S \) is continuous. We assume a Banach space \( E \) is real. We also denote by \( \mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{R} \) and \( \mathbb{R}^+ \) the sets of all integers, nonnegative integers, positive integers, real numbers and nonnegative real numbers, respectively. Let \( l^\infty(S) \) be the Banach space of all bounded real valued functions on \( S \) with the supremum norm, and let \( C_0(S) \) be the subspace of \( l^\infty(S) \) of all bounded continuous real valued functions on \( S \). Then for each \( s \in S \) and \( f \in C_0(S) \), we define an element \( r(s)f \) in \( C_0(S) \) by

\[
(r(s)f)(t) = f(t + s) \quad \text{for all } t \in S.
\]

An element \( \mu \) of \( C_0(S)^* \), where \( C_0(S)^* \) is the dual space of \( C_0(S) \), is called a mean on \( C_0(S) \) if \( \|\mu\| = \mu(1) = 1 \). As is known, \( \mu \) is a mean on \( C_0(S) \) if and only if \( \inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s) \) for each \( f \in C_0(S) \). A mean \( \mu \) on \( C_0(S) \) is invariant if \( \mu(r(s)f) = \mu(f) \) for all \( s \in S \) and \( f \in C_0(S) \). It is well known that there exists an invariant mean \( \mu \) on \( l^\infty(S) \); see Day [6]. Since \( C_0(S) \) is invariant under \( r(s) \), the restriction of \( \mu \) on \( C_0(S) \) is an invariant mean on \( C_0(S) \). A finite mean is an element of \( \text{co}\{\delta(s); s \in S\} \), where \( \delta(s)(f) = f(s) \) for each \( f \in l^\infty(S) \) and \( \text{co}A \) is the convex hull of \( A \). Depending on time and circumstances, the values of mean \( \mu \) at \( f \in C_0(S) \) will also be denoted by \( \mu(f) \) or \( \mu_s(f) \). A commutative semigroup \( S \) is a directed system when the binary relation is defined by \( s \leq t \) if and only if \( \{s\} \cup (S + s) \supset \{t\} \cup (S + t) \). We write \( x_n \to x \) (resp. \( x_n \rightharpoonup x \)) to indicate that the sequence \( \{x_n\} \) of vectors converges strongly (resp. weakly) to \( x \).

Let \( C \) be a closed subset of a Banach space \( E \). A mapping \( V \) of \( C \) into itself is called Lipschitzian if there exists \( K \geq 0 \) such that \( \|Vx -Vy\| \leq K\|x - y\| \) for every \( x, y \in C \). We define the Lipschitz norm of \( V \) by \( \|V\| := \sup\{\frac{\|Vx - Vy\|}{\|x - y\|}; x, y \in C, x \neq y\} \). If \( \|V\| \leq 1 \), then \( V \) is called nonexpansive. We denote by \( \text{Lip}(C) \), \( \text{Cont}(C) \) and \( L(E) \) the semitopological semigroup of all Lipschitzian self-mappings of \( C \), the semitopological semigroup of all nonexpansive self-mappings of \( C \) and the semitopological semigroup of all bounded linear mappings of \( E \), under composition and pointwise convergence topology, respectively. Let \( T: S \to \text{Lip}(C) \) or \( \text{Cont}(C) \) (resp. \( L(E) \)) be a representation, i.e., \( T(s + t) = T(s)T(t) \) for every \( s, t \in S \), and \( T(\cdot)x \) is continuous for every \( x \in C \) (resp. \( x \in E \)). We denote by \( \text{Fix}(T) \) the set of common fixed points of \( T(s), s \in S \). We also denote by \( B(E) = \{x \in E; \|x\| \leq 1\} \) the unit ball of \( E \). For two Banach spaces \( E \) and \( F \), \( L(E, F) \) is the set of all bounded linear mappings from \( E \) to \( F \).

