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SURFACES WITH HARMONIC INVERSE MEAN CURVATURE IN SPACE FORMS

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ABSTRACT. We define surfaces with harmonic inverse mean curvature in space forms and generalize a theorem due to Lawson by which surfaces of constant mean curvature in one space form isometrically correspond to those in another. We also obtain an immersion formula, which gives a deformation family for these surfaces.

INTRODUCTION

In classical differential geometry, surfaces of constant mean curvature have been studied for a long time. As a generalization of surfaces of constant mean curvature in $\mathbf{R}^3$, Bobenko introduced surfaces with harmonic inverse mean curvature and obtained an immersion formula for these surfaces ([B2]).

In this paper we shall define surfaces with harmonic inverse mean curvature in space forms and generalize a theorem due to Lawson by which surfaces of constant mean curvature in one space form isometrically correspond to those in another ([L], [P], [F]). We shall also obtain an immersion formula, which gives a deformation family for these surfaces.

§1. DEFINITION OF SURFACES WITH HARMONIC INVERSE MEAN CURVATURE

For $c = 0, \pm 1$ we define a 1-dimensional Riemannian manifold I_c by

$$I_c = \begin{cases} (\mathbf{R}, g_c) & \text{if } c = 0, 1, \\ ((-1,1), g_c) & \text{if } c = -1, \end{cases} \quad \text{where} \quad g_c = \frac{1}{(1+ct^2)^2} dt^2.$$

Let M be a Riemann surface and z a local holomorphic coordinate on M. Then we have the following:

Proposition 1.1. *$\varphi : M \to I_c$ is harmonic if and only if φ satisfies the following partial differential equation:*

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} - \frac{2c\varphi}{1+c\varphi^2} \left| \frac{\partial \varphi}{\partial z} \right|^2 = 0. \tag{1.1}$$

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Equation (1.1) can be directly solved:

$$(1.2)\qquad \begin{cases} \varphi = \frac{\sqrt{-1}(h-\bar{h})}{|h|^2} & c = 0, \\ \varphi = \frac{\sqrt{-1}(h-\bar{h})}{|h|^2-c} \text{ or } \frac{|h|^2-c}{\sqrt{-1}(h-\bar{h})} & c = \pm 1, \end{cases}$$

where $h : M \to \mathbf{C}$ is a holomorphic function such that $|h|^2 - c \neq 0$ or $\mathrm{Im} h \neq 0$ on M. In the following we assume that $\mathrm{Im} h \neq 0$, $|h|^2 - c \neq 0$ for $c = 0, \pm 1$ on M. Let $\mathfrak{M}^3(c)$ be the simply connected 3-dimensional space form of curvature c.

Definition. A conformal immersion $F : M \to \mathfrak{M}^3(c)$ is called a surface with harmonic inverse mean curvature in $\mathfrak{M}^3(c)$ if the inverse of its mean curvature satisfies the equation (1.2).

Remark. If we restrict to the case $c = 0$, this definition is given by Bobenko [B2]. For examples of surfaces with harmonic inverse mean curvature in $\mathfrak{M}^3(0) = \mathbf{R}^3$, see [B2], [CK].

§2. Differential equations of surfaces

We consider $\mathbf{R}^4$ with the scalar product given by

$$\langle a, b\rangle_c = c a_0 b_0 + \sum_{k=1}^{3} a_k b_k,$$

where

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbf{R}^4.$$

Then $\mathfrak{M}^3(c)$ is embedded in $\mathbf{R}^4$ by

$$\begin{aligned} &\mathfrak{M}^3(0) = \left\{x \in \mathbf{R}^4; x_0 = 0\right\}, \\ &\mathfrak{M}^3(1) = \left\{x \in \mathbf{R}^4; \langle x, x\rangle_1 = 1\right\}, \\ &\mathfrak{M}^3(-1) = \text{ a connected component of } \left\{x \in \mathbf{R}^4; \langle x, x\rangle_{-1} = -1\right\}. \end{aligned}$$

