

## ON THE MINIMALITY OF POWERS OF MINIMAL $\omega$ -BOUNDED ABELIAN GROUPS

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ABSTRACT. We describe the structure of totally disconnected minimal  $\omega$ -bounded abelian groups by reducing the description to the case of those of them which are subgroups of powers of the  $p$ -adic integers  $\mathbb{Z}_p$ . In this case the description is obtained by means of a functorial correspondence, based on Pontryagin duality, between topological and linearly topologized groups introduced by Tonolo. As an application we answer the question (posed in *Pseudocompact and countably compact abelian groups: Cartesian products and minimality*, Trans. Amer. Math. Soc. **335** (1993), 775–790) when arbitrary powers of minimal  $\omega$ -bounded abelian groups are minimal. We prove that the positive answer to this question is equivalent to non-existence of measurable cardinals.

### INTRODUCTION

All group topologies are assumed to be Hausdorff. We denote by  $\hat{G}$  the two-sided (Raïkov) completion of a topological group  $G$ . A topological group  $G$  is *precompact* if  $\hat{G}$  is compact, *pseudocompact* if every continuous real-valued function on  $G$  is bounded, *countably compact* if every open countable cover of  $G$  admits a finite subcover,  *$\omega$ -bounded* if every countable subset of  $G$  has compact closure, *minimal* if every continuous group isomorphism  $G \rightarrow H$  is open ([Ste]). Every compact group is minimal and  $\omega$ -bounded,  $\omega$ -bounded groups are countably compact, countably compact groups are pseudocompact. According to a deep theorem of Prodanov and Stoyanov minimal abelian groups are precompact ([PS]). In this paper we are interested mainly in  $\omega$ -bounded minimal groups.

The first example of a  $\omega$ -bounded minimal non-compact group was given by Comfort and Grant [CG]. The group they proposed was non-abelian, zero-dimensional, so in particular totally disconnected. An example of a totally disconnected minimal,  $\omega$ -bounded non-compact abelian group was given in [DS3, Theorem 1.5]. A connected example was given in [DS2]; it was again non-abelian. In fact, item (b) of the following theorem from [D4] shows that the connected minimal countably compact abelian groups are “frequently” compact. (A cardinal  $\alpha$  is *measurable* if there exists an ultrafilter on  $\alpha$  which is closed under countable intersections.)

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**Theorem A.** *Let  $G$  be a countably compact abelian group with connected component  $c(G)$ .*

- (a) *If  $c(G)$  is compact, then  $G$  is minimal iff  $G/c(G)$  is minimal.*
- (b) *If  $G$  is minimal and  $|c(G)|$  is not measurable, then  $c(G)$  is compact.*

It should be stressed that the assumption on  $|c(G)|$  in item (b) is very weak. In fact, the assumption that there exist no measurable cardinals is known to be consistent with ZFC (actually,  $[L=V]$  implies measurable cardinals do not exist [J]), while it is not known if their existence is consistent with ZFC. Examples of non-compact connected minimal  $\omega$ -bounded abelian groups of every measurable cardinality are given in [D4].

It was proved in [DS3] that all finite powers of a countably compact minimal abelian group are minimal. The aim of this paper is to answer the following question ([DS3, Question 1.10]) in the case of  $\omega$ -bounded groups.

**Question B.** Is  $G^\omega$  minimal for a countably compact minimal abelian group  $G$ ?

In the next theorem we answer positively Question B for minimal  $\omega$ -bounded totally disconnected abelian groups (Corollary 2) and we show that the answer may be negative in the connected case if measurable cardinals exist (Corollary 3).

**Main Theorem.** *Let  $G$  be an  $\omega$ -bounded minimal abelian group. Then  $G^\omega$  is minimal iff  $c(G)$  is compact.*

By this theorem  $G^\omega$  is minimal for a connected  $\omega$ -bounded minimal abelian group  $G$  iff  $G$  is compact.

The background of Question B is the important fact that if  $G^\omega$  is minimal for a countably compact minimal abelian group, then *all* powers of  $G$  are minimal ([DS3, Corollary 1.9]). This permits one to define the *critical power of minimality*  $\kappa(G)$  of a countably compact minimal abelian group  $G$  as 1 if all powers of  $G$  are minimal (i.e.,  $G^\omega$  is minimal), otherwise  $\kappa(G)$  is the least cardinal  $\lambda$  such that  $G^\lambda$  is not minimal, i.e.,  $\kappa(G) = \omega$ . The critical power of minimality  $\kappa(\mathcal{P})$  of a class  $\mathcal{P}$  of countably compact minimal abelian groups is defined as  $\sup\{\kappa(G) : G \in \mathcal{P}\}$ . This invariant was introduced and studied for larger classes of minimal abelian groups where it takes more than just two values, 1 and  $\omega$ , as in the countably compact case (see [St], [D1], [DS1],[DS3] or [DPS, §6.3]). For example, it was shown in [DS3] that under the assumption of Martin's axiom (MA) the critical power of minimality of the class of pseudocompact minimal abelian groups is  $2^\omega$ . Recently ([D6]), a pseudocompact minimal abelian group  $G$  was found with  $\kappa(G) = \omega_1$ ; moreover, under the assumption of MA, for every cardinal  $\lambda$  between  $\omega$  and  $2^\omega$  a pseudocompact minimal abelian group was found with  $\kappa(G) = \lambda$ . This shows that  $\kappa$  provides a good measure to distinguish between pseudocompactness and  $\omega$ -boundedness within the class of minimal abelian groups (see Corollary 2 and take into account that all pseudocompact minimal abelian groups mentioned above are also totally disconnected).

