

## EXISTENCE OF ERGODIC RETRACTION FOR NONCOMMUTATIVE SEMIGROUPS IN BANACH SPACES

OSAMU KADA

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ABSTRACT. We prove the existence of ergodic retraction for a noncommutative semigroup which is right Eberlein-weakly almost periodic in a uniformly convex Banach space.

### 1. INTRODUCTION

Let  $S$  be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each  $t \in S$ , mappings  $s \mapsto st$  and  $s \mapsto ts$  from  $S$  to  $S$  are continuous. We denote by  $\mathbb{Z}$ ,  $\mathbb{Z}^+$ ,  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{R}^+$  the sets of all integers, nonnegative integers, positive integers, real numbers and nonnegative real numbers, respectively. Let  $C$  be a subset of a real Banach space  $E$ . Then a mapping  $T : C \rightarrow C$  is called *nonexpansive* provided  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . The following nonlinear ergodic theorem for nonexpansive mappings (in a Hilbert space) was established by Baillon [1, 2]:

**Theorem 1.** *Let  $C$  be a nonempty closed convex subset of  $H$  and let  $T$  be a nonexpansive mapping of  $C$  into itself. If  $T$  has a fixed point, then the Cesáro means  $(1/n) \sum_{k=0}^{n-1} T^k x$  converge weakly as  $n \rightarrow \infty$  to a fixed point  $y$  of  $T$ , and if  $-C = C$ ,  $0 \in C$  and  $T$  is odd, then the Cesáro means converge strongly as  $n \rightarrow \infty$  to a fixed point  $y$  of  $T$ .*

In these cases, putting  $y = Px$  for each  $x \in C$ ,  $P$  is a nonexpansive retraction of  $C$  onto the set  $\text{Fix}(T)$  of fixed points of  $T$  such that  $PT^n = T^n P = P$  for all  $n \in \mathbb{Z}^+$ , and  $Px \in \text{cl co}\{T^n x; n \in \mathbb{Z}^+\}$  for each  $x \in C$ , where  $\text{cl co } A$  is the closure of the convex hull of  $A$ . We call such a retraction “ergodic retraction”. Takahashi [29] proved the existence of ergodic retraction for a noncommutative semigroup in a Hilbert space, and Hirano, Kido and Takahashi [12] proved it for commutative semigroups in a uniformly convex Banach space. Recently, we studied the operator  $T(\mu)$  on  $C$  when  $\mu$  is an invariant mean on  $C_b(S)$ , which is the candidate of the ergodic retraction ([16]). But it remains open whether there exists ergodic retraction for noncommutative semigroups in Banach spaces. On the other hand, Ruess and Summers [24, 26, 27] proved strong mean ergodic theorems for Eberlein-weakly almost periodic functions over  $S = \mathbb{R}^+$ , which generalized Baillon [2] and Bruck

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[4, 5]. Further, we generalized this for commutative semigroups and obtained some results about ergodic projections and retractions ([13]).

In this paper, we prove the existence of ergodic retraction for a noncommutative semigroup in a uniformly convex Banach space, assuming the semigroup is right Eberlein-weakly almost periodic.

## 2. PRELIMINARIES

Throughout this paper we assume a Banach space  $E$  is real. Let  $S$  be a semitopological semigroup with identity, let  $l^\infty(S)$  be the Banach space of all bounded real valued functions on  $S$  with the supremum norm, and let  $l^1(S)$  be the Banach space of all functions  $f : S \rightarrow \mathbb{R}$  such that  $\|f\|_1 = \sum_{s \in S} |f(s)| < \infty$  with norm  $\|\cdot\|_1$ . Then  $l^1(S)^* = l^\infty(S)$ . We also denote by  $C_b(S)$  the space of all bounded continuous real valued functions on  $S$ . Then for each  $s \in S$  and  $f \in l^\infty(S)$ , we define elements  $r(s)f$  and  $l(s)f$  in  $l^\infty(S)$  by  $(r(s)f)(t) = f(ts)$ , and  $(l(s)f)(t) = f(st)$ ,  $t \in S$ . Let  $X$  be a subspace of  $l^\infty(S)$  containing constant functions and translation invariant; i.e. for any  $s \in S$ ,  $r(s)X \subset X$  and  $l(s)X \subset X$ . Then an element  $\mu$  of  $X^*$ , where  $X^*$  is the dual space of  $X$ , is called a *mean* on  $X$  if  $\|\mu\| = \mu(1) = 1$ . As is known,  $\mu$  is a mean on  $X$  if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each  $f \in X$ . A mean  $\mu$  on  $X$  is invariant if  $\mu(r(s)f) = \mu(f)$  for all  $s \in S$  and  $f \in X$ . A finite mean is an element of  $\text{co}\{\delta(s); s \in S\}$ , where  $\delta(s)(f) = f(s)$  for each  $f \in l^\infty(S)$ . Depending on time and circumstances, the values of mean  $\mu$  at  $f \in X$  will also be denoted by  $\mu(f)$  or  $\mu_s(f(s))$ .

