STRONG ERGODIC THEOREMS FOR COMMUTATIVE SEMIGROUPS OF OPERATORS

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Abstract. We prove strong mean convergence theorems and the existence of ergodic projection and retraction for commutative semigroups of operators which is Eberlein-weakenly almost periodic.

Introduction

A strong mean convergence theorem for nonexpansive mappings was first established for odd mappings by Baillon [2], and it was generalized to that for asymptotically isometric mappings by Bruck [4]. After works of Miyadera and Kobayasi [21] and Oka [22], we proved the theorem for general commutative semigroups ([16]). On the other hand, Ruess and Summers [26, 28, 29] proved the strong convergence theorem by different methods, such as the following.

A bounded and continuous function \( f \) from \( \mathbb{R}^+ \) to a Banach space \( E \) is called Eberlein-weakenly almost periodic if its orbit by translation \( \{ r(s)f; s \in \mathbb{R}^+ \} \) is relatively weakly compact in the Banach space of all bounded and continuous functions from \( \mathbb{R}^+ \) to \( E \) with supremum norm. Ruess and Summers showed that if \( f \) is Eberlein-weakenly almost periodic, then \( (1/t) \int_0^t f(s + h)ds \) converges strongly as \( t \to \infty \) to a point \( z \in E \) uniformly in \( h \in \mathbb{R}^+ \) (they proved for strongly regular kernels). Consequently, for a nonexpansive semigroup \( \{ T(s); s \in \mathbb{R}^+ \} \) on a closed convex subset \( C \) of a uniformly convex Banach space \( E \), if \( T(\cdot)x \) is asymptotically isometric for \( x \in C \), then \( (1/t) \int_0^t T(s + h)ds \) converges strongly as \( t \to \infty \) to a common fixed point of \( T(s), s \in S \), uniformly in \( h \in \mathbb{R}^+ \).

In section 2, we generalize their results to a commutative semigroup of operators, and give a result of the existence of ergodic projection and retraction. In section 3, we use results of Ruess and Summers ([26]) on Eberlein-almost periodicity for nonexpansive semigroups on \( \mathbb{R}^+ \) (resp. \( \mathbb{Z}^+ \)), adapted to the \( d \)-dimensional case, to apply our results to a pair of commuting nonexpansive maps in uniformly convex Banach spaces.

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1. Preliminaries

Throughout this paper, $S$ denotes a commutative semitopological semigroup with identity, i.e., a commutative semigroup with a Hausdorff topology such that for each $t \in S$, the mapping $s \mapsto s + t$ from $S$ to $S$ is continuous. We assume a Banach space $E$ is real. We also denote by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{R}$ and $\mathbb{R}^+$ the sets of all integers, nonnegative integers, positive integers, real numbers and nonnegative real numbers, respectively. Let $l^\infty(S)$ be the Banach space of all bounded real valued functions on $S$ with the supremum norm, and let $C_b(S)$ be the subspace of $l^\infty(S)$ of all bounded continuous real valued functions on $S$. Then for each $s \in S$ and $f \in C_b(S)$, we define an element $r(s)f$ in $C_b(S)$ by

$$(r(s)f)(t) = f(t + s) \quad \text{for all } t \in S.$$  

An element $\mu$ of $C_b(S)^*$, where $C_b(S)^*$ is the dual space of $C_b(S)$, is called a mean on $C_b(S)$ if $\|\mu\| = \mu(1) = 1$. As is known, $\mu$ is a mean on $C_b(S)$ if and only if $\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$ for each $f \in C_b(S)$. A mean $\mu$ on $C_b(S)$ is invariant if $\mu(r(s)f) = \mu(f)$ for all $s \in S$ and $f \in C_b(S)$. It is well known that there exists an invariant mean $\mu$ on $l^\infty(S)$; see Day [6]. Since $C_b(S)$ is invariant under $r(s)$, the restriction of $\mu$ on $C_b(S)$ is an invariant mean on $C_b(S)$. A finite mean is an element of $\text{co}\{\delta(s); s \in S\}$, where $\delta(s)(f) = f(s)$ for each $f \in l^\infty(S)$ and $\text{co} A$ is the convex hull of $A$. Depending on time and circumstances, the values of mean $\mu$ at $f \in C_b(S)$ will also be denoted by $\mu(f)$ or $\mu_s(f(s))$. A commutative semigroup $S$ is a directed system when the binary relation is defined by $s \leq t$ if and only if $\{s\} \cup (S + s) \supset \{t\} \cup (S + t)$. We write $x_n \to x$ (resp. $x_n \rightharpoonup x$) to indicate that the sequence $\{x_n\}$ of vectors converges strongly (resp. weakly) to $x$.