In view of Theorem A our Main Theorem entails:

**Corollary 1.** *Let  $G$  be an  $\omega$ -bounded minimal abelian group such that  $|c(G)|$  is not measurable. Then  $\kappa(G) = 1$ , i.e. all powers of  $G$  are minimal.*

By the Main Theorem  $G^\omega$  is minimal for every totally disconnected  $\omega$ -bounded minimal abelian group  $G$ . Hence we have:

**Corollary 2.** *The critical power of minimality of the class of totally disconnected  $\omega$ -bounded minimal abelian groups is equal to 1.*

This corollary reduces the study of the critical power of minimality of the larger class of *all*  $\omega$ -bounded minimal abelian groups to that of the connected ones. The proof of the next corollary is given in §2.

**Corollary 3.** *Let  $\alpha$  denote the critical power of minimality of the class of all  $\omega$ -bounded minimal abelian groups. Then:*

- (a)  *$\alpha$  is equal to the critical power of minimality of the class of  $\omega$ -bounded minimal connected abelian groups,*
- (b) *under the assumption that there exist no measurable cardinals,  $\alpha$  is equal to 1,*
- (c) *under the assumption that there exist measurable cardinals,  $\alpha$  is equal to  $\omega$ .*

In other words, Corollary 3 says that the assertion “ $\alpha = 1$ ” is equivalent to non-existence of measurable cardinals.

The proof of the Main Theorem is obtained from Theorem A in two steps. The first one is a process of localization (described in §1) that permits one to describe locally the minimality of the powers (Fact 1.4) and reduces the study of totally disconnected  $\omega$ -bounded minimal abelian groups to the particular case of minimal  $\omega$ -bounded subgroups of powers of  $\mathbb{Z}_p$  (Theorem 1.3 and Lemma 1.5). The second step is the following lemma that can be considered as the core of the proof of the Main Theorem.

**Main Lemma.** *Let  $\alpha$  be a cardinal number and let  $p$  be a prime number. Then for every dense  $\omega$ -bounded minimal subgroup  $G$  of  $\mathbb{Z}_p^\alpha$  there exists  $k \in \mathbb{N}$  such that  $p^k \mathbb{Z}_p^\alpha \subseteq G$ .*

The proof of the Main Lemma exploits a functorial correspondence based on Pontryagin duality (see §2). This correspondence was developed by the second author to solve a longstanding problem of Warner on the existence of a finest equivalent linear module topology ([T1]) and to study minimality and total minimality in topological modules covered by compact submodules ([T2]). Through this correspondence (see Lemma 2.1) the Main Lemma is translated in an equivalent question on linear topologies on direct sums of the Prüfer group  $\mathbb{Z}(p^\infty)$ .

The proof of the next corollary of the Main Lemma concerning the structure of totally disconnected  $\omega$ -bounded minimal abelian groups is given in §2.

**Corollary 4.** *Let  $G$  be a totally disconnected  $\omega$ -bounded minimal abelian group. Then there exists a compact subgroup  $N$  of  $G$  such that the quotient group  $G/N$  is a direct product (provided with the product topology) of  $\omega$ -bounded torsion minimal groups  $G_p$ .*

We denote by  $\mathbb{N}$  and  $\mathbb{P}$  the sets of naturals and primes, respectively, by  $\mathbb{Z}$  the integers, by  $\mathbb{R}$  the reals, by  $\mathbb{Z}_p$  the  $p$ -adic integers ( $p \in \mathbb{P}$ ). For undefined symbols or notions see [E], [HR] or [J].

## 1. THE LOCALIZATION – QUASI- $p$ -TORSION ELEMENTS

The following minimality criterion of Banaschewski-Prodanov-Stephenson is very important. A subgroup  $H$  of a topological group  $G$  is *essential* if for every non-trivial closed normal subgroup  $N$  of  $G$  the intersection  $H \cap N$  is non-trivial.