Let \( S \) be a topological space and let \( E \) be a Banach space. Then we denote by \( C_0(S, E) \) the Banach space of all bounded continuous mappings from \( S \) to \( E \) with supremum norm, and by \( C_0(C(S, E)) \) the set of all elements \( f \in C_0(S, E) \) such that \( f(S) := \{f(s); s \in S\} \) is relatively weakly compact. It is obvious that \( C_0(C(S, E)) \) is a linear subspace of \( C_0(S, E) \). Let \( \kappa: E \to C_0(S, E) \) be a mapping such that for \( x \in E \) and \( s \in S \), \( \kappa(x)(s) = x \). Since \( \kappa \) is a norm preserving isomorphism of \( E \) into \( C_0(S, E) \), we consider \( E \) as a subspace of \( C_0(S, E) \).

For any mean \( \mu \) on \( C_0(S) \), we define a “vector valued mean” \( \tau(\mu) \) homomorphically, i.e., \( \tau(\mu) \) is an element of \( L(C_0(C(S, E)), E) \) such that \( \tau(\mu)x = x \) for each \( x \in E \), and \( \|\tau(\mu)\| = 1 \). More generally, we give a definition as follows; see [16].
Definition 1.1. Let $E$ be a Banach space. For $\mu \in C_b(S)^*$ and $f \in C_C(S,E)$, let $x_{\mu,f}^*: x^* \mapsto \mu(f(x),x^*)$ be an element of $E^{**}$. Then $x_{\mu,f}^* \in E$. We define an element $\tau = \tau^E \in L(C_b(S)^*,L(C_C(S,E),E))$ by $\tau(\mu)f = x_{\mu,f}^*$.

Remark 1.2. The following holds; see [16]. (i) $\tau$ is injective and $\|\tau\| \leq 1$; (ii) $\tau(\mu)x = \mu(1)x$ for all $x \in E$, and if $\mu$ is a mean on $C_b(S)$, then $\|\tau(\mu)\| = 1$; (iii) $\tau$ maps the point evaluation $\varepsilon(s)$ to the point evaluation $\varepsilon(s)$, $s \in S$, where $\varepsilon(s)f = f(s)$ for $f \in C_C(S,E)$; (iv) $\tau(r(s)^*\mu) = r(s)^*\tau(\mu)$.

Let $T: S \to \text{Lip}(C)$ be a representation such that $T(\cdot)x \in C_C(S,E)$ for some $x \in C$. Then we shall denote $\tau(\mu)(T(\cdot)x)$ by $T(\mu)x$, which is an element of $E$.

2. Ergodic theorems for weakly almost periodic functions

In this section we prove strong mean convergence theorems and a result about the existence of ergodic projection and retraction of Eberlein-weakly almost periodic functions for commutative semigroups.

Definition 2.1. Let $E$ be a Banach space. A function $f \in C_b(S,E)$ is called Eberlein-weakly almost periodic if $\{r(s)f; s \in S\}$ is a relatively weakly compact subset of $C_b(S,E)$. We denote by $W(S,E)$ the set of all of Eberlein-weakly almost periodic functions. Let $UC_b(S,E)$ be the set of all bounded uniformly continuous functions from $S$ to $E$, i.e., the set of all $f \in C_b(S,E)$ such that the map $s \mapsto r(s)f$ from $S$ to $C_b(S,E)$ is continuous. Let $C$ be a closed subset of $E$. A representation $T: S \to \text{Lip}(C)$ is called Eberlein-weakly almost periodic on a subset $D$ of $C$ if for any $x \in D$, $T(\cdot)x \in W(S,E)$.

Remark 2.2. (a) $W(S,E)$ and $UC_b(S,E)$ are closed translation invariant linear subspaces of $C_b(S,E)$. (b) We do not know whether $W(S,E) \subset UC_b(S,E)$. This holds when $S = \mathbb{R}^+$; see [29, Proposition 2.1].

Definition 2.3. Let $\{\mu_\alpha; \alpha \in A\}$ be a net of means on $C_b(S)$. Then we call $\{\mu_\alpha\}$ strongly asymptotically invariant (cf. [30], [24]) if $\lim_{s \to \infty} \|\mu_\alpha - r(s)^*\mu_\alpha\| = 0$ for every $s \in S$.