Let $C$ be a closed subset of a Banach space $E$. A mapping $V$ of $C$ into itself is called Lipschitzian if there exists $K \geq 0$ such that $\|Vx - Vy\| \leq K\|x - y\|$ for every $x, y \in C$. We define the Lipschitz norm of $V$ by $\|V\| := \sup\{\|Vx - Vy\|; x, y \in C, x \neq y\}$. If $\|V\| \leq 1$, then $V$ is called nonexpansive. We denote by $\text{Lip}(C)$, $\text{Cont}(C)$ and $L(E)$ the semitopological semigroup of all Lipschitzian self-mappings of $C$, the semitopological semigroup of all nonexpansive self-mappings of $C$ and the semitopological semigroup of all bounded linear mappings of $E$, under composition and pointwise convergence topology, respectively. Let $T : S \to \text{Lip}(C)$ or $\text{Cont}(C)$ (resp. $L(E)$) be a representation, i.e., $T(s + t) = T(s)T(t)$ for every $s, t \in S$, and $T(\cdot)x$ is continuous for every $x \in C$ (resp. $x \in E$). We denote by $\text{Fix}(T)$ the set of common fixed points of $T(s), s \in S$. We also denote by $B(E) = \{x \in E; \|x\| \leq 1\}$ the unit ball of $E$. For two Banach spaces $E$ and $F$, $L(E, F)$ is the set of all bounded linear mappings from $E$ to $F$.

Let $S$ be a topological space and let $E$ be a Banach space. Then we denote by $C_b(S, E)$ the Banach space of all bounded continuous mappings from $S$ to $E$ with supremum norm, and by $C_C(S, E)$ the set of all elements $f \in C_b(S, E)$ such that $f(S) := \{f(s); s \in S\}$ is relatively weakly compact. It is obvious that $C_C(S, E)$ is a linear subspace of $C_b(S, E)$. Let $\kappa : (S, E) \to C_b(S, E)$ be a mapping such that for $x \in E$ and $s \in S$, $\kappa(x)(s) = x$. Since $\kappa$ is a norm preserving isomorphism of $E$ into $C_b(S, E)$, we consider $E$ as a subspace of $C_b(S, E)$.

For any mean $\mu$ on $C_b(S)$, we define a “vector valued mean” $\tau(\mu)$ homomorphically, i.e., $\tau(\mu)$ is an element of $L(C_C(S, E), E)$ such that $\tau(\mu)x = x$ for each $x \in E$, and $\|\tau(\mu)\| = 1$. More generally, we give a definition as follows; see [16].
Definition 2.1. Let $E$ be a Banach space. For $\mu \in C_b(S)^*$ and $f \in C_C(S,E)$, let $x_{\mu,f}^*: x^* \mapsto \mu(f(x), x^*)$ be an element of $E^{**}$. Then $x_{\mu,f}^* \in E$. We define an element $\tau = \tau^E \in L(C_b(S)^*, L(C_C(S,E), E))$ by $\tau(\mu)f = x_{\mu,f}^*$.

Remark 1.2. The following holds; see [16]. (i) $\tau$ is injective and $\|\tau\| \leq 1$; (ii) $\tau(\mu)x = \mu(1)x$ for all $x \in E$, and if $\mu$ is a mean on $C_b(S)$, then $\|\tau(\mu)\| = 1$; (iii) $\tau$ maps the point evaluation $\delta(s)$ to the point evaluation $\epsilon(s)$, $s \in S$, where $\epsilon(s)f = f(s)$ for $f \in C_C(S,E)$; (iv) $\tau(r(s)^*\mu) = r(s)^*\tau(\mu)$.