**1.1 Theorem** ([B], [P], [Ste]). *Let  $G$  be a topological group and  $H$  be a dense subgroup of  $G$ . Then  $H$  is minimal iff  $G$  is minimal and  $H$  is essential in  $G$ .*

**1.2 Definition** ([St]). For  $p \in \mathbb{P}$  and a topological abelian group  $G$  an element  $x \in G$  is *quasi- $p$ -torsion* if  $\langle x \rangle$  is either a finite  $p$ -group or equipped with the induced topology is isomorphic to  $(\mathbb{Z}, \tau_p)$ , where  $\tau_p$  is the  $p$ -adic topology of  $\mathbb{Z}$ .

The set  $td_p(G)$  of all quasi- $p$ -torsion elements of  $G$  is a subgroup of  $G$  ([DPS, Chap.4]).

Let us recall that  $\omega$ -bounded groups are precompact and quasi- $p$ -torsion precompact abelian groups are totally disconnected ([D2]).

We show next that in analogy with the compact case, every  $\omega$ -bounded minimal totally disconnected abelian group  $G$  decomposes into a direct product of its quasi- $p$ -torsion subgroups  $td_p(G)$ . This settles the classification of the totally disconnected minimal  $\omega$ -bounded abelian groups by reduction to the case of quasi- $p$ -torsion minimal  $\omega$ -bounded abelian groups. The counterpart of the following theorem for the case of countably compact groups is given in [D4]. Here we give a proof in the  $\omega$ -bounded case which is different, entirely based on the corresponding property of the compact totally disconnected abelian groups.

**1.3 Theorem.** *Let  $G$  be a  $\omega$ -bounded totally disconnected abelian group. Then for each  $p \in \mathbb{P}$  the subgroup  $td_p(G)$  is closed and*

$$(1) \quad G = \prod_{p \in \mathbb{P}} td_p(G)$$

*with the product topology. Moreover,  $G$  is minimal iff each  $td_p(G)$  is minimal.*

*Proof.* It is known that for a compact totally disconnected abelian group  $K$  the subgroup  $td_p(K)$  is closed for each  $p \in \mathbb{P}$  and  $K = \prod_{p \in \mathbb{P}} td_p(K)$  with the product topology (see [DPS, Example 4.1.3 (a)]). Applying this fact to the compact group  $K = \hat{G}$  we get  $\hat{G} = \prod_{p \in \mathbb{P}} td_p(\hat{G})$  with the product topology. Moreover, for each  $p \in \mathbb{P}$  the subgroup  $td_p(G)$  of  $G$  is closed as  $td_p(G) = G \cap td_p(\hat{G})$ . To prove (1) let us first observe that  $G$ , being  $\omega$ -bounded, is covered by compact subgroups, and for every one of them, say  $H$ , we have  $H = \prod_{p \in \mathbb{P}} td_p(H) \subseteq \prod_{p \in \mathbb{P}} td_p(G)$  since  $td_p(H) \subseteq td_p(G)$ . This proves the inclusion  $\subseteq$  in (1). To prove the other inclusion observe that  $\bigoplus td_p(G) \subseteq G$ , hence an arbitrary element  $x = (x_p) \in \prod_{p \in \mathbb{P}} td_p(G)$  can be presented as an element of the closure (taken in  $\hat{G}$ ) of a countably generated subgroup of  $G$ . Since countably generated subgroups of  $G$  are contained in compact subgroups of  $G$ , this gives  $x \in G$ . So the inclusion  $\supseteq$  in (1) is proved as well.

The last assertion follows from Corollary 6.1.3 of [DPS] (see also [D1]).  $\square$

For a topological abelian group  $G$ , a prime number  $p$  and  $k \in \mathbb{N}$ ,  $G$  is *strongly  $p$ -dense (of degree  $k$ )* iff there exists  $k \in \mathbb{N}$  with  $p^k td_p(\hat{G}) \subseteq G$  (see [D1, Definition 1] or [DS1, p. 586]). For a quasi- $p$ -torsion abelian group  $G$  this condition simplifies to  $p^k \hat{G} \subseteq G$ . Non-compact strongly  $p$ -dense quasi- $p$ -torsion  $\omega$ -bounded minimal abelian groups exist in profusion [DS3, Corollary 1.6].

The following fact explains the relation of strong  $p$ -density to minimality of powers.

**1.4 Fact** ([St], [D1, Corollaire 7]). *Let  $G$  be a minimal abelian group. Then all powers of  $G$  are minimal iff  $G$  is strongly  $p$ -dense for every prime number  $p$ .*

It follows from the minimality criterion, that the completion  $\hat{G}$  of a torsion-free minimal abelian group  $G$  is torsion-free. Since torsion-free quasi- $p$ -torsion compact abelian groups are of the form  $\mathbb{Z}_p^\alpha$ , the torsion-free quasi- $p$ -torsion  $\omega$ -bounded minimal abelian groups are subgroups of powers of  $\mathbb{Z}_p$ . Hence our Main Lemma says that *a torsion-free quasi- $p$ -torsion  $\omega$ -bounded minimal abelian group must be strongly  $p$ -dense*. Actually, by means of the localization developed in Theorem 1.3 and Lemma 1.5 we prove more: *every totally disconnected  $\omega$ -bounded minimal abelian group is strongly  $p$ -dense for every prime  $p$* .