Remark 2.4. (a) Since $S$ is commutative, a strongly asymptotically invariant net of finite means on $C_b(S)$ always exists; see Day [6]. (b) The following are examples of strongly asymptotically invariant net of means.

(i) Let $S = \mathbb{Z}^+$. Then putting $\mu_n(f) = (1/n) \sum_{k=0}^{n-1} f(k)$ for $f \in C_b(\mathbb{Z}^+) = l^\infty(\mathbb{Z}^+), \{\mu_n; n \in \mathbb{N}\}$ is an asymptotically invariant net of means.

(ii) Let $S = \mathbb{Z}^+$. Then putting $\mu_s(f) = (1-s) \sum_{k=0}^{\infty} s^k f(k)$ for $f \in C_b(\mathbb{Z}^+) = l^\infty(\mathbb{Z}^+), \{\mu_s; s \in (0,1)\}$ is an asymptotically invariant net of means.

(iii) Let $S = (\mathbb{Z}^+)^2$. Then putting $\mu_n(f) = (1/n^2) \sum_{i,j=0}^{n-1} f(i,j)$ for $f \in C_b((\mathbb{Z}^+)^2) = l^\infty((\mathbb{Z}^+)^2), \{\mu_n; n \in \mathbb{N}\}$ is an asymptotically invariant net of means.

(iv) Let $S = \mathbb{R}^+$. Then putting $\mu_s(f) = (1/s) \int_0^s f(t) dt$ for $f \in C_b(\mathbb{R}^+), \{\mu_s; s \in \mathbb{R}^+\}$ is an asymptotically invariant net of means.

(v) Let $S = \mathbb{R}^+$. Then putting $\mu_s(f) = s \int_0^\infty e^{-st} f(t) dt$ for $f \in C_b(\mathbb{R}^+), \{\mu_s; s \in \mathbb{R}^+\}$ is an asymptotically invariant net of means.

Definition 2.5 (Eberlein [9]). Let $E$ be a Banach space. $\{T_\alpha; \alpha \in A\}$ is called a system of almost invariant integrals of a representation $U: S \to L(E)$ on a subset
D of $E$ if the following hold:

I. $(T_\alpha) \subset L(E), \sup_{\alpha \in A} \|T_\alpha\| < \infty$;
II. for any $z \in E$ and $\alpha \in A$, $T_\alpha z \in \text{clco}\{U(s)z; s \in S\}$;
III. for any $s \in S$ and $x \in D$,
(a) $\lim_\alpha \|U(s)T_\alpha x - T_\alpha x\| = 0$,
(b) $\lim_\alpha \|T_\alpha U(s)x - x\| = 0$.

We give an example of a system of almost invariant integrals:

**Proposition 2.6.** Let $E$ be a Banach space and let $U(s) = \tau(s), s \in S$, be the translation operator on $F := W(S,E) \cap UC_b(S,E)$. Let $\{\mu_{\alpha}\}$ be a strongly asymptotically invariant net of means on $C_0(S)$, and define $(T_\alpha) \subset L(F)$ by $(T_\alpha f)(t) = \tau(t)^*\mu_{\alpha}f$ for $f \in F$ and $t \in S$ (see Definition 1.1 for $\tau$). Then $(T_\alpha)$ is a system of asymptotically invariant integrals of a representation $U : S \to L(F)$ on $F$.

We note that since $W(S,E) \subset C_0(S,E)$, $T_\alpha$ is well defined. For this proof, we give a lemma and a proposition.