Let $T : S \to \operatorname{Lip}(C)$ be a representation such that $T(\cdot)x \in C_C(S,E)$ for some $x \in C$. Then we shall denote $\tau(\mu)(T(\cdot)x)$ by $T(\mu)x$, which is an element of $E$.

2. Ergodic theorems for weakly almost periodic functions

In this section we prove strong mean convergence theorems and a result about the existence of ergodic projection and retraction of Eberlein-weakly almost periodic functions for commutative semigroups.

Definition 2.2. Let $E$ be a Banach space. A function $f \in C_b(S,E)$ is called Eberlein-weakly almost periodic if $\{r(s)f; s \in S\}$ is a relatively weakly compact subset of $C_b(S,E)$. We denote by $W(S,E)$ the set of all of Eberlein-weakly almost periodic functions. Let $UC_b(S,E)$ be the set of all bounded uniformly continuous functions from $S$ to $E$, i.e., the set of all $f \in C_b(S,E)$ such that the map $s \mapsto r(s)f$ from $S$ to $C_b(S,E)$ is continuous. Let $C$ be a closed subset of $E$. A representation $T : S \to \operatorname{Lip}(C)$ is called Eberlein-weakly almost periodic on a subset $D$ of $C$ if for any $x \in D$, $T(\cdot)x \in W(S,E)$.

Remark 2.2. (a) $W(S,E)$ and $UC_b(S,E)$ are closed translation invariant linear subspaces of $C_b(S,E)$.

(b) We do not know whether $W(S,E) \subset UC_b(S,E)$. This holds when $S = \mathbb{R}^+$; see [29, Proposition 2.1].

Definition 2.3. Let $\{\mu_\alpha; \alpha \in A\}$ be a net of means on $C_b(S)$. Then we call $\{\mu_\alpha\}$ strongly asymptotically invariant (cf. [30], [24]) if $\lim_{s} \|\mu_\alpha - r(s)^*\mu_\alpha\| = 0$ for every $s \in S$.

Remark 2.4. (a) Since $S$ is commutative, a strongly asymptotically invariant net of finite means on $C_b(S)$ always exists; see Day [6].

(b) The following are examples of strongly asymptotically invariant net of means.

(i) Let $S = \mathbb{Z}^+$. Then putting $\mu_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)$ for $f \in C_b(\mathbb{Z}^+) = l^\infty(\mathbb{Z}^+)$, $\{\mu_n; n \in \mathbb{N}\}$ is an asymptotically invariant net of means.

(ii) Let $S = \mathbb{Z}^+$. Then putting $\mu_s(f) = (1 - s)\sum_{k=0}^{\infty} s^k f(k)$ for $f \in C_b(\mathbb{Z}^+) = l^\infty(\mathbb{Z}^+)$, $\{\mu_s; s \in (0,1)\}$ is an asymptotically invariant net of means.

(iii) Let $S = (\mathbb{Z}^+)^2$. Then putting $\mu_n(f) = (1/n^2)\sum_{i,j=0}^{n-1} f(i,j)$ for $f \in C_b((\mathbb{Z}^+)^2) = l^\infty((\mathbb{Z}^+)^2)$, $\{\mu_n; n \in \mathbb{N}\}$ is an asymptotically invariant net of means.

(iv) Let $S = \mathbb{R}^+$. Then putting $\mu_s(f) = \int_0^s f(t) dt$ for $f \in C_b(\mathbb{R}^+)$, $\{\mu_s; s \in \mathbb{R}^+\}$ is an asymptotically invariant net of means.

(v) Let $S = \mathbb{R}^+$. Then putting $\mu_s(f) = s \int_0^\infty e^{-st} f(t) dt$ for $f \in C_b(\mathbb{R}^+)$, $\{\mu_s; s \in \mathbb{R}^+\}$ is an asymptotically invariant net of means.