Let $F : M \to \mathfrak{M}^3(c)$ be a conformal immersion, let $e^u dz d\bar{z}$ be the induced metric on M, and let N be the unit normal to M. Direct computation shows that the Gauss-Codazzi equations have the following form (see [B1]):

$$(\mathrm{GC})_c \qquad \begin{cases} u_{z\bar{z}} + \frac{1}{2}(H^2 + c)e^u - 2|Q|^2 e^{-u} = 0, \\ Q_{\bar{z}} = \frac{1}{2} H_z e^u, \end{cases}$$

where $Q = \langle F_{zz}, N\rangle_c$, $\langle F_{z\bar{z}}, N\rangle_c = \frac{1}{2} H e^u$.

In the following we assume that M is simply connected. We put

$$\mathcal{C}_H = \{F : M \to \mathfrak{M}^3(c); F \text{ is a conformal immersion with mean curvature } H\}/\mathrm{Iso}_0\left(\mathfrak{M}^3(c)\right),$$

where $\mathrm{Iso}_0\left(\mathfrak{M}^3(c)\right)$ is the identity component of the isometry group of $\mathfrak{M}^3(c)$. Then we have

$$\mathcal{C}_H \cong \{(u, Q); (u, Q) \text{ is a solution of } (\mathrm{GC})_c\}.$$

§3. An immersion formula: The case $H = \frac{|h|^2 - c}{\sqrt{-1}(h-\bar{h})}$

In this section we put $H = H_c = \frac{|h|^2 - c}{\sqrt{-1}(h-\bar{h})}$.

Theorem 3.1.

$$\mathcal{C}_{H_0} \cong \mathcal{C}_{H_1} \cong \mathcal{C}_{H_{-1}}.$$

Proof. Let (u, Q) be a solution of $(\mathrm{GC})_c$ for $c = \pm 1$. We put

$$e^{u'} = \left| \frac{h^2 - c}{h^2} \right|^2 e^u, \quad Q' = \frac{h^2 - c}{h^2} Q.$$

Then it is easy to see that (u', Q') is a solution of $(\mathrm{GC})_0$. Conversely, we can easily construct a solution of $(\mathrm{GC})_c$ for $c = \pm 1$ from one of $(\mathrm{GC})_0$. □

Remark. Theorem 3.1 is considered as a generalization of a theorem due to Lawson by which surfaces of constant mean curvature in one space form isometrically correspond to those in another ([L], [P], [F]).

Let (u, Q) be a solution of $(\mathrm{GC})_0$. For a 2×2 matrix-valued function Φ on M, we consider the following equations:

$$\text{(3.1)} \qquad \begin{cases} \Phi_z = U\Phi, \\ \Phi_{\bar{z}} = V\Phi, \end{cases}$$

where

$$U = \begin{pmatrix} \frac{1}{2}u_z & -Qe^{-\frac{1}{2}u} \\ \frac{1}{2}\lambda H_0 e^{\frac{1}{2}u} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -\frac{1}{2}\lambda^{-1} H_0 e^{\frac{1}{2}u} \\ \bar{Q}e^{-\frac{1}{2}u} & \frac{1}{2}u_{\bar{z}} \end{pmatrix},$$

$$\lambda = \left(1 + \frac{2}{\bar{h}}\mu\right) \Big/ \left(1 + \frac{2}{h}\mu\right), \quad \mu \in \mathbf{C}.$$

Note that the integrability conditions of (3.1),

$$U_{\bar{z}} - V_z + [U, V] = 0,$$

are equivalent to $(\mathrm{GC})_0$. We put

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and use the following identification according to c:

$$x = -\sqrt{-1} \sum_{k=1}^{3} x_k \sigma_k \longleftrightarrow x = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^4 \quad \text{if } c = 0,$$

$$x = x_0 E + \sqrt{-1} \sum_{k=1}^{3} x_k \sigma_k \longleftrightarrow x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^4 \quad \text{if } c = 1,$$

$$x = x_0 E + \sum_{k=1}^{3} x_k \sigma_k \longleftrightarrow x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^4 \quad \text{if } c = -1.$$