Now we reduce the study of quasi- $p$ -torsion  $\omega$ -bounded minimal abelian groups to the study of those which are torsion-free, i.e., subgroups of powers of  $\mathbb{Z}_p$ .

**1.5 Lemma.** *Fix a prime  $p$ . Let  $G$  be a minimal quasi- $p$ -torsion  $\omega$ -bounded abelian group. Then, with  $\alpha = w(G)$ , there exist a dense minimal  $\omega$ -bounded subgroup  $G_1$  of  $\mathbb{Z}_p^\alpha$  and a compact subgroup  $N$  of  $G_1$  such that:*

- (a) *the quotient group  $G_1/N$  is isomorphic to  $G$ ;*
- (b)  *$G$  and  $G_1$  have the same degree of strong  $p$ -density.*

*Proof.* Since  $\hat{G}$  is a compact  $\mathbb{Z}_p$ -module, there exists a continuous surjective homomorphism  $f : K = \mathbb{Z}_p^\alpha \rightarrow \hat{G}$  (see [DPS]). Then  $G_1 = f^{-1}(G)$  is a dense  $\omega$ -bounded subgroup of  $K$  ([DS2, Lemma 2.1]). Minimality of  $G_1$  follows from Theorem 1.1 and the fact that  $G_1$  is essential in  $K$ .

Set  $N = \ker f$ ; then  $G \cong G_1/N$  by the Sulley-Grant lemma ([DPS, Lemma 4.3.2]). This proves (a).

To prove (b) note that  $p^k \hat{G} \subseteq G$  is equivalent to  $p^k \mathbb{Z}_p^\alpha \subseteq G_1$ . □

## 2. PROOF OF THE MAIN RESULTS

We give briefly an outline of the functorial correspondence between precompact groups covered by their compact subgroups and linearly topologized groups developed in [T1] and [T2]. Let  $\mathbb{T}$  be the circle group  $\mathbb{R}/\mathbb{Z}$  endowed with the usual topology. The Pontryagin duality associates to each compact group  $K$  the abstract group  $K^* = \text{Chom}(K, \mathbb{T})$  of continuous homomorphisms of  $K$  in  $\mathbb{T}$ , to each abstract group  $X$  the group  $X^* = \text{Hom}(X, \mathbb{T})$  of homomorphisms of  $X$  in  $\mathbb{T}$ , endowed with the pointwise-convergence topology, and to each homomorphism, of compact or abstract groups, its transpose.

Let  $G$  be a precompact abelian group such that every element of  $G$  is contained in a compact subgroup of  $G$ . Then the family  $\mathcal{M} = \{M_\lambda : \lambda \in \Lambda\}$  of compact separable subgroups of  $G$  (i.e., the subgroups which have a dense countable subset) is directed with respect to inclusion and  $G = \bigcup_\lambda M_\lambda$ . Let  $X$  be the Pontryagin dual of the compact completion  $\hat{G}$  of  $G$ . For each  $M \in \mathcal{M}$ , consider the annihilator  $\text{Ann}(M) = \{\xi \in X : \xi(M) = 0\}$ . The family  $\{\text{Ann}(M) : M \in \mathcal{M}\}$  is a filter base of subgroups of  $X$ ; hence it defines a linear topology  $\tau$  on  $X$ , i.e. a group topology which has a base of neighbourhoods of zero consisting of subgroups.

The idea is to express properties of  $G$  in terms of properties of  $(X, \tau)$ .

**Lemma 2.1.** *Let  $G$ ,  $X$  and  $\tau$  be as above.*

- (i) *The group  $G$  is strongly  $p$ -dense with degree  $k$  if and only if for every subgroup  $L$  of  $X$  with  $X/L \leq \mathbb{Z}(p^\infty)$  the subgroup  $(L : p^k) = \{x \in X : p^k x \in L\}$  is  $\tau$ -closed.*
- (ii) *The group  $G$  is minimal if and only if  $(X, \tau)$  has no dense proper subgroups.*

- (iii) *The group  $G$  is  $\omega$ -bounded if and only if the space  $(X, \tau)$  is a  $P$ -space, i.e., countable intersections of  $\tau$ -open sets are  $\tau$ -open.*