**Lemma 2.7** (Ruess and Summers [27, proof of Theorem 2.1]). Let $E$ be a Banach space. Let us equip the closed unit ball $B(E^*)$ of $E^*$, and the set of point evaluations $\delta(s) := \{\delta(s); s \in S\} \subset C_b(S)^*$ on $C_b(S)$, with the topologies induced by the $w^*$-topology and the strong topology, respectively. Then putting $\Phi(f)((\delta(s),x^*)) = \langle f(s),x^* \rangle$ for $f \in C_b(S,E)$ and $(\delta(s),x^*) \in \delta(S) \times B(E^*)$, $\Phi$ is a norm preserving isomorphism from $C_b(S,E)$ onto $C_b(\delta(S) \times B(E^*))$.

**Proposition 2.8.** Let $E$ be a Banach space and let $\mu$ be a mean on $C_0(S)$. Then for any $f \in F := W(S,E) \cap UC_b(S,E)$, $\tau(\tau(\cdot)^*\mu)f \in \text{clco}\{r(s)f; s \in S\} \subset F$.

**Proof.** First we prove $\|\tau(\tau(t)^*\mu)f - \tau(\tau(s)^*\mu)f\| = \|\tau(\mu)(\tau(t)f) - \tau(\mu)(\tau(s)f)\| 
\leq \|\tau(t)f - \tau(s)f\|,$
and if $f \in UC_b(S,E)$, the assertion follows. Assume $\tau(\tau(\cdot)^*\mu)f \not\in A_f := \text{clco}\{r(s)f; s \in S\}$. We consider $C_b(S,E)$ as a subspace of $C_b(\delta(S) \times B(E^*))$ by Lemma 2.7. Since $A_f$ is weakly compact, $A_f$ is closed in the pointwise convergence topology. So, by the separation theorem, there exists a functional $T$ in the dual of $C_b(S,E)$ with respect to the pointwise topology on $\delta(S) \times B(E^*)$ such that $T(\tau(\tau(\cdot)^*\mu)f) < \inf\{T(g); g \in A_f\}$. Put $T = \sum_{i=1}^n \delta(s_i) \otimes x_i^*, n \in \mathbb{N}, s_i \in S, x_i^* \in E^*$. Then we have

$$\sum_{i=1}^n (\tau(r(s_i)^*\mu)f, x_i^*) < \inf\left\{\sum_{i=1}^n \langle g(s_i), x_i^* \rangle; g \in A_f\right\} \leq \inf_{s \in S} \left\{\sum_{i=1}^n \langle (r(s)f)(s_i), x_i^* \rangle \right\}$$
\leq \mu_s \left(\sum_{i=1}^n \langle (r(s)f)(s_i), x_i^* \rangle \right) = \sum_{i=1}^n \mu_s \langle (r(s)f)(s_i), x_i^* \rangle$

$$= \sum_{i=1}^n \langle r(s_i)^*\mu(f, x_i^*) \rangle = \sum_{i=1}^n \langle \tau(r(s_i)^*\mu)f, x_i^* \rangle,$$
which is a contradiction. Hence we have $\tau(\tau(t)^*\mu)f \in A_f$. By Remark 2.2 (a), we have $A_f \subset F$. \qed
Proof of Proposition 2.6. By Proposition 2.8, Definition 2.5 II is satisfied, and $T_\alpha F \subset F$ for any $\alpha \in A$. Clearly $T_\alpha$ is linear, and for any $f \in F$,
\[
\|T_\alpha f\| = \sup_t |T_\alpha f(t)| = \sup_t |\tau*(r(t)^* \mu_\alpha)f| \\
\leq \sup_t \|\tau*(r(t)^* \mu_\alpha)\| \|f\| \leq \|f\|,
\]
implying $\|T_\alpha\| \leq 1$. Hence we have $T_\alpha \in L(F)$. So, Definition 2.5 I is satisfied.