Definition 2.5 (Eberlein [9]). Let $E$ be a Banach space. $\{T_\alpha; \alpha \in A\}$ is called a system of almost invariant integrals of a representation $U : S \to L(E)$ on a subset
D of E if the following hold:
I. \( \{ T_\alpha \} \subset L(E), \sup_{\alpha \in A} \| T_\alpha \| < \infty; \)
II. for any \( z \in E \) and \( \alpha \in A, T_\alpha z \in \text{clco}\{ U(s)z; s \in S \}; \)
III. for any \( s \in S \) and \( x \in D, \)
   (a) \( \lim_{\alpha} \| U(s)T_\alpha x - T_\alpha x \| = 0, \)
   (b) \( \lim_{\alpha} \| T_\alpha U(s)x - T_\alpha x \| = 0. \)

We give an example of a system of almost invariant integrals:

**Proposition 2.6.** Let \( E \) be a Banach space and let \( U(s) = r(s), s \in S, \) be the translation operator on \( F := W(S, E) \cap UC_b(S, E) \). Let \( \{ \mu_\alpha \} \) be a strongly asymptotically invariant net of means on \( C_b(S) \), and define \( \{ T_\alpha \} \subset L(F) \) by \( (T_\alpha f)(t) = \tau(r(t)^* \mu_\alpha) f \) for \( f \in F \) and \( t \in S \) (see Definition 1.1 for \( \tau \)). Then \( \{ T_\alpha \} \) is a system of asymptotically invariant integrals of a representation \( U: S \to L(F) \) on \( F \).

We note that since \( W(S, E) \subset C_c(S, E) \), \( T_\alpha \) is well defined. For this proof, we give a lemma and a proposition.

**Lemma 2.7** (Ruess and Summers [27, proof of Theorem 2.1]). Let \( E \) be a Banach space. Let us equip the closed unit ball \( B(E^*) \) of \( E^* \), and the set of point evaluations \( \delta(S) := \{ \delta(s); s \in S \} \subset C_b(S)^* \) on \( C_b(S) \), with the topologies induced by the \( w^* \)-topology and the strong topology, respectively. Then putting \( \Phi(f)((\delta(s), x^*)) = \langle f(s), x^* \rangle \) for \( f \in C_b(S, E) \) and \( (\delta(s), x^*) \in \delta(S) \times B(E^*) \), \( \Phi \) is a norm preserving isomorphism from \( C_b(S, E) \) into \( C_b(\delta(S) \times B(E^*)) \).

**Proposition 2.8.** Let \( E \) be a Banach space and let \( \mu \) be a mean on \( C_b(S) \). Then for any \( f \in F := W(S, E) \cap UC_b(S, E) \), \( \tau(r(\cdot)^* \mu) f \in \text{clco}\{ r(s)f; s \in S \} \subset F \).

Proof. First we prove \( \tau(r(\cdot)^* \mu) f \in C_b(S, E) \). For any \( t, s \in S \),
\[
\| \tau(r(t)^* \mu) f - \tau(r(s)^* \mu) f \| = \| \tau(\mu)(r(t)f) - \tau(\mu)(r(s)f) \| \\
\leq \| r(t)f - r(s)f \|,
\]
and since \( f \in UC_b(S, E) \), the assertion follows. Assume \( \tau(r(\cdot)^* \mu) f \notin A_f := \text{clco}\{ r(s)f; s \in S \} \). We consider \( C_b(S, E) \) as a subspace of \( C_b(\delta(S) \times B(E^*)) \) by Lemma 2.7. Since \( A_f \) is weakly compact, \( A_f \) is closed in the pointwise convergence topology. So, by the separation theorem, there exists a functional \( T \) in the dual of \( C_b(S, E) \) with respect to the pointwise topology on \( \delta(S) \times B(E^*) \) such that \( T(\tau(r(\cdot)^* \mu) f) < \inf\{ T(g); g \in A_f \} \). Put \( T = \sum_{i=1}^{n} \delta(s_i) \otimes x_i^* \), \( n \in N, s_i \in S, x_i^* \in E^* \). Then we have
\[
\sum_{i=1}^{n} \langle (r(s_i)^* \mu) f, x_i^* \rangle < \inf\left\{ \sum_{i=1}^{n} \langle g(s_i), x_i^* \rangle; g \in A_f \right\} \leq \inf_{s \in S} \left\{ \sum_{i=1}^{n} \langle (r(s)f)(s_i), x_i^* \rangle \right\} \\
\leq \mu_s \left( \sum_{i=1}^{n} \langle (r(s)f)(s_i), x_i^* \rangle \right) = \sum_{i=1}^{n} \mu_s \langle (r(s)f)(s_i), x_i^* \rangle \\
= \sum_{i=1}^{n} (r(s_i)^* \mu)(f, x_i^*) = \sum_{i=1}^{n} (\tau(r(s_i)^* \mu)f, x_i^*),
\]
which is a contradiction. Hence we have \( \tau(r(t)^* \mu) f \in A_f \). By Remark 2.2 (a), we have \( A_f \subset F \).

