SURFACES WITH HARMONIC INVERSE MEAN CURVATURE IN SPACE FORMS

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Abstract. We define surfaces with harmonic inverse mean curvature in space forms and generalize a theorem due to Lawson by which surfaces of constant mean curvature in one space form isometrically correspond to those in another. We also obtain an immersion formula, which gives a deformation family for these surfaces.

Introduction

In classical differential geometry, surfaces of constant mean curvature have been studied for a long time. As a generalization of surfaces of constant mean curvature in \( \mathbb{R}^3 \), Bobenko introduced surfaces with harmonic inverse mean curvature and obtained an immersion formula for these surfaces ([B2]).

In this paper we shall define surfaces with harmonic inverse mean curvature in space forms and generalize a theorem due to Lawson by which surfaces of constant mean curvature in one space form isometrically correspond to those in another ([L], [P], [F]). We shall also obtain an immersion formula, which gives a deformation family for these surfaces.

§1. Definition of surfaces with harmonic inverse mean curvature

For \( c = 0, \pm 1 \) we define a 1-dimensional Riemannian manifold \( I_c \) by

\[
I_c = \begin{cases} 
(R, g_c) & \text{if } c = 0, 1, \\
((-1), g_c) & \text{if } c = -1,
\end{cases}
\]

where \( g_c = \frac{1}{(1 + ct^2)^2} dt^2 \).

Let \( M \) be a Riemann surface and \( z \) a local holomorphic coordinate on \( M \). Then we have the following:

Proposition 1.1. \( \varphi : M \to I_c \) is harmonic if and only if \( \varphi \) satisfies the following partial differential equation:

\[
\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} - \frac{2c \varphi}{1 + c \varphi^2} \left| \frac{\partial \varphi}{\partial z} \right|^2 = 0.
\]
Equation (1.1) can be directly solved:

\[
\begin{cases}
\phi = \frac{\sqrt{-1(h-h)}}{|h|^2} & c = 0, \\
\phi = \frac{\sqrt{-1(h-h)}}{|h|^2-c} \text{ or } \frac{|h|^2-c}{\sqrt{-1(h-h)}} & c = \pm 1,
\end{cases}
\]

where \( h : M \to \mathbb{C} \) is a holomorphic function such that \( |h|^2 - c \neq 0 \) or \( \text{Im} h \neq 0 \) on \( M \). In the following we assume that \( \text{Im} h \neq 0, |h|^2 - c \neq 0 \) for \( c = 0, \pm 1 \) on \( M \). Let \( \mathcal{M}^3(c) \) be the simply connected 3-dimensional space form of curvature \( c \).

**Definition.** A conformal immersion \( F : M \to \mathcal{M}^3(c) \) is called a surface with harmonic inverse mean curvature in \( \mathcal{M}^3(c) \) if the inverse of its mean curvature satisfies the equation (1.2).

**Remark.** If we restrict to the case \( c = 0 \), this definition is given by Bobenko [B2]. For examples of surfaces with harmonic inverse mean curvature in \( \mathcal{M}^3(0) = \mathbb{R}^3 \), see [B2], [CK].

§2. Differential equations of surfaces

We consider \( \mathbb{R}^4 \) with the scalar product given by

\[
\langle a, b \rangle_c = ca_0b_0 + \sum_{k=1}^{3} a_kb_k,
\]

where

\[
a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad b = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^4.
\]

Then \( \mathcal{M}^3(c) \) is embedded in \( \mathbb{R}^4 \) by

\[
\begin{align*}
\mathcal{M}^3(0) &= \{ x \in \mathbb{R}^4; x_0 = 0 \}, \\
\mathcal{M}^3(1) &= \{ x \in \mathbb{R}^4; \langle x, x \rangle_1 = 1 \}, \\
\mathcal{M}^3(-1) &= \text{a connected component of } \{ x \in \mathbb{R}^4; \langle x, x \rangle_{-1} = -1 \}.
\end{align*}
\]