For $f \in F$,
\[
\|U(s)T_\alpha f - T_\alpha f\| = \|r(s)\tau*(r(t)^* \mu_\alpha)f - \tau*(r(t)^* \mu_\alpha)f\| \\
= \sup_t \|\tau*(r(t+s)^* \mu_\alpha)f - \tau*(r(t)^* \mu_\alpha)f\| \\
= \sup_t \|\tau*(r(t)^* \mu_\alpha - \mu_\alpha)(r(t)f)\| \\
\leq \|r(s)^* \mu_\alpha - \mu_\alpha\| \|f\|.
\]
Hence, III (a) is satisfied. Since $T_\alpha U(s)f = \tau*(r(s)^* \mu_\alpha)(r(s)f) = \tau*(s)(r(s)^* \mu_\alpha)f = \tau^*(s+r)^* \mu_\alpha)f = r(s)^* \tau(r(s)^* \mu_\alpha)f = U(s)T_\alpha f$, III (b) is satisfied. This completes the proof.

From Eberlein [9] and Kido and Takahashi [18], we have the following theorem.

Theorem 2.9. Let $E$ be a Banach space and let $\{T_\alpha\}$ be a system of almost invariant integrals of representation $U : S \to L(E)$ on $\{x\}$ for some point $x \in E$. Assume $\sup_{s \in S} \|U(s)\| < \infty$ and a weak cluster point $y$ of $\{T_\alpha x\}$ exists. Then $T_\alpha x$ converges strongly to $y$. In this case, if $\{U(s)x; s \in S\}$ is relatively weakly compact, then $y = U(\mu)x$ for every invariant mean on $C_b(S)$, and $\{U(\mu)x\} = \clco\{U(s)x; s \in S\} \cap \text{Fix}(U)$.

Definition 2.10. Let $C$ be a closed subset of a Banach space $E$, and let $D$ be a subset of $C$. A mapping $P : C \to D$ is called a retraction if $P^2 = P$. Furthermore, assume $C$ and $D$ are linear subspaces of $E$, and $P$ is linear. Then $P$ is called a projection.

Put $G = W(S, E)$ and $\tau^* = \tau^G \in L(C_b(S)^*, L(C_C(S, G), G))$, where $\tau^G$ is as in Definition 1.1. The following theorem is a strong ergodic theorem for Eberlein-weakly almost periodic representations.

Theorem 2.11. Let $E$ be a Banach space and let $U : S \to L(W(S, E))$ be the translating representation. Let $\mu$ be an invariant mean on $C_b(S)$. Then the following hold:

(a) For each invariant mean $\mu$ on $C_b(S)$, $U(\mu)$ is a nonexpansive projection from $W(S, E)$ onto $\text{Fix}(U) = E$ such that $U(\mu)U(s) = U(s)U(\mu) = U(\mu)$ for every $s \in S$ and $\{U(\mu)f\} = E \cap \clco\{U(s)f; s \in S\}$ for every $f \in W(S, E)$;
(b) $U(\mu)f = \tau'(\mu)(U(\cdot)f) = \tau(\mu)f$ for every $f \in W(S, E)$;
(c) for any strongly asymptotically invariant net of means $\{\mu_\alpha; \alpha \in A\}$ on $C_b(S)$ and $f \in F := W(S, E) \cap UC_b(S, E)$, $\tau(h)^* \mu_\alpha)f$ converges strongly to $y \in E$ uniformly in $h \in S$. Here $y = U(\mu)f$ for every invariant mean $\mu$ on $C_b(S)$ and is the only point in $E \cap \clco\{r(s)f; s \in S\}$.