$S$  is called *right reversible* if any two closed left ideals of  $S$  have nonvoid intersection. In this case,  $(S, \leq)$  is a directed system when the binary relation “ $\leq$ ” on  $S$  is defined by  $s \leq t$  if and only if  $\{s\} \cup \text{cl}(Ss) \supset \{t\} \cup \text{cl}(St)$ ,  $s, t \in S$ .

*Remark 2.1.* Let  $S$  be a semitopological semigroup such that the topology is normal and  $C_b(S)$  has a right (resp. left) invariant mean. Then  $S$  is right (resp. left) reversible; see Lau and Takahashi [21].

Let  $C$  be a closed subset of a Banach space  $E$ . A mapping  $V$  of  $C$  into itself is called *Lipschitzian* if there exists  $K \geq 0$  such that  $\|Vx - Vy\| \leq K\|x - y\|$  for every  $x, y \in C$ . We define the *Lipschitz norm* of  $V$  by

$$\|V\| := \sup \left\{ \frac{\|Vx - Vy\|}{\|x - y\|}; x, y \in C, x \neq y \right\}.$$

If  $\|V\| \leq 1$ , then  $V$  is called *nonexpansive*. We denote by  $\text{Lip}(C)$ ,  $\text{Cont}(C)$ ,  $\text{Aff}(C)$  and  $L(E)$ , the semitopological semigroup of all Lipschitzian self-mappings of  $C$ , the semitopological semigroup of all nonexpansive self-mappings of  $C$ , the semitopological semigroup of all affine self-mappings of  $C$  and the semitopological semigroup of all bounded linear mappings of  $E$ , under composition and pointwise convergence topology, respectively. Let  $T : S \rightarrow \text{Lip}(C)$ ,  $\text{Cont}(C)$  or  $L(E)$  be a *representation* (resp. *anti-representation*), i.e.,  $T(s+t) = T(s)T(t)$  (resp.  $T(st) = T(t)T(s)$ ) for every  $s, t \in S$ , and  $T(\cdot)x$  is continuous for every  $x \in C$  or  $E$ . We denote by  $\text{Fix}(T)$  the *set of common fixed points* of  $T(s)$ ,  $s \in S$ , and by  $O(x) = \{T(s)x; s \in S\}$  the orbit from  $x \in C$ . We also denote by  $B(E) = \{x \in E; \|x\| \leq 1\}$  the unit ball of  $E$ . For two Banach spaces  $E$  and  $F$ ,  $L(E, F)$  is the set of all bounded linear mappings from  $E$  to  $F$ .

First we give a lemma.

**Lemma 2.2.** *Let  $S$  be a semitopological semigroup such that there exists a left invariant mean  $\mu$  on  $C_b(S)$ . Let  $C$  be a closed convex subset of a Banach space  $E$  and let  $D$  be a bounded subset of  $C$ . Let  $T : S \rightarrow \text{Cont}(C)$  be a representation such that  $\{T(s)x_0; s \in S\}$  is bounded for some  $x_0 \in C$ . Then there exists a bounded closed convex subset  $M$  of  $C$ , which is  $T(s)$ -invariant for every  $s \in S$  and includes  $K := \{T(s)x; s \in S, x \in D\}$ .*