*Proof.* (i) Consider the algebraic monomorphism  $X/L \rightarrow \mathbb{Z}(p^\infty)$ . The Pontryagin duality produces a surjective continuous homomorphism  $\psi : \mathbb{Z}_p \rightarrow \text{Ann } L$ . The element  $x = \psi(1)$  belongs to  $td_p(\hat{G})$  and  $(x) = \overline{\langle x \rangle} = \text{Ann } L$ . If  $G$  is strongly  $p$ -dense with degree  $k$ , then  $p^k x \in G$ , hence  $(p^k x) = p^k(x) \leq G$ . Now  $p^k(x)$  is a separable compact subgroup of  $G$ , hence

$$\text{Ann } p^k(x) = (\text{Ann}(x) : p^k) = (L : p^k)$$

is  $\tau$ -open, hence  $\tau$ -closed. Conversely, given  $x \in td_p(\hat{G})$ , let us consider the surjective continuous epimorphism  $\psi : \mathbb{Z}_p \rightarrow (x)$ , defined by  $\psi(1) = x$ . The Pontryagin duality produces the monomorphism

$$(x)^* = X / \text{Ann}(x) \rightarrow \mathbb{Z}(p^\infty).$$

By hypothesis  $(\text{Ann}(x) : p^k) = \text{Ann } p^k(x)$  is  $\tau$ -closed; since  $X/[\text{Ann } p^k(x)] \leq \mathbb{Z}(p^\infty)$  is Hausdorff and  $\mathbb{Z}(p^\infty)$  is finitely cogenerated,  $\text{Ann } p^k(x)$  is also  $\tau$ -open. Then  $p^k(x)$  is contained in a (separable compact) subgroup of  $G$ ; in particular  $p^k x$  belongs to  $G$ .

(ii) Also the family of annihilators of all compact subgroups of  $G$  defines a linear topology  $\sigma$  on  $X$ , clearly finer than  $\tau$ . By Proposition 2.2 (see the proof of point i)) of [T2], it is possible to verify that  $\sigma$  and  $\tau$  are *equivalent*, i.e. they determine the same closed subgroups of  $X$ . In [T2, Theorem 3.1] the second author has proved that  $G$  is essential in  $\hat{G}$  if and only if  $(X, \sigma)$  has no dense proper subgroups. Since  $\tau$  and  $\sigma$  are equivalent,  $G$  is essential in  $\hat{G}$  if and only if  $(X, \tau)$  has no dense proper subgroups. Then we conclude by Theorem 1.1.

(iii) Clearly, the topology  $\tau$  is a  $P$ -topology if and only if the family  $\mathcal{M}$  of compact separable subgroups of  $G$  is  $\omega$ -directed, i.e., for every countable family  $\{L_i : i \in \mathbb{N}\}$  of  $\mathcal{M}$  there exists  $L \in \mathcal{M}$  containing all  $L_i$ . Let us prove that this happens if and only if the group  $G$  is  $\omega$ -bounded. Let  $G$  be  $\omega$ -bounded and let  $\{L_i : i \in \mathbb{N}\}$  be a family in  $\mathcal{M}$ . For each  $i \in \mathbb{N}$ , denote by  $J_i$  a countable dense subset of  $L_i$ . Then the set  $\bigcup_{i \in \mathbb{N}} J_i$  is countable and  $L = \overline{\bigcup_{i \in \mathbb{N}} J_i} \in \mathcal{M}$  contains every  $L_i$ . Conversely, suppose the family  $\mathcal{M}$  is  $\omega$ -directed. Let  $J$  be a countable subset of  $G$ . Then  $\overline{\langle j \rangle} \in \mathcal{M}$  for each  $j \in J$ . By our hypothesis there exists a compact subgroup  $L \in \mathcal{M}$  that contains  $\overline{\langle j \rangle}$  for every  $j \in J$ . Hence the closure of  $J$  in  $G$  is compact.  $\square$

*Proof of the Main Lemma.* Let  $G$  be a dense  $\omega$ -bounded minimal subgroup of  $\mathbb{Z}_p^\alpha$ . Then by [DS3, Lemma 3.1] every element of  $G$  is contained in a compact subgroup of  $G$ . The Pontryagin dual of  $\mathbb{Z}_p^\alpha$  is  $\mathbb{Z}(p^\infty)^{(\alpha)}$ , so that by Lemma 2.1, after dualizing, our Main Lemma takes the following form. Consider  $\mathbb{Z}(p^\infty)^{(\alpha)}$  equipped with a linear Hausdorff topology  $\tau$  satisfying the following conditions:

- i) there exist no proper  $\tau$ -dense subgroups of  $(\mathbb{Z}(p^\infty)^{(\alpha)}, \tau)$ ;
- ii) countable intersections of  $\tau$ -open sets are  $\tau$ -open.