Proof. (a) We note that $U(\cdot)f \in C_C(S, W(S, E))$ for $f \in W(S, E)$. So, $U(\mu)f = \tau'(\mu)(U(\cdot)f) \in W(S, E)$ is well defined. From [18, Theorem 2, Lemma 4], the assertion follows.
(b) Let \( f \in W(S, E) \). By Lemma 2.7, we consider \( C_b(S, E) \) as a subspace of \( C_b(\delta(S) \times B(E^*)) \). Hence, by (a), \( U(\mu)f \in \text{Fix}(U) = E \subset W(S, E) \subset C_b(S, E) \subset C_b(\delta(S) \times B(E^*)) \). For any \( x^* \in B(E^*) \) and \( t \in S \),
\[
\langle U(\mu)f, x^* \rangle = \langle (U(\mu)f)(t), x^* \rangle \quad (\text{considering } U(\mu)f \in C_b(S, E))
\]
\[
= \langle (U(\mu)f)((\delta(t), x^*)) \rangle \quad (\text{considering } U(\mu)f \in C_b(\delta(S) \times B(E^*)))
\]
\[
= \langle (U(\mu)f, (\delta(t), x^*)) \rangle \quad (\text{considering } U(\mu)f \in W(S, E) \text{ and since})
\]
\[
(\delta(t), x^*): g \mapsto g((\delta(t), x^*)) = (g(t), x^*),
\]
\[
g \in W(S, E), \text{ is in } W(S, E)^*
\]
\[
= \mu_s(U(s)f, (\delta(t), x^*))
\]
\[
= \mu_s((U(s)f)(t), x^*) = \mu_s(f(t + s), x^*)
\]
\[
= \mu_s(f(t), x^*) = \tau(\mu)f, x^*).
\]

Since \( x^* \in B(E^*) \) is arbitrary, we have \( \tau^*(\mu)(U(\cdot)f) = U(\mu)f = \tau(\mu)f \).

(c) From Proposition 2.8 we see that there exists a weak cluster point \( g \) of \( \{\tau(r(\cdot)^*\mu_\alpha); \alpha \in A\} \). From Proposition 2.6 and Theorem 2.9, \( \tau(r(\cdot)^*\mu_\alpha)f \) converges strongly to \( g = U(\mu)f \in \text{Fix}(U) = E \) in the supremum norm and \( \{\tau(r(\cdot)^*\mu_\alpha)\} = E \cap \text{clco}\{r(s)f; s \in S\} \). So, the assertion follows.

\[\square\]

**Corollary 2.12.** Let \( C \) be a closed subset of a Banach space \( E \) and let \( T: S \to \text{Lip}(C) \) be an Eberlein-weakly almost periodic representation on a subset \( D \) of \( C \) such that \( \sup_{s \in S}\|T(s)\| < \infty \). We assume \( \text{Fix}(T) \subset D \). Let \( U(s), s \in S, \) be the translation operator on \( W(S, E) \) and let \( \mu \) be an invariant mean on \( C_b(S, E) \). Then the following hold:

(a) For any strongly asymptotically invariant net of means \( \{\mu_\alpha\} \) on \( C_b(S, E) \) and \( x \in D, T(r(h)^*\mu_\alpha)x \) converges strongly to \( y \in E \) uniformly in \( h \in S \). Here \( y = U(\mu)(T(\cdot)x) \) and is the only point in \( E \cap \text{clco}\{r(s)f; s \in S\} \);

(b) if \( C \) is a closed convex subset of a uniformly convex Banach space \( E \) and \( T: S \to \text{Cont}(C) \), putting \( Px = U(\mu)(T(\cdot)x) \) for \( x \in D, P \) is a nonexpansive retraction from \( D \) onto \( \text{Fix}(T) \) such that \( PT(s) = T(s)P = P \) for every \( s \in S \) and \( Px = \text{clco}(T(s)x); s \in S \) for every \( x \in D \).

**Proof.** Putting \( f = T(\cdot)x, f \in W(S, E) \cap UC_b(S, E) \). So, (a) follows from Theorem 2.11 (c), and (b) follows from Theorem 2.11 (b) and [12, Theorem 3].

\[\square\]

**Remark 2.13.** By a result of [14], we see that Theorem 2.11 (c) and Corollary 2.12 (a) hold for strongly regular net \( \{\mu_\alpha\} \) which is a generalization of strongly regular kernel (see [23, 26]). Here a net \( \{\mu_\alpha\} \subset C_b(S)^* \) is called strongly regular ([12]) if it satisfies the following conditions: (a) \( \sup_\alpha \|\mu_\alpha\| < \infty \); (b) \( \lim_\alpha \mu_\alpha(1) = 1 \); (c) \( \lim_\alpha \|\mu_\alpha - r(\cdot)^*\mu_\alpha\| = 0 \) for every \( s \in S \).