*Proof.* Put  $a = \sup_{s \in S} \|T(s)x_0\|, b = \sup_{x \in D} \|x\|$ , and define the set

$$M := \{z \in C; \mu_s \|T(s)x_0 - z\| \leq a + b\}.$$

Then we can easily see that  $M$  is as in the assertion. □

Let  $S$  be a topological space, let  $E$  be a Banach space  $E$  and let  $C_b(S, E)$  be the Banach space of all bounded continuous mappings from  $S$  to  $E$  with supremum norm, and let  $C_C(S, E)$  be the set of all elements  $f \in C_b(S, E)$  such that  $f(S) := \{f(s); s \in S\}$  is relatively weakly compact. Then  $C_C(S, E)$  is a linear subspace of  $C_b(S, E)$ . For any mean  $\mu$  on  $C_b(S)$ , we define a “vector valued mean”  $\tau(\mu)$ , homomorphically; i.e.,  $\tau(\mu)$  is an element of  $L(C_C(S, E), E)$  such that  $\tau(\mu)x = x$  for all constant functions  $x \in E$ , and  $\|\tau(\mu)\| = 1$ . More generally, we give definitions as follows; see [18].

**Definition 2.3.** Let  $S$  be a topological space and let  $E$  be a Banach space. For  $\mu \in C_b(S)^*$  and  $f \in C_C(S, E)$ , let  $x_{\mu, f}^{**} : x^* \mapsto \mu_s \langle f(s), x^* \rangle$  be an element of  $E^{**}$ . Then  $x_{\mu, f}^{**} \in E$ . We define an element  $\tau = \tau^E \in L(C_b(S)^*, L(C_C(S, E), E))$  by  $\tau(\mu)f = x_{\mu, f}^{**}$ .

*Remark 2.4.* The following holds; see [18]. (i) If  $\mu$  is a mean on  $C_b(S)$ , then  $\tau(\mu)f \in \text{cl co}\{f(s); s \in S\}$ ; (ii)  $\tau$  is injective and  $\|\tau\| \leq 1$ ; (iii)  $\tau(\mu)x = \mu(1)x$  for all  $x \in E$ , and if  $\mu$  is a mean on  $C_b(S)$ , then  $\|\tau(\mu)\| = 1$ ; (iv)  $\tau$  maps the point evaluation  $\delta(s)$  to the point evaluation  $\varepsilon(s)$ ,  $s \in S$ , where  $\varepsilon(s)f = f(s)$  for  $f \in C_C(S, E)$ ; (v)  $\tau(r(s)^*\mu) = r(s)^*\tau(\mu)$ .

Let  $T : S \rightarrow \text{Lip}(C)$  be a representation such that  $T(\cdot)x \in C_C(S, E)$  for some  $x \in C$ . Then we shall denote  $\tau(\mu)(T(\cdot)x)$  by  $T(\mu)x$ , which is an element of  $E$ .

*Remark 2.5.* From (ii) and (iii), for any  $\{a_i\} \subset \mathbb{R}, \{s_i\} \subset S, f \in C_C(S, E)$ , we get  $\tau(\sum_{i=1}^n a_i \delta(s_i))f = \sum_{i=1}^n a_i \tau(\delta(s_i))f = \sum_{i=1}^n a_i \varepsilon(s_i)f = \sum_{i=1}^n a_i f(s_i)$ .

We give two lemmas.

**Lemma 2.6.** *Let  $f \in C_C(S, E)$  and let  $\mu$  be a left (resp. right) invariant mean on  $C_b(S)$ . Then  $\tau(\mu)f \in \bigcap_{s \in S} \text{cl co}\{f(st); t \in S\}$  (resp.  $\tau(\mu)f \in \bigcap_{s \in S} \text{cl co}\{f(ts); t \in S\}$ ). If  $\mu$  is an invariant mean on  $C_b(S)$ ,  $\tau(\mu)f \in \bigcap_{s, q \in S} \text{cl co}\{f(stq); t \in S\}$ .*

*Proof.* For any  $s \in S$ , considering  $l(s)f$  for  $f$  in Remark 2.4(i), we have  $\tau(\mu)f = \tau(\mu)(l(s)f) \in \text{cl co}\{(l(s)f)(t); t \in S\} = \text{cl co}\{f(st); t \in S\}$ . So, the first assertion follows. The other assertions are proved similarly. □

**Lemma 2.7.** *Let  $\mu$  be a right invariant mean on  $C_b(S)$  and let  $\lambda$  be a finite mean. Putting  $f(t) = T(l(t)^*\lambda)x$  for every  $t \in S$  and  $x \in E$ , we have  $T(\mu)x = \tau(\mu)f$ .*