Let \( F : M \to \mathcal{M}^3(c) \) be a conformal immersion, let \( e^u dzd\bar{z} \) be the induced metric on \( M \), and let \( N \) be the unit normal to \( M \). Direct computation shows that the Gauss-Codazzi equations have the following form (see [B1]):

\[
(GC)_c \quad \begin{cases} u_{z\bar{z}} + \frac{1}{2}(H^2 + c)e^u - 2|Q|^2 e^{-u} = 0, \\
Q_{\bar{z}} = \frac{1}{2}He^u,
\end{cases}
\]

where \( Q = \langle F_{zz}, N \rangle_c, \langle F_{z\bar{z}}, N \rangle_c = \frac{1}{2}He^u \).

In the following we assume that \( M \) is simply connected. We put

\[
\mathcal{C}_H = \{ F : M \to \mathcal{M}^3(c); F \text{ is a conformal immersion} \}
\]

with mean curvature \( H \)/\( \text{Iso}_0(\mathcal{M}^3(c)) \),

where \( \text{Iso}_0(\mathcal{M}^3(c)) \) is the identity component of the isometry group of \( \mathcal{M}^3(c) \). Then we have

\[
\mathcal{C}_H \cong \{ (u, Q); (u, Q) \text{ is a solution of } (GC)_c \}.
\]
§3. AN IMMERSION FORMULA: THE CASE $H = \frac{|h|^2 - c}{\sqrt{1 - (h^2 - c)}}$

In this section we put $H = H_c = \frac{|h|^2 - c}{\sqrt{1 - (h^2 - c)}}$.

Theorem 3.1.

$$\mathcal{C}_{H_0} \cong \mathcal{C}_{H_1} \cong \mathcal{C}_{H_{-1}}.$$  

Proof. Let $(u, Q)$ be a solution of $(GC)_c$ for $c = \pm 1$. We put

$$e^{u'} = \left| \frac{h^2 - c}{h^2} \right|^2 e^u, \quad Q' = \frac{h^2 - c}{h^2} Q.$$

Then it is easy to see that $(u', Q')$ is a solution of $(GC)_0$. Conversely, we can easily construct a solution of $(GC)_c$ for $c = \pm 1$ from one of $(GC)_0$. \hfill \Box

Remark. Theorem 3.1 is considered as a generalization of a theorem due to Lawson by which surfaces of constant mean curvature in one space form isometrically correspond to those in another ([L], [P], [F]).

Let $(u, Q)$ be a solution of $(GC)_0$. For a $2 \times 2$ matrix-valued function $\Phi$ on $M$, we consider the following equations:

(3.1) \[ \begin{cases} \Phi_z = U\Phi, \\ \Phi_{\bar{z}} = V\Phi, \end{cases} \]

where

$$U = \left( \begin{array}{cc} \frac{1}{2} u_z - Qe^{-\frac{1}{2} u} & 0 \\ \frac{1}{2} \lambda H_0 e^{\frac{1}{2} u} & 0 \end{array} \right), \quad V = \left( \begin{array}{cc} 0 & -\frac{1}{2} \lambda^{-1} H_0 e^{\frac{1}{2} u} \\ Qe^{-\frac{1}{2} u} & \frac{1}{2} \lambda H_0 e^{\frac{1}{2} u} \end{array} \right),$$

$$\lambda = \left( 1 + \frac{2}{H} \mu \right) / \left( 1 + \frac{2}{H} \mu \right), \quad \mu \in \mathbb{C}.$$  

Note that the integrability conditions of (3.1),

$$U_{\bar{z}} - V_z + [U, V] = 0,$$

are equivalent to $(GC)_0$. We put

$$E = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \sigma_2 = \left( \begin{array}{cc} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{array} \right), \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

and use the following identification according to $c$:

$$x = -\sqrt{-1} \sum_{k=1}^{3} x_k \sigma_k \longrightarrow x = \left( \begin{array}{c} 0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right) \in \mathbb{R}^4 \quad \text{if } c = 0,$$

$$x = x_0 E + \sqrt{-1} \sum_{k=1}^{3} x_k \sigma_k \longrightarrow x = \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right) \in \mathbb{R}^4 \quad \text{if } c = 1,$$
$$x = x_0 E + \sum_{k=1}^{3} x_k \sigma_k \iff x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^4 \text{ if } c = -1.$$ 

