

SINGULAR MEASURES WITH ABSOLUTELY CONTINUOUS CONVOLUTION SQUARES ON LOCALLY COMPACT GROUPS

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ABSTRACT. Saeki's result states that on any locally compact nondiscrete group there exist continuous singular measures, with respect to the left Haar measure, μ with $\mu * \mu$ in L^p for all p , $1 \leq p < \infty$. This paper gives a new and short proof of this using Rademacher-Riesz products.

1. INTRODUCTION

In 1938, Wiener and Wintner [9], [3], p.209, proved that there exists a singular probability measure μ on $[-\pi, \pi]$ such that $\mu * \mu$ is absolutely continuous. In 1966, Hewitt and Zuckerman [4] showed that every nondiscrete locally compact abelian group supports such a measure (absolutely continuous with respect to Haar measure) and the Lebesgue-Radon-Nikodým derivative of $\mu * \mu$ is in L^p for all p , $1 \leq p < \infty$. Hewitt has asked whether such measures exist on every (locally) compact group [7], p.217. Ragozin [7] proved that every connected compact simple Lie group has a measure with absolutely continuous square. In 1977 Saeki [8], p. 403, proved Hewitt's question. We also note two papers on this subject. One by Karanikas and Koumandos [6] which makes use of the Rademacher-Riesz products, and another by Dooley and Gupta [2] which makes use of the theory of compact Lie groups. In this paper we give a new and short proof of Saeki's result. Our ideas have been adapted from [6]; see also [1] and [5]. Section 2 contains a brief summary, for the reader's convenience, of some results and techniques from [6] that we shall need to use. Our proof of the main theorem appears in section 3.

2. TERMINOLOGY

Let G be a locally compact, nondiscrete metrizable group and E a compact subset of G with left Haar measure equal to one. We define a system of Rademacher functions $(r_n)_{n=1}^{\infty}$ as follows: We divide E into two disjoint subsets, E_{11}, E_{12} of equal measure and define $r_1(x) = 1$ for $x \in E_{11}$ and $r_1(x) = -1$ for $x \in E_{12}$. Similarly, we divide each of these two subsets E_{11}, E_{12} to define r_2 , etc. We define the Walsh functions $(w_n)_{n=0}^{\infty}$ by the relations:

$$w_n = r_{j_1} r_{j_2} \dots r_{j_p}, \quad n = 2^{j_1-1} + 2^{j_2-1} + \dots + 2^{j_p-1}, \quad j_1 < j_2 < \dots < j_p.$$

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The existence is given by the following Lemma:

Lemma 1. *On any nondiscrete metrizable group we can find a set E and a partition into sets of equal measure, $E_{n,k}$, $1 \leq k \leq 2^n$, such that:*

- (i) $\max_{1 \leq k \leq 2^n} \text{diam } E_{n,k} \rightarrow 0$, as $n \rightarrow \infty$, ($\text{diam } E_{n,k}$ denotes the diameter of $E_{n,k}$ with respect to the metric of G).
- (ii) *The Walsh system is complete in $L^2(E)$.*
- (iii) *If $f, g \in L^1(G)$, then for any $p \in [1, \infty)$, $\lim_{n \rightarrow \infty} \int f w_n = 0$, $\lim_{n \rightarrow \infty} \|f*(g w_n)\|_p = 0$ and $\lim_{n \rightarrow \infty} \|(g w_n)*f\|_p = 0$.*

Proof. For (i) and (ii) see [6], Lemma 2.1, and for (iii) see the proof of Lemma 3.2 in [6]. □

We have also (see [1], [5], [6]) that

Lemma 2. *Let $f_n(x) = \prod_{j=1}^n (\chi_E(x) + a_j r_j(x))$, where χ_E is the characteristic function of E and $|a_j| \leq 1$. Then*

- (i) *The sequence f_n converges weak* to some μ in $M(G)$, the convolution algebra of bounded complex-valued Borel measures on G .*
- (ii) *The measure μ is singular if and only if $\sum_{n=1}^{\infty} a_n^2 = \infty$.*
- (iii) *The measure μ is continuous if and only if $\sum_{n=1}^{\infty} (1 - |a_n|) = \infty$.*