Similar computation as in [B1] gives the following:

Theorem 3.2. (i) (*Bobenko* [B2]) *Let* $\Phi = \Phi(z, \bar{z}, \mu)$, $\mu \in \mathbf{R}$, *be a* $\mathbf{R}_+ SU(2)$*-valued solution of* (3.1) *such that* $\det \Phi$ *is independent of* μ. *Then* $\Phi^{-1} \frac{\partial}{\partial \mu} \Phi$ *is a surface with harmonic inverse mean curvature in* $\mathfrak{M}^3(0)$.

(ii) *Let* $\Phi_1 = \Phi(z, \bar{z}, \mu_1)$, $\Phi_2 = \Phi(z, \bar{z}, \mu_2)$, μ_1, $\mu_2 \in \mathbf{R}$, *with* $\mu_1 \neq \mu_2$ *be* $\mathbf{R}_+ SU(2)$*-valued solutions of* (3.1) *such that* $\det \Phi_1 = \det \Phi_2$. *Then* $\Phi_1^{-1} \Phi_2$ *is a surface with harmonic inverse mean curvature in* $\mathfrak{M}^3(1)$.

(iii) *Let* $\Phi = \Phi(z, \bar{z}, \mu)$, $\mu \in \sqrt{-1}\mathbf{R}$, *be a solution of* (3.1) *such that* $\det \Phi \in \mathbf{R} \setminus \{0\}$. *Then* $\Phi^{-1} \sigma_2 \bar{\Phi} \sigma_2$ *is a surface with harmonic inverse mean curvature in* $\mathfrak{M}^3(-1)$.

Remark. (i) In Theorem 3.2 μ plays a role of deformation parameter for surfaces with harmonic inverse mean curvature.

(ii) If we replace $\mathfrak{M}^3(c)$ with the 3-dimensional pseudo-Riemannian space form of curvature c, we can define spacelike surfaces with harmonic inverse mean curvature. Then similar arguments as above can be applied to these surfaces.

§4. An immersion formula: The case $H = \frac{\sqrt{-1}(h-\bar{h})}{|h|^2 - c}$, $c = \pm 1$

In the following we put $H = \frac{\sqrt{-1}(h-\bar{h})}{|h|^2 - c}$, $c = \pm 1$. As in the previous section, we consider the following equations:

$$\begin{cases} \Phi_z = U\Phi, \\ \Phi_{\bar{z}} = V\Phi, \end{cases} \tag{4.1}$$

where

$$U = \begin{pmatrix} \frac{1}{4} u_z & -Q e^{-\frac{1}{2}u} \\ \frac{1}{2}\lambda (H_\alpha - c) e^{\frac{1}{2}u} & -\frac{1}{4} u_z \end{pmatrix},$$

$$V = \begin{pmatrix} -\frac{1}{4} u_{\bar{z}} & -\frac{1}{2}\bar{\lambda}(H_\alpha + c) e^{\frac{1}{2}u} \\ \bar{Q} e^{-\frac{1}{2}u} & \frac{1}{4} u_{\bar{z}} \end{pmatrix},$$

$$\lambda = \frac{1}{\bar{\alpha}} \frac{h^2 - c}{\alpha^2 h^2 - c}, \quad H_\alpha = \frac{\sqrt{-1}(\alpha h - \bar{\alpha}\bar{h})}{|h|^2 - c},$$

$$\alpha \in \{\mu \in \mathbf{C}; |\mu| = 1\}.$$

Note that the integrability conditions of (4.1),

$$U_{\bar{z}} - V_z + [U, V] = 0,$$

are equivalent to $(\mathrm{GC})_c$. Using (4.1), we can obtain an immersion formula in a similar manner to that of Theorem 3.2.

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