For every subgroup  $L$  of  $\mathbb{Z}(p^\infty)^{(\alpha)}$  with  $[\mathbb{Z}(p^\infty)^{(\alpha)}]/L \cong \mathbb{Z}(p^\infty)$  there exists  $n = n_L \in \mathbb{N}$  such that  $(L : p^n)$  is  $\tau$ -open. This is the number  $n$  determined by the  $\tau$ -closure  $\bar{L}$  of  $L$ : by (i) it cannot be the whole  $\mathbb{Z}(p^\infty)^{(\alpha)}$ , so that  $\bar{L}/L$  must be isomorphic to a proper subgroup, say  $\mathbb{Z}(p^n)$ , of Prüfer's group. We have to prove that *the number  $n$  can be chosen independently of  $L$ .*

Let us denote by  $H_n$  the unique subgroup of  $\mathbb{Z}(p^\infty)$  of order  $p^n$ . Suppose there exist subgroups  $L_1, \dots, L_n, \dots$  of  $\mathbb{Z}(p^\infty)^{(\alpha)}$  such that  $[\mathbb{Z}(p^\infty)^{(\alpha)}]/L_i \cong \mathbb{Z}(p^\infty)$  and such that the orders  $|\overline{L_i}/L_i| = p^{n_i}$  form a strictly increasing sequence of natural numbers. It is not restrictive to suppose  $|\overline{L_i}/L_i| = p^i$ . The subgroups  $L_i$  are kernels of morphisms

$$f_i : \mathbb{Z}(p^\infty)^{(\alpha)} \rightarrow \mathbb{Z}(p^\infty).$$

Since

$$\text{Hom}(\mathbb{Z}(p^\infty)^{(\alpha)}, \mathbb{Z}(p^\infty)) \cong [\text{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty))]^\alpha$$

and

$$\text{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong \mathbb{Z}_p,$$

for each  $f_i$  we have some  $\mu_i = (\mu_{i,x})_{x \in \alpha}$  belonging to  $\mathbb{Z}_p^\alpha$  such that

$$f_i((a_x)_{x \in \alpha}) = \sum_{x \in \alpha} \mu_{i,x} a_x.$$

Since the only Hausdorff linear topology on  $\mathbb{Z}(p^\infty)$  is the discrete one, every  $\overline{L_i}$  is open. Then, by the hypothesis ii) on  $\tau$ ,  $W = \bigcap_{i \in \mathbb{N}} \overline{L_i}$  is an open subgroup of  $\mathbb{Z}(p^\infty)^{(\alpha)}$ . Clearly, for each  $i \in \mathbb{N}$  and each open subgroup  $U \leq W$ , we have  $L_i + U = \overline{L_i}$ . Since  $\overline{L_i}/L_i \cong H_i$ , for each open subgroup  $U \leq W$

$$f_i(\overline{L_i}) = f_i(L_i + U) = f_i(L_i) + f_i(U) = f_i(U) = H_i$$

hold. For  $u \in \mathbb{Z}(p^\infty)^{(\alpha)}$  denote by  $|u|$  the order of  $u$ . For each  $i \in \mathbb{N}$ , there exists a natural number  $\nu(i) > 0$  such that for every open subgroup  $U$  there exists  $u \in U$  with  $|u| < \nu(i)$  such that  $f_i(u)$  generates  $H_i$ . Otherwise, for each  $n \in \mathbb{N}$  there would be an open subgroup  $U_n$  such that for no element  $u \in U_n$  with  $|u| < n$ , does  $f_i(u)$  generate  $H_i$ ; then,  $U_\infty = \bigcap_{n \in \mathbb{N}} U_n$  would be an open subgroup with  $f_i(U_\infty)$  properly contained in  $H_i$ : a contradiction.

Now for  $i \in \mathbb{N}$  define  $\psi(i) = \max\{\nu(i^2), i + 1\}$ . Since

$$(2) \quad \psi(1) < \psi^2(1) < \dots < \psi^j(1) < \dots,$$

the series

$$\mu_1 + \sum_{j \geq 1} p^{\psi^j(1)} \mu_{[\psi^j(1)]^2}$$

defines an element  $\overline{\mu} \in \mathbb{Z}_p^\alpha$ .

Let  $L$  be the kernel of the morphism  $f$  associated to  $\overline{\mu}$ , i.e.,

$$f : \mathbb{Z}(p^\infty)^{(\alpha)} \rightarrow \mathbb{Z}(p^\infty), \quad (a_x)_{x \in \alpha} \mapsto \sum_x \overline{\mu}_x a_x.$$

Let us prove that  $L$  is a proper dense subgroup of  $\mathbb{Z}(p^\infty)^{(\alpha)}$  contradicting the hypotheses.