*Proof.* Let  $\lambda = \sum_{i=1}^n a_i \delta(s_i)$ . Then by Remark 2.5,  $T(l(t)^* \lambda)x = \sum_{i=1}^n a_i T(ts_i)x$ , hence

$$\begin{aligned} \tau(\mu)f &= \tau(\mu)(T(l(\cdot)^* \lambda)x) = \tau(\mu) \left( \sum_{i=1}^n a_i T(\cdot s_i)x \right) \\ &= \sum_{i=1}^n a_i \tau(\mu)(T(\cdot s_i)x) = \sum_{i=1}^n a_i \tau(\mu)(T(\cdot)x) \\ &= \tau(\mu)(T(\cdot)x) = T(\mu)x. \end{aligned}$$

Therefore  $T(\mu)x = \tau(\mu)f$ .  $\square$

### 3. EXISTENCE OF ERGODIC RETRACTION

In this section, we prove that there exists the ergodic retraction for a noncommutative semigroup which is right Eberlein-weakly almost periodic. First, we give the definition of vector valued weakly almost periodic functions; see Ruess and Summers [24].

**Definition 3.1.** Let  $S$  be a semitopological semigroup and let  $E$  be a Banach space. We say a function  $f \in C_b(S, E)$  is *right (resp. left) Eberlein-weakly almost periodic* if  $\{r(s)f; s \in S\}$  (resp.  $\{l(s)f; s \in S\}$ ) is a relatively weakly compact subset of  $C_b(S, E)$ . If  $f \in C_b(S, E)$  is right and left Eberlein-weakly almost periodic, then we say  $f$  is *Eberlein-weakly almost periodic*. We denote by  $WR(S, E)$  (resp.  $WL(S, E), W(S, E)$ ) the space of all of right Eberlein- (resp. left Eberlein-, Eberlein-) weakly almost periodic functions. Let  $RUC(S, E)$  be the set of all *right bounded uniformly continuous* functions from  $S$  to  $E$ , i.e., the set of all  $f \in C_b(S, E)$  such that the map  $s \mapsto r(s)f$  from  $S$  to  $C_b(S, E)$  is continuous. Let  $C$  be a closed subset of  $E$ . We say a representation  $T : S \rightarrow \text{Cont}(C)$  is *right Eberlein- (resp. left Eberlein-, Eberlein-) weakly almost periodic* on a subset  $D$  of  $C$  if for any  $x \in D, T(\cdot)x \in WR(S, E)$  (resp.  $WL(S, E), W(S, E)$ ).

**Proposition 3.2.** (a)  $WR(S, E), WL(S, E)$  and  $W(S, E)$  are closed translation invariant linear subspaces of  $C_b(S, E)$ , and (b)  $RUC(S, E)$  is a closed left translation invariant linear subspace of  $C_b(S, E)$ .

*Proof.* (a) (i) Closeness of  $WR(S, E), WL(S, E)$  and  $W(S, E)$ : this is proved just as in the proof of Berglund, Junghenn and Milnes [3, Theorem 4.2.5].

(ii) Translation invariance of  $WR(S, E), WL(S, E)$  and  $W(S, E)$ : For  $f \in WR(S, E)$  and  $t \in S$ , since  $\{r(s)r(t)f; s \in S\} \subset \{r(s)f; s \in S\}$ , we see that  $r(t)f \in WR(S, E)$ . So,  $WR(S, E)$  is right translation invariant.

Let  $f \in WR(S, E)$  and  $t \in S$ . Since  $\text{cl}\{r(s)l(t)f; s \in S\} = \text{cl}\{l(t)r(s)f; s \in S\} = l(t)\text{cl}\{r(s)f; s \in S\}$ , and  $l(t)$  is weak to weak continuous,  $\text{cl}\{r(s)l(t)f; s \in S\}$  is weakly compact. So, we have  $l(t)f \in WR(S, E)$ , and  $WR(S, E)$  is left translation invariant. Similarly, we see  $WL(S, E)$  and  $W(S, E)$  are translation invariant.