**Definition 2.14.** Let \( C \) be a closed subset of a Banach space \( E \). A representation \( T: S \to \text{Lip}(C) \) is called strongly asymptotic regular on a subset \( D \) of \( C \) if for any \( x \in D, \lim_s \|T(s + t)x - T(s)x\| = 0 \) for every \( t \in S \).

**Corollary 2.15.** Let \( C \) be a closed subset of a Banach space \( E \) and let \( U(s), s \in S, \) be the translation operator on \( W(S, E) \). Assume a representation \( T: S \to \text{Lip}(C) \) is Eberlein-weakly almost periodic and strongly asymptotic regular on a subset \( D \) of \( C \) such that \( K := \sup_{s \in S}\|T(s)\| < \infty \). Let \( \mu \) be an invariant mean on \( C_b(S) \).
Then the following hold:

(a) \( T(\mu) \) is a Lipschitzian retraction from \( D \) onto \( \text{Fix}(T) \) with \( \|T(\mu)\| \leq K \) such that \( T(\mu)T(s) = T(s)T(\mu) = T(\mu) \) for every \( s \in S \) and \( T(\mu)x \in \text{clo}\{T(s)x; s \in S\} \) for every \( x \in D \);

(b) for any \( x \in D \), \( T(t+s)x \) converges strongly to \( y \in \text{Fix}(T) \) uniformly in \( t \in S \).

Here \( y = U(\mu)(T(\cdot)x) = T(\mu)x \).

**Proof.** It is easy to see that \( \{U(t); t \in S\} \) is a system of almost invariant integrals of a representation \( U : S \rightarrow L(W(S,E)) \) on \( \{T(\cdot)x; x \in D\} \). So, from Theorem 2.9 and Theorem 2.11 (b), for any \( x \in D \), \( T(t+s)x \rightarrow U(\mu)(T(\cdot)x) = \tau(\mu)(T(\cdot)x) = T(\mu)x \) uniformly in \( t \in S \). Then for any \( q \in S \), \( T(q)T(t+s)x \rightarrow T(q)T(\mu)x \), implying \( T(\mu)x = T(q)T(\mu)x \). So, \( T(\mu)x \in \text{Fix}(T) \). The other assertions are easy. \( \square \)

### 3. A weakly almost periodic representation

In the previous section, we proved the strong ergodic theorem for Eberlein-weakly almost periodic representations. Russ and Summers [26] proved that if a representation \( T : S \rightarrow \text{Cont}(C) \) is asymptotically isometric, then \( T \) is Eberlein-weakly almost periodic when \( S = \mathbb{R}^+ \) or \( \mathbb{Z}^+ \), and \( E \) is uniformly convex. Their method of proof works as well for the \( d \)-dimensional case.

First, we give a definition.

**Definition 3.1.** A representation \( T : S \rightarrow \text{Cont}(C) \) is called **asymptotically isometric** on \( D \) if for any \( x, y \in D \), \( \lim_s \|T(s+h)x - T(s+k)y\| \) exists uniformly over \( h, k \in S \).

Let \( S \) be \((\mathbb{Z}^+)^d \) or \((\mathbb{R}^+)^d \), \( d \in \mathbb{N} \), and let \( \| \cdot \| \) be the restriction to \( S \) of any norm on \((\mathbb{R}^d)^d \). We call a representation \( T : S \rightarrow \text{Cont}(C) \) **strongly asymptotically isometric** on a subset \( D \) of \( C \) if for any \( x, y \in D \),

\[
\lim_{\|s\| \to \infty} \|T(s+h)x - T(s)y\| \quad \text{exists uniformly over } h \in S.
\]

**Remark 3.2.** The following hold; see [16].

1. Assume \( S \) is totally ordered. Then \( T \) is asymptotically isometric on \( D \) if and only if for any \( x, y \in D \), \( \lim_s \|T(s+h)x - T(s+k)y\| \) exists uniformly over \( h \in S \); see Bruck [4] and Oka [22].