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Definition 2.10. Let \( \| \) imply \( \| F \) is linear, and for any \( f \in F \),
\[
\| T_\alpha f \| = \sup_t |T_\alpha f(t)| = \sup_t |\tau(r(t)^* \mu_\alpha) f|
\leq \sup_t \|\tau(r(t)^* \mu_\alpha)\| \|f\| \leq \|f\|,
\]
implying \( \| T_\alpha \| \leq 1 \). Hence we have \( T_\alpha \in L(F) \). So, Definition 2.5 I is satisfied.

For \( f \in F \),
\[
\| U(s)T_\alpha f - T_\alpha f \| = \| r(s)\tau(r(\cdot)^* \mu_\alpha) f - \tau(r(\cdot)^* \mu_\alpha) f \|
= \sup_t \|\tau(r(t + s)^* \mu_\alpha) f - \tau(r(t)^* \mu_\alpha) f\|
= \sup_t \|\tau(r(s)^* \mu_\alpha) f - \tau(r(t)^* \mu_\alpha) f\|
\leq \|r(s)^* \mu_\alpha - \mu_\alpha\| \|f\|.
\]

Hence, III (a) is satisfied. Since \( T_\alpha U(s)f = \tau(r(s)^* \mu_\alpha) f = \tau(r(s)^* \mu_\alpha) f = \tau(r(s)^* \mu_\alpha) f = U(s)T_\alpha f \), III (b) is satisfied. This completes the proof.

From Eberlein [9] and Kido and Takahashi [18], we have the following theorem.

Theorem 2.9. Let \( E \) be a Banach space and let \( \{T_\alpha\} \) be a system of almost invariant integrals of representation \( U : S \to L(E) \) on \( \{x\} \) for some point \( x \in E \). Assume \( \sup_{s \in S} \|U(s)\| < \infty \) and a weak cluster point \( y \) of \( \{T_\alpha x\} \) exists. Then \( T_\alpha x \) converges strongly to \( y \). In this case, if \( \{U(s)x; s \in S\} \) is relatively weakly compact, then \( y = U(\mu)x \) for every invariant mean on \( C_b(S) \), and \( \{U(\mu)x\} = \text{clco}\{U(s)x; s \in S\} \cap \text{Fix}(U) \).

Definition 2.10. Let \( C \) be a closed subset of a Banach space \( E \), and let \( D \) be a subset of \( C \). A mapping \( P : C \to D \) is called a retraction if \( P^2 = P \). Furthermore, assume \( C \) and \( D \) are linear subspaces of \( E \), and \( P \) is linear. Then \( P \) is called a projection.

Put \( G = W(S,E) \) and \( \tau' = \tau'^G \in L(C_b(S)^*, L(C_C(S,G),G)) \), where \( \tau'^G \) is as in Definition 1.1. The following theorem is a strong ergodic theorem for Eberlein-weakly almost periodic representations.