3. THE MAIN THEOREM

Theorem 1. *Let G be a nondiscrete locally compact metrizable group. Then there exists a continuous singular measure $\mu \in M(G)$ with absolutely continuous convolution square and the Lebesgue-Radon-Nikodým derivative of $\mu * \mu$ is in $L^p(G)$ for all p , $1 \leq p < \infty$.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a sequence such that $\sum_{n=1}^{\infty} a_n^2 = \infty$, $\sum_{n=1}^{\infty} a_{n+1}^4 \left(\sum_{k=1}^n a_k^2\right)^q < \infty$, for any $q \in \mathbb{N}$ and $|a_n| \leq 1$. We shall show that there exists a subsequence $(r_{m_k})_{k=1}^{\infty}$ of Rademacher functions such that if $f_n(x) = \prod_{k=1}^n (\chi_E(x) + a_k r_{m_k}(x))$, then the weak* limit μ of f_n is a continuous singular measure $\mu \in M(G)$, with absolutely continuous convolution square and the Lebesgue-Radon-Nikodým derivative of $\mu * \mu$ is in $L^p(G)$ for all p , $1 \leq p < \infty$. From Lemma 2 we have that μ is a continuous singular measure. It is easy to show that $f_n * f_n \rightarrow \mu * \mu$, $n \rightarrow \infty$, weak* in $M(G)$ and so from completeness of L^p it is enough to show that $f_n * f_n$ is a Cauchy sequence in L^p . It is also enough to prove this only for p an even integer. For simplicity of notation we write

$$t_n = f_n * f_n, \quad g_n = (f_n r_{m_{n+1}}) * f_n, \quad h_n = f_n * (f_n r_{m_{n+1}}), \\ s_n = (f_n r_{m_{n+1}}) * (f_n r_{m_{n+1}}), \quad n \in \mathbb{N}.$$

We shall estimate the norm $\|t_n - t_m\|_p$, $n > m$. Schwarz's inequality yields

$$\|t_n - t_m\|_p^p = \left\| \sum_{i=m}^{n-1} (t_{i+1} - t_i) \right\|_p^p = \left\| \sum_{i=m}^{n-1} [a_{i+1}(g_i + h_i) + a_{i+1}^2 s_i] \right\|_p^p$$

$$\begin{aligned}
&= \sum_{\nu=0}^p \binom{p}{\nu} \int \left[\sum_{i=m}^{n-1} a_{i+1}(g_i + h_i) \right]^{p-\nu} \left[\sum_{i=m}^{n-1} a_{i+1}^2 s_i \right]^\nu \\
&\leq \sum_{\nu=0}^p \binom{p}{\nu} \left(\int \left[\sum_{i=m}^{n-1} a_{i+1}(g_i + h_i) \right]^{2(p-\nu)} \right)^{\frac{1}{2}} \left(\int \left[\sum_{i=m}^{n-1} a_{i+1}^2 s_i \right]^{2\nu} \right)^{\frac{1}{2}}.
\end{aligned}$$

For $\nu \geq 1$ we have

$$\begin{aligned}
\int \left[\sum_{i=m}^{n-1} a_{i+1}^2 s_i \right]^{2\nu} &= \sum_{i=m}^{n-1} a_{i+1}^{4\nu} \int s_i^{2\nu} \\
&\quad + \sum_{k=m+1}^{n-1} \sum_{l=1}^{2\nu-1} \binom{2\nu}{l} \int \left[(a_{k+1}^2 s_k)^l \left(\sum_{i=m}^{k-1} a_{i+1}^2 s_i \right)^{2\nu-l} \right].
\end{aligned}$$

Using Lemma 1(iii) and Fubini's theorem we find $(r_{m_k})_{k>1}$ such that

$$(1) \quad \max \left\{ \left| \int \left[s_k \left(\sum_{i=m}^{k-1} a_{i+1}^2 s_i \right)^{2\nu-1} \right] \right|, \quad 1 \leq \nu \leq k, \quad 1 \leq m \leq k-1 \right\} \leq \frac{1}{k^2}$$

and, at the same time, using Lemma 1(iii),

$$(2) \quad \max \{ \|g_i\|_k, \|h_i\|_k, \quad k=1, \dots, 2i \} < \frac{1}{2i^3}, \quad i=1, 2, \dots$$

Thus, for $p < m < n$, using Minkowski's inequality we obtain

$$\begin{aligned}
\|t_n - t_m\|_p^p &\leq \frac{1}{m^{2p}} + \sum_{\nu=1}^p \binom{p}{\nu} \frac{1}{m^{2(p-\nu)}} \left[\sum_{i=m}^{n-1} a_{i+1}^{4\nu} \int s_i^{2\nu} + 2\nu \sum_{k=m+1}^{n-1} \frac{1}{k^2} \right. \\
&\quad \left. + \sum_{k=m+1}^{n-1} \sum_{l=2}^{2\nu-1} \binom{2\nu}{l} \int \left[(a_{k+1}^2 s_k)^l \left(\sum_{i=m}^{k-1} a_{i+1}^2 s_i \right)^{2\nu-l} \right] \right]^{\frac{1}{2}}.
\end{aligned}$$