Similar computation as in [B1] gives the following:

**Theorem 3.2.** (i) (Bobenko [B2]) Let $\Phi = \Phi(z, \bar{z}, \mu), \mu \in \mathbb{R}$, be a $\mathbb{R}_+ SU(2)$-valued solution of (3.1) such that $\det \Phi$ is independent of $\mu$. Then $\Phi^{-1} \frac{\partial}{\partial \mu} \Phi$ is a surface with harmonic inverse mean curvature in $\mathbb{M}^3(0)$.

(ii) Let $\Phi_1 = \Phi(z, \bar{z}, \mu_1), \Phi_2 = \Phi(z, \bar{z}, \mu_2), \mu_1, \mu_2 \in \mathbb{R}$, with $\mu_1 \neq \mu_2$ be $\mathbb{R}_+ SU(2)$-valued solutions of (3.1) such that $\Phi_1 = \det \Phi_2$. Then $\Phi_1^{-1} \Phi_2$ is a surface with harmonic inverse mean curvature in $\mathbb{M}^3(1)$.

(iii) Let $\Phi = \Phi(z, \bar{z}, \mu), \mu \in \sqrt{-1} \mathbb{R}$, be a solution of (3.1) such that $\det \Phi \in \mathbb{R} \setminus \{0\}$. Then $\Phi^{-1} \sigma_2 \Phi \sigma_2$ is a surface with harmonic inverse mean curvature in $\mathbb{M}^3(-1)$.

**Remark.** (i) In Theorem 3.2 $\mu$ plays a role of deformation parameter for surfaces with harmonic inverse mean curvature.

(ii) If we replace $\mathbb{M}^3(c)$ with the 3-dimensional pseudo-Riemannian space form of curvature $c$, we can define spacelike surfaces with harmonic inverse mean curvature. Then similar arguments as above can be applied to these surfaces.

§4. An immersion formula: The case $H = \frac{\sqrt{-1}(h - \bar{h})}{|h|^2 + c}$, $c = \pm 1$

In the following we put $H = \frac{\sqrt{-1}(h - \bar{h})}{|h|^2 + c}, c = \pm 1$. As in the previous section, we consider the following equations:

\begin{equation}
\begin{cases}
\Phi_z = U \Phi, \\
\Phi_{\bar{z}} = V \Phi,
\end{cases}
\end{equation}

where

$$U = \begin{pmatrix}
\frac{1}{4} u_z & -Q e^{-\frac{1}{2} u} \\
-\lambda(H_{\alpha} - c)e^{\frac{1}{2} u} & -\frac{1}{4} u_z
\end{pmatrix},$$

$$V = \begin{pmatrix}
-\frac{1}{4} u_{\bar{z}} & -\frac{1}{2} \bar{\lambda}(H_{\bar{\alpha}} + c)e^{\frac{1}{2} u} \\
\bar{Q} e^{-\frac{1}{2} u} & \frac{1}{4} u_{\bar{z}}
\end{pmatrix},$$

$$\lambda = \frac{1}{\alpha} \frac{h^2 - c}{\alpha^2 h^2 - c}, \quad H_{\alpha} = \frac{\sqrt{-1}(\alpha h - \bar{\alpha} \bar{h})}{|h|^2 - c},$$

$$\alpha \in \{\mu \in \mathbb{C}; |\mu| = 1\}.$$ 

Note that the integrability conditions of (4.1),

$$U_{\bar{z}} - V_z + [U, V] = 0,$$

are equivalent to $(GC)_c$. Using (4.1), we can obtain an immersion formula in a similar manner to that of Theorem 3.2.
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