Theorem 2.11. Let \( E \) be a Banach space and let \( U : S \to L(W(S,E)) \) be the translating representation. Let \( \mu \) be an invariant mean on \( C_b(S) \). Then the following hold:

a) For each invariant mean \( \mu \) on \( C_b(S) \), \( U(\mu) \) is a nonexpansive projection from \( W(S,E) \) onto \( \text{Fix}(U) = E \) such that \( U(\mu)U(s) = U(s)U(\mu) = U(\mu) \) for every \( s \in S \) and \( \{U(\mu)f\} = E \cap \text{clco}\{U(s)f; s \in S\} \) for every \( f \in W(S,E) \);

b) \( U(\mu)f = \tau'(\mu)(U(\cdot)f) = \tau'(\mu)f \) for every \( f \in W(S,E) \);

c) for any strongly asymptotically invariant net of means \( \{\mu_\alpha; \alpha \in A\} \) on \( C_b(S) \) and \( f \in F := W(S,E) \cap U(C_b(S,E)), \tau(\tau(h)^* \mu_\alpha)f \) converges strongly to \( y \) in \( E \) uniformly in \( h \in S \). Here \( y = U(\mu)f \) for every invariant mean \( \mu \) on \( C_b(S) \) and is the only point in \( E \cap \text{clco}\{r(s)f; s \in S\} \).

Proof. (a) We note that \( U(\cdot)f \in C_C(S,W(S,E)) \) for \( f \in W(S,E) \). So, \( U(\mu)f = \tau'(\mu)(U(\cdot)f) \in W(S,E) \) is well defined. From [18, Theorem 2, Lemma 4], the assertion follows.
Since $x^* \in B(E^*)$ is arbitrary, we have $\tau(\mu)(U(\cdot)f) = U(\mu)f = \tau(\mu)f$.

(c) From Proposition 2.8 we see that there exists a weak cluster point $g = U(\mu)f \in \text{Fix}(U) = E$ in the supremum norm and $\{\tau(r(\cdot)^*\mu_\alpha)f ; \alpha \in A\}$. From Proposition 2.6 and Theorem 2.9, $\tau(r(\cdot)^*\mu_\alpha)f$ converges strongly to $g = U(\mu)f \in \text{Fix}(U) = E$ in the supremum norm and $\{\mu_\alpha\} = E \cap \text{clco}\{r(s)f ; s \in S\}$. So, the assertion follows.

**Corollary 2.12.** Let $C$ be a closed subset of a Banach space $E$ and let $T: S \to \text{Lip}(C)$ be an Eberlein-weakly almost periodic representation on a subset $D$ of $C$ such that $\sup_{s \in S} \|T(s)\| < \infty$. We assume $\text{Fix}(T) \subset D$. Let $U(s)$, $s \in S$, be the translation operator on $W(S, E)$ and let $\mu$ be an invariant mean on $C_b(S)$. Then the following hold:

(a) For any strongly asymptotically invariant net of means $\{\mu_\alpha\}$ on $C_b(S)$ and $x \in D$, $T(r(h)^*\mu_\alpha)x$ converges strongly to $y \in E$ uniformly in $h \in S$. Here $y = U(\mu)(T(\cdot)x)$ and is the only point in $E \cap \text{clco}\{r(s)f ; s \in S\}$;

(b) if $C$ is a closed convex subset of a uniformly convex Banach space $E$ and $T: S \to \text{Cont}(C)$, putting $Pf = U(\mu)(T(\cdot)x)$ for $x \in D$, $P$ is a nonexpansive retraction from $D$ onto $\text{Fix}(T)$ such that $PT(s) = T(s)P = P$ for every $s \in S$ and $Pf \in \text{clco}\{T(s)x ; s \in S\}$ for every $x \in D$.

**Proof.** Putting $f = T(\cdot)x$, $f \in W(S, E) \cap UC_b(S, E)$. So, (a) follows from Theorem 2.11 (c), and (b) follows from Theorem 2.11 (b) and [12, Theorem 3].