2. Let \( C \) be a closed convex subset of a Hilbert space \( E \). Then \( T \) is asymptotically isometric on a subset \( D \) of \( C \) and \( 0 \in \text{Fix}(T) \) if and only if there exists a function \( \varepsilon(\cdot, \cdot) : S \times S \rightarrow \mathbb{R} \) such that \( \lim_{s,t \to \infty} \varepsilon(s,t) = 0 \) and for any \( x, y \in D \), \( q \in S \), \( \|T(s+q)x + T(t+q)y\|^2 \leq \|T(s)x + T(t)y\|^2 + \varepsilon(s,t) \).

3. Assume there exist a subnet \( \{s_\alpha\} \) of \( S \) and \( x, y \in C \) such that \( T(s_\alpha)x \) converges strongly to \( y \). Then \( T \) is asymptotically isometric on \( \{T(s)x; s \in S\} \cup \text{Fix}(T) \).

4. Let \( C \) be a closed convex subset of a Hilbert space \( E \) and each \( T(s), s \in S \), is affine. Then \( T \) is asymptotically isometric on \( C \).

**Theorem 3.3** (Russ and Summers [27]). Let \( S \) be \((\mathbb{Z}^+)^d \) or \((\mathbb{R}^+)^d \), \( d \in \mathbb{Z}^+ \). Let \( C \) be a closed convex subset of a uniformly convex Banach space \( E \) and let \( x \in C \). Let \( T : S \rightarrow \text{Cont}(C) \) be a strongly asymptotically isometric representation on \( \{x\} \).

Then the following are equivalent:

(a) \( T(\cdot)x \) is Eberlein-weakly almost periodic;

(b) \( \{T(s)x; s \in S\} \) is relatively weakly compact.
Out of the corollaries which we can get by applying Remark 2.13 and Remark 2.4 (b), we give only the following:

**Corollary 3.4.** Let $C$ be a closed convex subset of a uniformly convex Banach space $E$, and let $V, W \in \text{Cont}(C)$ be such that $V W = W V$ and $\text{Fix} V \cap \text{Fix} W \neq \emptyset$. Let $x \in C$ and assume $\lim \| (s_1, s_2) \| \to \infty \| V^{s_1+h_1} W^{s_2+h_2} x - V^{s_1} W^{s_2} x \|$ exists uniformly over $h_1, h_2 \in \mathbb{Z}^+$. Then putting $f(s) = V^{s_1} W^{s_2} x$ for $s = (s_1, s_2) \in (\mathbb{Z}^+)^2$, $f$ is an Eberlein-weakly almost periodic function from $(\mathbb{Z}^+)^2$ to $E$.

Furthermore, assume $\lim_{s_1, s_2 \to \infty} \| V^{s_1+t_1} W^{s_2+t_2} x - V^{s_1} W^{s_2} x \| = 0$ for every $t_1, t_2 \in \mathbb{Z}^+$. Then $V^{s_1+t_1} W^{s_2+t_2} x$ converges strongly as $s_1, s_2 \to \infty$ to a common fixed point of $V$ and $W$ uniformly over $t_1, t_2 \in \mathbb{Z}^+$.

**Proof.** Define $T : (\mathbb{Z}^+)^2 \to \text{Cont}(C)$ by $T((s_1, s_2)) = V^{s_1} W^{s_2} x$ for $(s_1, s_2) \in (\mathbb{Z}^+)^2$, $x \in C$. Let $\mu_n$ be that of Remark 2.4 (b)(iii). Then $T(h) = \mu_n x = (1/n^2) \sum_{i,j=1}^{n-1} V^{i+h_1} W^{j+h_2} x$ for $h = (h_1, h_2) \in (\mathbb{Z}^+)^2$. So, the assertions follow from Remark 2.4 (b)(iii), Theorem 3.3, Corollary 2.12 and Corollary 2.15.

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