(b) Closeness of  $RUC(S, E)$ : Let  $f \in \text{cl} RUC(S, E)$ , and let  $\{f_\alpha\} \subset RUC(S, E)$  be such that  $f_\alpha \rightarrow f$ . We show  $f \in RUC(S, E)$ . For this it is sufficient to show the following: let  $\{s_\beta\}$  be a net of  $S$  such that  $s_\beta \rightarrow s$ . Then  $r(s_\beta)f \rightarrow r(s)f$ . For any  $\varepsilon > 0$ , there exists  $\alpha = \alpha(\varepsilon)$  such that

$$\|f - f_\alpha\| \leq \varepsilon.$$

Then there exists  $\beta_0 = b_0(\alpha)$  such that for any  $\beta \geq \beta_0$ ,

$$\|r(s_\beta)f_\alpha - r(s)f_\alpha\| \leq \varepsilon.$$

Then we have

$$\begin{aligned} \|r(s_\beta)f - r(s)f\| &\leq \|r(s_\beta)f - r(s_\beta)f_\alpha\| + \|r(s_\beta)f_\alpha - r(s)f_\alpha\| + \|r(s)f_\alpha - r(s)f\| \\ &\leq \varepsilon + 2\|f - f_\alpha\| \leq 3\varepsilon. \end{aligned}$$

This implies  $r(s_\beta)f \rightarrow r(s)f$ .

(ii) Left translation invariance of  $RUC(S, E)$ : Let  $f \in RUC(S, E)$  and  $q \in S$ . Then we have

$$\begin{aligned} \|r(s)l(q)f - r(t)l(q)f\| &= \|l(q)r(s)f - l(q)r(t)f\| \\ &\leq \|r(s)f - r(t)f\|, \end{aligned}$$

so that  $l(q)f \in RUC(S, E)$ . □

*Remark 3.3.* Let  $C$  be a closed convex subset of a uniformly convex Banach space  $E$  and let  $T : S \rightarrow \text{Cont}(C)$  be a representation such that  $\text{Fix}(T) \neq \emptyset$ . Then we know that when  $S = (\mathbb{Z}^+)^d$  or  $(\mathbb{R}^+)^d$ ,  $d \in \mathbb{N}$ , if the representation  $T$  is strongly asymptotically isometric on a subset  $D$  of  $C$ , then  $T$  is Eberlein-weakly almost periodic on  $D$ . Here  $T$  is called strongly asymptotically isometric on  $D$  if the following hold: Let  $\|\cdot\|$  be the restriction to  $S$  of any norm on  $(\mathbb{R})^d$ . For any  $x, y \in D$ ,  $\lim_{\|s\| \rightarrow \infty} \|T(s+h)x - T(s)y\|$  exists uniformly over  $h \in S$  (see [13, 24]).

As in the proof of Ruess and Summers [27, Proposition 2.1], we have the following lemma.

**Lemma 3.4.**  $f \in WR(S, E)$  (resp.  $WL(S, E)$ ) if and only if  $\{\langle f(t), x^* \rangle; t \in S, x^* \in B(E^*)\}$  (resp.  $\{\langle f(\cdot t), x^* \rangle; t \in S, x^* \in B(E^*)\}$ ) is relatively weakly compact in  $C_b(S)$ .

*Proof.* By Milnes [22],  $f \in WR(S, E)$  if and only if for any sequences  $\{(t_m, x_m^*)\}$  in  $S \times B(E^*)$  and  $\{s_n\}$  in  $S$ ,

$$\lim_n \lim_m \langle f(t_m s_n), x_m^* \rangle = \lim_m \lim_n \langle f(t_m s_n), x_m^* \rangle,$$

whenever both iterated limits exists. From this, the assertion follows. The second assertion is proved similarly. □

Let  $C$  be a closed convex subset of a Banach space  $E$ , and let  $T$  be a nonexpansive mapping on  $C$  and  $\varepsilon > 0$ . Then we define

$$F_\varepsilon(T) = \{x \in C; \|x - Tx\| \leq \varepsilon\}.$$

The following is Bruck's result [5].

**Lemma 3.5.** Let  $C$  be a bounded closed convex subset of a uniformly convex Banach space  $E$ . Then for any  $\varepsilon > 0$  and any nonexpansive mapping  $T$  on  $C$ , there exists  $\delta > 0$  such that

$$\text{cl co } F_\delta(T) \subset F_\varepsilon(T).$$

The following lemma is a noncommutative version of a result of Hirano, Kido and Takahashi [11]; see [16].