**Remark 2.13.** By a result of [14], we see that Theorem 2.11 (c) and Corollary 2.12 (a) hold for strongly regular net $\{\mu_\alpha\}$ which is a generalization of strongly regular kernel (see [23, 26]). Here a net $\{\mu_\alpha\} \subset C_b(S)^*$ is called strongly regular ([12]) if it satisfies the following conditions: (a) $\sup_\alpha \|\mu_\alpha\| < \infty$; (b) $\lim_\alpha \mu_\alpha(1) = 1$; (c) $\lim_\alpha \|\mu_\alpha - r(s)^*\mu_\alpha\| = 0$ for every $s \in S$.

**Definition 2.14.** Let $C$ be a closed subset of a Banach space $E$. A representation $T: S \to \text{Lip}(C)$ is called strongly asymptotic regular on a subset $D$ of $C$ if for any $x \in D$, $\lim_t \|T(s + t)x - T(s)x\| = 0$ for every $t \in S$.

**Corollary 2.15.** Let $C$ be a closed subset of a Banach space $E$ and let $U(s)$, $s \in S$, be the translation operator on $W(S, E)$. Assume a representation $T: S \to \text{Lip}(C)$ is Eberlein-weakly almost periodic and strongly asymptotic regular on a subset $D$ of $C$ such that $K := \sup_{s \in S} \|T(s)\| < \infty$. Let $\mu$ be an invariant mean on $C_b(S)$. 


Then the following hold:

(a) $T(\mu)$ is a Lipschitzian retraction from $D$ onto $\text{Fix}(T)$ with $\|T(\mu)\| \leq K$ such that $T(\mu)p = T(s)T(\mu) = T(\mu)$ for every $s \in S$ and $T(\mu)x \in \text{clco}\{T(s)x; s \in S\}$ for every $x \in D$;

(b) for any $x \in D$, $T(t+s)x$ converges strongly to $y \in \text{Fix}(T)$ uniformly in $t \in S$. Here $y = U(\mu)(T(\cdot)x) = T(\mu)x$.

Proof. It is easy to see that $\{U(t); t \in S\}$ is a system of almost invariant integrals of a representation $U : S \rightarrow L(W(S,E))$ on $\{T(\cdot)x; x \in D\}$. So, from Theorem 2.9 and Theorem 2.11 (b), for any $x \in D$, $T(t+s)x \rightarrow U(\mu)(T(\cdot)x) = (\tau(\mu)| T(\cdot)x) = T(\mu)x$ uniformly in $t \in S$. Then for any $q \in S$, $T(q)T(t+s)x \rightarrow T(q)T(\mu)x$, implying $T(\mu)x = T(q)T(\mu)x$. So, $T(\mu)x \in \text{Fix}(T)$. The other assertions are easy. □

3. A WEAKLY ALMOST PERIODIC REPRESENTATION

In the previous section, we proved the strong ergodic theorem for Eberlein-weakly almost periodic representations. Ruess and Summers [26] proved that if a representation $T : S \rightarrow \text{Cont}(C)$ is asymptotically isometric, then $T$ is Eberlein-weakly almost periodic when $S = \mathbb{R}^+$ or $\mathbb{Z}^+$, and $E$ is uniformly convex. Their method of proof works as well for the $d$-dimensional case.

First, we give a definition.

Definition 3.1. A representation $T : S \rightarrow \text{Cont}(C)$ is called asymptotically isometric on $D$ if for any $x, y \in D$, $\lim_{h,k \in S} \|T(s+h)x - T(s+k)y\|$ exists uniformly over $h, k \in S$.

Let $S$ be $(\mathbb{Z}^+)^d$ or $(\mathbb{R}^+)^d$, $d \in \mathbb{N}$, and let $\| \cdot \|$ be the restriction to $S$ of any norm on $(\mathbb{R})^d$. We call a representation $T : S \rightarrow \text{Cont}(C)$ strongly asymptotically isometric on a subset $D$ of $C$ if for any $x, y \in D$,

$$\lim_{\|s\| \rightarrow \infty} \|T(s+h)x - T(s)y\| \text{ exists uniformly over } h \in S.$$

Remark 3.2. The following hold: see [16].

(1) Assume $S$ is totally ordered. Then $T$ is asymptotically isometric on $D$ if and only if for any $x, y \in D$, $\lim_{s, t \rightarrow \infty} \|T(s+h)x - T(s+h)y\|$ exists uniformly over $h \in S$; see Bruck [4] and Oka [22].