Applying Schwarz's and Minkowski's inequalities successively, in the last term of the above, we observe that, in view of our hypothesis on the sequence $(a_n)_{n=1}^\infty$ and since

$$(3) \quad |s_n| \leq |t_n|, \quad n \in \mathbb{N},$$

it suffices to prove that $\sup_n \|t_n\|_p < c_p$, where c_p is a constant which depends only on p . We have

$$\begin{aligned}
\|t_{n+1}\|_p^p &= \|t_n + a_{n+1}(g_n + h_n) + a_{n+1}^2 s_n\|_p^p \\
&= \sum_{k_2=0}^p \sum_{k_1=0}^{k_2} \binom{p}{k_2} \binom{k_2}{k_1} C_{p,n}(k_1, k_2),
\end{aligned}$$

where

$$C_{p,n}(k_1, k_2) = a_{n+1}^{2p-k_2-k_1} \int t_n^{k_1} (g_n + h_n)^{k_2-k_1} s_n^{p-k_2}.$$

If $k_1 < k_2$, then by Lemma 1(iii) and (3) we find $r_{m_{n+1}}$ such that

$$(4) \quad \max \{ |C_{p,n}(k_1, k_2)|, \quad 0 \leq k_1 < k_2 \leq p, \quad 2 \leq p \leq n+1 \} \leq \frac{a_{n+1}^4}{3^{n+1}} \min_{2 \leq k \leq n+1} \|t_n\|_k^k.$$

If $k_1 = k_2$, then we have

$$C_{p,n}(k_1, k_1) = a_{n+1}^{2(p-k_1)} \int t_n^{k_1} s_n^{p-k_1}.$$

If $k_1 \leq p-2$, then using (3) we have

$$\begin{aligned} \sum_{k_1=0}^{p-2} \binom{p}{k_1} |C_{p,n}(k_1, k_1)| &\leq \sum_{k_1=0}^{p-2} \binom{p}{k_1} a_{n+1}^{2(p-k_1)} \left| \int t_n^{k_1} g_n^{p-k_1} \right| \\ &\leq a_{n+1}^4 (2^p - (p+1)) \|t_n\|_p^p. \end{aligned}$$

If $k_1 = p-1$, then Fubini's theorem and Lemma 1(iii) show that we can find $r_{m_{n+1}}$ such that

$$(5) \quad \max_{2 \leq p \leq n+1} |C_{p,n}(p-1, p-1)| \leq a_{n+1}^4 \min_{2 \leq k \leq n+1} \|t_n\|_k^k.$$

Finally, for $k_1 = p$ we have

$$|C_{p,n}(p, p)| \leq \|t_n\|_p^p.$$

Subsequently, we can find $r_{m_{n+1}}$ satisfying (4) and (5) such that

$$\|t_{n+1}\|_p^p \leq (1 + 2^p a_{n+1}^4) \|t_n\|_p^p, \quad 2 \leq p \leq n+1.$$

This implies that $\sup_n \|t_n\|_p < c_p$. It is clear that we can find a subsequence $(r_{m_k})_{k=1}^\infty$, using induction, satisfying the relations (1), (2), (4) and (5) simultaneously. This completes the proof. \square

Corollary 1. *The result of Theorem 1 holds for any nondiscrete locally compact group.*

Proof. As in [6]; see also [10].

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REFERENCES

- [1] A. Bisbas and C. Karanikas, On the continuity of measures, *Applicable Analysis*, Vol 48 (1993), 23-35. MR **95e**:43001
- [2] A. H. Dooley and S. K. Gupta, Continuous singular measures with absolutely continuous convolution squares, *Proceedings of the American Math. Soc.* Vol. 124, 10, (1996), 3115 - 3122. MR **96m**:43011
- [3] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Springer-Verlag, New York 1979. MR **81d**:43001
- [4] E. Hewitt and H. Zuckerman, Singular measures with absolutely continuous convolution squares, *Proc. Cambridge Phil. Soc.* 62 (1966), 339-420; 63, (1967), 367-368. MR **33**:1655
- [5] C. Karanikas, Examples of Riesz products-type measures on metrizable groups, *Boll. Un. Math. Ital.* 7, 4-A (1990), 331-341. MR **92d**:43002
- [6] C. Karanikas and S. Koumandos, Continuous singular measures with absolutely continuous convolution squares on locally compact groups, *Illinois J. of Math.* 35 (3) (1991), 490-495. MR **92b**:43004
- [7] D. L. Ragozin, Central measures on compact simple Lie groups, *J. of Functional Analysis* 10 (1972), 212-229. MR **49**:5715
- [8] S. Saeki, Singular measures having absolutely continuous convolution powers, *Illinois J. of Math.* 21 (1977), 395-412. MR **58**:6719

- [9] N. Wiener and A. Wintner, Fourier-Stieltjes transforms and singular infinite convolutions, Amer. J. Math., vol. 60 (1938), 513-522.
- [10] J. H. Williamson, A theorem on algebras of measures on topological groups, Proc. Edinburgh Math. Soc., vol. 11 (1958/59), 195 - 206. MR **22**:2851

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