**Lemma 3.6.** *Let  $C$  be a closed subset of a Banach space  $E$ , let  $S$  be a right reversible semitopological semigroup, and let  $T : S \rightarrow \text{Cont}(C)$  be a representation. Assume  $\text{Fix}(T) \neq \emptyset$ . Then for any  $x \in C$  and any finite mean  $\lambda \in \text{co}\{\delta(s); s \in S\}$ ,*

$$\lim_t \|T(s)T(l(t)^*\lambda)x - T(l(st)^*\lambda)x\| = 0 \quad \text{uniformly over } s \in S.$$

Let  $\langle F, G \rangle$  be a duality of vector spaces. We do not assume it is separated. Let  $\sigma(F, G)$  be the weak topology on  $F$  generated by  $G$ , and let  $\tau(F, G)$  be the Mackey topology on  $F$ , i.e., the topology of uniform convergence on all  $\sigma(G, F)$ -compact convex circled subsets of  $G$ .

**Lemma 3.7.** *Let  $X$  be a translation invariant closed subspace of  $l^\infty(S)$  containing constants, and assume that  $X$  has a left invariant mean  $\mu$ . Then there exist nets  $\{\phi_\alpha\}$  and  $\{\varphi_\alpha\}$  of finite means such that for any  $s \in S$ ,  $l(s)^*\phi_\alpha - \phi_\alpha$  and  $l(s)^*\varphi_\alpha - \mu$  converge to 0 in the  $\tau(l^1(S), X)$  topology and  $\tau(X^*, X)$  topology, respectively.*

*Proof.* The first assertion is proved in [16]. The second assertion is proved similarly. Indeed, by the Mackey-Arens theorem, the continuous dual of  $(X^*, \tau(X^*, X))$  is also  $X$ . Let  $E$  be the product  $(X^*, \tau(X^*, X))^S$ . Let  $\Phi$  be the set of finite means and define a map  $T : \Phi \rightarrow E$  given by

$$T(\phi)(s) = l(s)^*\phi - \mu, \quad \phi \in \Phi, s \in S.$$

Then  $T(\Phi)$  is a convex subset of  $E$ . There exists a net  $\{\lambda_\alpha\} \subset \Phi$  such that  $\langle f, l(s)^*\lambda_\alpha - \mu \rangle \rightarrow 0$  for each  $s \in S$  and  $f \in X$ . Hence  $T(\lambda_\alpha) \rightarrow 0$  in the weak topology of  $E$ , i.e.,  $0 \in$  the weak closure of  $T(\Phi)$ . Hence by Mazur's theorem,  $0 \in$  the strong closure of  $T(\Phi)$ , that is, there exists a net  $\{\varphi_\alpha\}$  of finite means such that  $l(s)^*\varphi_\alpha - \mu$  converges to 0 in the  $\tau(X^*, X)$  topology for each  $s \in S$ .  $\square$

Now we can prove our main theorem on the existence of ergodic retraction for a noncommutative semigroup which is right Eberlein-weakly almost periodic in a uniformly convex Banach space.

**Theorem 3.8.** *Let  $S$  be a normal semitopological semigroup such that there exists a left invariant mean on  $C_b(S)$ . Let  $C$  be a closed convex subset of a uniformly convex Banach space  $E$ , and let  $T : S \rightarrow \text{Cont}(C)$  be a right Eberlein-weakly almost periodic representation on a subset  $D$  of  $C$ . Assume  $\text{Fix}(T) \neq \emptyset$ . Then for each right invariant mean  $\mu$  on  $C_b(S)$ , if it exists,  $T(\mu)$  is a nonexpansive retraction from  $D$  onto  $\text{Fix}(T)$  such that  $T(\mu)T(s) = T(s)T(\mu) = T(\mu)$  for each  $s \in S$  and  $T(\mu)x \in \text{clco } O(x)$  for each  $x \in C$ .*