(2) Let $C$ be a closed convex subset of a Hilbert space $E$. Then $T$ is asymptotically isometric on a subset $D$ of $C$ and $0 \in \text{Fix}(T)$ if and only if there exists a function $\varepsilon(\cdot, \cdot) : S \times S \rightarrow \mathbb{R}$ such that $\lim_{s, t \rightarrow \infty} \varepsilon(s, t) = 0$ and for any $x, y \in D$, $q \in S$, $\|T(s+q)x + T(t+q)y\| \leq \|T(s)x + T(t)y\| + \varepsilon(s, t)$.

(3) Assume there exist a subnet $\{s_\alpha\}$ of $S$ and $x, y \in C$ such that $T(s_\alpha)x$ converges strongly to $y$. Then $T$ is asymptotically isometric on $\{T(s)x; s \in S\} \cup \text{Fix}(T)$.

(4) Let $C$ be a closed convex subset of a Hilbert space $E$ and each $T(s), s \in S$, is affine. Then $T$ is asymptotically isometric on $C$.

Theorem 3.3 (Ruess and Summers [27]). Let $S$ be $(\mathbb{Z}^+)^d$ or $(\mathbb{R}^+)^d$, $d \in \mathbb{Z}^+$. Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ and let $x \in C$. Let $T : S \rightarrow \text{Cont}(C)$ be a strongly asymptotically isometric representation on $\{x\}$. Then the following are equivalent:

(a) $T(\cdot)x$ is Eberlein-weakly almost periodic;

(b) $\{T(s)x; s \in S\}$ is relatively weakly compact.
Out of the corollaries which we can get by applying Remark 2.13 and Remark 2.4 (b), we give only the following:

**Corollary 3.4.** Let $C$ be a closed convex subset of a uniformly convex Banach space $E$, and let $V, W \in \text{Cont}(C)$ be such that $VW = WV$ and $\text{Fix} V \cap \text{Fix} W = \emptyset$. Let $x \in C$ and assume $\lim_{t \to \infty} ||V^{s_1+h_1}W^{s_2+h_2}x - V^{s_1}W^{s_2}x||$ exists uniformly over $h_1, h_2 \in \mathbb{Z}^+$. Then putting $f(s) = V^{s_1}W^{s_2}x$ for $s = (s_1, s_2) \in (\mathbb{Z}^+)^2$, $f$ is an Eberlein-weakly almost periodic function from $(\mathbb{Z}^+)^2$ to $E$.

And $(1/n^2) \sum_{i,j=1}^{n-1} V^{i+h_1}W^{j+h_2}x$ converges strongly as $n \to \infty$ to a common fixed point of $V$ and $W$ uniformly over $h_1, h_2 \in \mathbb{Z}^+$.

Furthermore, assume $\lim_{s_1, s_2 \to \infty} ||V^{s_1+t_1}W^{s_2+t_2}x - V^{s_1}W^{s_2}x|| = 0$ for every $t_1, t_2 \in \mathbb{Z}^+$. Then $V^{s_1+t_1}W^{s_2+t_2}x$ converges strongly as $s_1, s_2 \to \infty$ to a common fixed point of $V$ and $W$ uniformly in $t_1, t_2 \in \mathbb{Z}^+$.

**Proof.** Define $T : (\mathbb{Z}^+)^2 \to \text{Cont}(C)$ by $T((s_1, s_2))x = V^{s_1}W^{s_2}x$ for $(s_1, s_2) \in (\mathbb{Z}^+)^2$, $x \in C$. Let $\mu_n$ be that of Remark 2.4 (b)(iii). Then $T(r(h)^*\mu_n)x = (1/n^2) \sum_{i,j=1}^{n-1} V^{i+h_1}W^{j+h_2}x$ for $h = (h_1, h_2) \in (\mathbb{Z}^+)^2$. So, the assertions follow from Remark 2.4 (b)(iii), Theorem 3.3, Corollary 2.12 and Corollary 2.15.

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**References**


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