For every  $U \leq W$  and every  $j \in \mathbb{N}$  we have

$$p^{\psi^j(1)} f_{[\psi^j(1)]^2}(U) = p^{\psi^j(1)} H_{[\psi^j(1)]^2} = H_{[\psi^j(1)]^2 - \psi^j(1)}.$$

Moreover, there exists  $u^{(j)} \in U$  with  $|u^{(j)}| < \nu([\psi^j(1)]^2)$ , such that  $p^{\psi^j(1)} f_{[\psi^j(1)]^2}(u^{(j)})$  generates  $H_{[\psi^j(1)]^2 - \psi^j(1)}$ . Now we prove that also  $f(u^{(j)})$  generates  $H_{[\psi^j(1)]^2 - \psi^j(1)}$ .

Indeed,

$$\begin{aligned}
 f(u^{(j)}) &= \sum_x \bar{\mu}_x u_x^{(j)} = \sum_x (\mu_{1,x} + \sum_{i \geq 1} p^{\psi^i(1)} \mu_{[\psi^i(1)]^2, x}) u_x^{(j)} \\
 &= \left[ \sum_x (\mu_{1,x} + \sum_{i=1}^{j-1} (p^{\psi^i(1)} \mu_{[\psi^i(1)]^2, x}) u_x^{(j)}) \right] + \sum_x (p^{\psi^j(1)} \mu_{[\psi^j(1)]^2, x}) u_x^{(j)} \\
 &\quad + \left[ \sum_x \left( \sum_{i \geq j+1} (p^{\psi^i(1)} \mu_{[\psi^i(1)]^2, x}) u_x^{(j)} \right) \right].
 \end{aligned}$$

The first summand is contained in  $H_{[\psi^{j-1}(1)]^2 - \psi^{j-1}(1)}$ , the second one generates  $H_{[\psi^j(1)]^2 - \psi^j(1)}$  and the third one vanishes since for each  $l \geq 1$  we have

$$\psi^{j+l}(1) = \psi^l(\psi^j(1)) > \psi(\psi^j(1)) \geq \nu([\psi^j(1)]^2) > |u^{(j)}|.$$

By (2) the sequence  $[\psi^j(1)]^2 - \psi^j(1)$ ,  $j \in \mathbb{N}$ , is strictly increasing (note that the function  $x \mapsto x^2 - x$  is strictly increasing for  $x > 1$ ). Hence the family  $\{f(u^{(j)}) : j \in \mathbb{N}\}$  generates  $\mathbb{Z}(p^\infty)$ .

We have proved in this way that for each open subgroup  $U$  contained in  $W$ ,  $f(U) = \mathbb{Z}(p^\infty)$ . Therefore  $L$  is dense – a contradiction.  $\square$

*Proof of Main Theorem.* Since  $C = c(G)$  is connected, its compact completion  $\hat{C}$  is connected, hence divisible ([HR]). Now the minimality of  $G^\omega$  yields that  $C^\omega$  is minimal, as a closed subgroup of the minimal group  $G^\omega$ . By [DS3, Corollary 1.9] this gives  $\kappa(C) = 1$ , hence, by Fact 1.4,  $C$  is strongly  $p$ -dense for every prime  $p$ . The divisibility of  $\hat{C}$  yields that  $td_p(\hat{C})$  is divisible for every prime  $p$  ([DPS, Proposition 4.1.2]). Hence  $td_p(\hat{C}) \subseteq C$  for every prime  $p$ . Then  $C$  is totally minimal ([DPS, Theorem 4.3.7]). Since  $C$  is also countably compact, it follows from [DS2] that  $C$  is compact.

Suppose now that  $C$  is compact and let  $\alpha$  be a cardinal. The quotient  $H = G/C$  is totally disconnected and  $\omega$ -bounded; moreover  $c(G^\alpha) = C^\alpha$  and  $H^\alpha \cong G^\alpha/c(G^\alpha)$ . By Theorem A,  $H = G/C$  is a minimal group. If we prove that  $H^\alpha$  is minimal, Theorem A will imply that the group  $G^\alpha$  is minimal as well. Hence it suffices to consider only the totally disconnected case.

From now on we assume  $G$  is an  $\omega$ -bounded totally disconnected minimal abelian group. According to Fact 1.4 the minimality of all powers  $G^\alpha$  for such group  $G$  follows from the strong  $p$ -density of  $G$  for every prime  $p$ . By Theorem 1.3 the subgroup  $td_p(G)$  of quasi- $p$ -torsion elements of  $G$  is still minimal and  $\omega$ -bounded, moreover,  $td_p(G)$  coincides with  $td_p(\hat{G})$ . Therefore, strong  $p$ -density for  $G$  is equivalent with strong  $p$ -density for the quasi- $p$ -torsion group  $td_p(G)$ . By Lemma 1.5 it suffices to consider quasi- $p$ -torsion groups that are subgroups of powers of  $\mathbb{Z}_p$ . In such a case the strong  $p$ -density of  $G$  was established in our Main Lemma.  $\square$

*Proof of Corollary 3.* (a) If  $G^\omega$  is not minimal for some  $\omega$ -bounded abelian group  $G$ , then  $c(G)$  is not compact by Main Theorem. Hence  $c(G)$  is a connected  $\omega$ -bounded minimal abelian group that is not compact, hence  $c(G)^\omega$  is not minimal again by Main Theorem.