*Proof.* By Lemma 2.2, we may assume  $C$  is bounded. Fix any  $s \in S$  and  $\varepsilon > 0$ . By Lemma 3.5, there exists  $\delta > 0$  such that

$$(3.1) \quad \text{clco } F_\delta(T(s)) \subset F_\varepsilon(T(s)).$$

Let  $\lambda$  be a left invariant mean. Since  $T(\cdot)x \in WR(S, E)$ , by Lemma 3.4 and Krein's theorem we have that the convex, circled hull of  $\{\langle T(t)x, x^* \rangle; t \in S, x^* \in B(E^*)\}$  is relatively weakly compact. So, by Lemma 3.7 there exists a finite mean  $\lambda_0$  such that the following holds:

$$\sup\{|\langle (\lambda_0 - \lambda)(T(t)x, x^*) \rangle; t \in S, x^* \in B(E^*)\} \leq \delta/4.$$

Then, for any  $t \in S$ ,

$$\begin{aligned} & \|T(l(st)^*\lambda_0)x - T(l(t)^*\lambda_0)x\| \\ &= \|\tau(l(st)^*\lambda_0 - l(t)^*\lambda_0)(T(\cdot)x)\| \\ &\leq \|\tau(l(st)^*\lambda_0 - \lambda)(T(\cdot)x)\| + \|\tau(l(t)^*\lambda_0 - \lambda)(T(\cdot)x)\| \\ &= \sup\{(\lambda_0 - \lambda)\langle T(st \cdot)x, x^* \rangle; x^* \in B(E^*)\} \\ &\quad + \sup\{(\lambda_0 - \lambda)\langle T(t \cdot)x, x^* \rangle; x^* \in B(E^*)\} \\ &\leq \delta/4 + \delta/4 = \delta/2. \end{aligned}$$

By Remark 2.1 and Lemma 3.6, there exists  $t_0 \in S$  such that

$$\|T(s)T(l(t)^*\lambda_0)x - T(l(st)^*\lambda_0)x\| \leq \delta/2 \quad \text{for all } t \geq t_0.$$

Put  $f(t) = T(l(t)^*\lambda_0)x$ . Then for any  $t \geq t_0$ ,

$$\begin{aligned} & \|T(s)f(t) - f(t)\| \\ &= \|T(s)T(l(t)^*\lambda_0)x - T(l(t)^*\lambda_0)x\| \\ &\leq \|T(s)T(l(t)^*\lambda_0)x - T(l(st)^*\lambda_0)x\| + \|T(l(st)^*\lambda_0)x - T(l(t)^*\lambda_0)x\| \\ &\leq \delta/2 + \delta/2 = \delta. \end{aligned}$$

So, we have

$$(3.2) \quad f(t) \in F_\delta(T(s)) \quad \text{for all } t \geq t_0.$$

By Lemma 2.7, Lemma 2.6, (3.2) and (3.1), we have  $T(\mu)x = \tau(\mu)f \in \text{cl co}\{f(t); t \geq t_0\} \subset \text{cl co } F_\delta(T(s)) \subset F_\varepsilon(T(s))$ . Since  $\varepsilon > 0$  and  $s \in S$  are arbitrary, we obtain  $T(\mu)x \in \text{Fix}(T)$ . So, for any  $s \in S$ ,  $T(s)T(\mu) = T(\mu)$ . Since  $\mu$  is right invariant, we have, using Remark 2.4,

$$\begin{aligned} T(\mu)T(s)x &= \tau(\mu)(T(\cdot)T(s)x) = \tau(\mu)(T(\cdot s)x) \\ &= \tau(r(s)^*\mu)(T(\cdot)x) = \tau(\mu)(T(\cdot)x) \\ &= T(\mu)x, \end{aligned}$$

and hence  $T(\mu)T(s) = T(\mu)$  for every  $s \in S$ . Since  $T(\mu)^2x = \tau(\mu)(T(\cdot)T(\mu)x) = \tau(\mu)T(\mu)x = T(\mu)x$ , we obtain  $T(\mu)^2 = T(\mu)$ , which means  $T(\mu)$  is a retraction. From Lemma 2.6, we have  $T(\mu)x \in \text{cl co } O(x)$ .

$$\begin{aligned} \|T(\mu)x - T(\mu)y\| &= \|\tau(\mu)(T(\cdot)x - T(\cdot)y)\| \leq \|T(\cdot)x - T(\cdot)y\| \\ &\leq \|x - y\|, \end{aligned}$$

so that  $T(\mu)$  is nonexpansive. This completes the proof.  $\square$

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DEPARTMENT OF INFORMATION SCIENCES, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA,  
MEGURO-KU, TOKYO 152, JAPAN  
E-mail address: kada@math.sci.ynu.ac.jp