(b) Follows from Corollary 2 and Theorem A.

(c) For the connected  $\omega$ -bounded minimal abelian group  $G$  of measurable weight constructed in [D4] the power  $G^\omega$  is not minimal according to Main Theorem since  $G$  is not compact.  $\square$

*Proof of Corollary 4.* Let  $f : K = \prod_{p \in \mathbb{P}} \mathbb{Z}_p^\sigma \rightarrow \hat{G}$  be a continuous surjective homomorphism (see [DPS]). Then  $H = f^{-1}(G)$  is a dense  $\omega$ -bounded minimal subgroup of  $K$ . Set  $K_p = \mathbb{Z}_p^\sigma$ ; then for each  $p$  the subgroup  $H_p = H \cap K_p$  of  $K_p$  is an essential  $\omega$ -bounded subgroup of  $K_p$ . So by the Main Theorem there exists a natural  $k_p$  such that  $p^{k_p} K_p \subseteq H_p$  for each  $p$ . Then for  $N_1 = \prod p^{k_p+1} K_p$  and  $N = f(N_1)$  obviously  $G/N \cong K/N_1 \cong \prod_p K_p/(N_1 \cap K_p)$  and each quotient  $K_p/(N_1 \cap K_p)$  is obviously a compact torsion abelian group. Moreover, every  $f(H_p)$  is minimal as a dense subgroup of  $L_p = K_p/p^{k_p+1} K_p \cong \mathbb{Z}(p^{k_p+1})^\sigma$  containing the socle  $f(p^{k_p} K_p)$  of  $L_p$ .  $\square$

It follows from this corollary and Theorem A that the connected component  $C$  of an  $\omega$ -bounded minimal abelian group  $G$  of non-measurable cardinality is compact and contained in a compact subgroup  $N$  of  $G$  such that the quotient  $G/N$  is a direct product (provided with the product topology) of  $\omega$ -bounded torsion minimal groups.

We finish by showing a possible way of extension of our Main Lemma to the countably compact case. The following result on approximation by large  $\omega$ -bounded subgroups was proved in [D4]:

**2.2 Theorem.** *Every minimal torsion-free connected countably compact abelian group  $G$  contains a minimal connected  $\omega$ -bounded subgroup  $G_\omega$  such that its closure  $\overline{G_\omega}$  is a  $G_\delta$ -subgroup of  $G$ .*

We do not know if one can replace “connected” by “totally disconnected” in this theorem. In case the answer is positive, then the Main Lemma implies that for every prime  $p$ , a torsion-free quasi- $p$ -torsion countably compact minimal abelian group  $G$  must be strongly  $p$ -dense. In fact,  $\hat{G} \cong \mathbb{Z}_p^\alpha$ . Let  $O_n$ ,  $n \in \mathbb{N}$ , be open subgroups of  $\hat{G}$  such that for  $V = \bigcap_n O_n$  we have  $\overline{G_\omega} = G \cap V$ . Since  $G$  is  $G_\delta$ -dense in  $\hat{G}$  (as  $G$  is pseudocompact, cf. [CR]), it follows that  $\overline{G_\omega}$  is dense in  $V$ , i.e.,  $V$  is the completion of  $G_\omega$ . By the Main Lemma there exists  $k \in \mathbb{N}$  such that  $p^k V \subseteq G_\omega \subseteq G$ . By the definition of the Tychonov topology of  $\mathbb{Z}_p^\alpha = \hat{G}$ , there exists a countable  $D \subseteq \alpha$ , such that  $U = \{0\} \times \mathbb{Z}_p^{\alpha \setminus D} \subseteq V$ . Since  $U \subseteq V$ , obviously  $p^k U \subseteq G$ . Now consider the closed subgroup  $G_1 := G \cap \mathbb{Z}_p^D$  of  $G$ . By the minimality of  $G$ , it is essential in  $\mathbb{Z}_p^D$ . Since  $G_1$  is compact (as a metrizable subgroup of a countably compact group), it is closed in  $\mathbb{Z}_p^D$  as well. Now the essentiality means that the quotient group  $\mathbb{Z}_p^D/G_1$  is torsion. Since it is compact abelian, it must have a finite exponent  $p^m$ . Thus  $p^m \mathbb{Z}_p^D \subseteq G$ . Then, with  $n = \max\{k, m\}$ , we get  $p^n \hat{G} \subseteq G$ .

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