PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 10, Pages 3091–3094
S 0002-9939(99)04854-6
Article electronically published on April 28, 1999

A COUNTABLE NONDISCRETE TOPOLOGICAL FIELD WITHOUT NONTRIVIAL CONVERGENT SEQUENCES

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(Communicated by Alan Dow)

Abstract. A construction of a space in the title is presented, answering a question asked by A. Arkhangel′skii and D. Shakhmatov.

§1. Introduction

In [S2] D. Shakhmatov attributes the following question to A. Arkhangel′skii:

Question 1.1. Does there exist a nondiscrete countable topological field without nontrivial convergent sequences?

The following argument[1] shows how to settle a similar question for topological groups. For an arbitrary infinite abelian group G, following van Douwen (see [vD]), denote by $G^{\#}$ the group G endowed with the maximal totally bounded topology. The fact that $G^{\#}$ has no nontrivial convergent sequences follows from [vD, Theorem 1.1.3(c)]. The motivation for asking the question above lies in the fact that the topological field structure puts much more restriction on the topological properties of subsets of the field than the topological group structure (see [S2]). Thus it is pointed out in [S2] that in a topological field where no point is G_δ (such as an ultrapower of the real line with the order topology) all convergent sequences are trivial.

In this paper we prove the following theorem which affirmatively answers the question above:

Theorem 1.2. *Any infinite countable field admits a nondiscrete topology without nontrivial convergent sequences.*

We use standard notation. In what follows $\mathbb{F}$ denotes a countable metrizable nondiscrete commutative field, $\mathcal{C}$ is the family of all compact subsets of $\mathbb{F}$, and $\mathcal{B}$ is a countable base of open sets for the topology of $\mathbb{F}$. Set $\mathcal{B}^+ = \{\, \overline{U} : U \in \mathcal{B}, 0 \notin \overline{U} \,\}$. We will assume that the base $\mathcal{B}$ has the property that $\mathcal{B}^+$ is closed under taking $^{-1}$, i.e. if $U \in \mathcal{B}^+$, then $U^{-1} \in \mathcal{B}^+$. If $K_0, K_1 \subseteq \mathbb{F}$, then $K_0 \cdot K_1 = \{\, k_0 \cdot k_1 : k_0 \in K_0, k_1 \in K_1 \,\} \subseteq \mathbb{F}$, the sets $K_0 + K_1$ and K_0^{-1} are defined similarly. Note that it follows from the definition above that $\varnothing^{-1} = \varnothing$, $K + \varnothing = K \cdot \varnothing = \varnothing$ for any

Received by the editors March 25, 1997 and, in revised form, December 12, 1997.
1991 *Mathematics Subject Classification.* Primary 54H13; Secondary 54A20.
Key words and phrases. Topological field, convergent sequence.
[1]The author is thankful to the referee for suggesting this construction.

$K \subseteq \mathbb{F}$. We write $S \to x$ if $S \cup \{x\}$ is homeomorphic to a compact space with a unique nonisolated point x.

If X is a topological space and $\mathcal{K}$ is a family of subsets of X, then it is said that $\mathcal{K}$ *determines the topology of* X if every $H \subseteq X$ such that $H \cap K$ is closed for every $K \in \mathcal{K}$ is closed in X. We use the following standard facts (see [M]):

(1) if the topology of X is determined by a countable family $\mathcal{K}_X$ of compact subsets of X and the topology of Y is determined by a countable family $\mathcal{K}_Y$ of compact subsets of Y, then the topology of $X \times Y$ is determined by the family $\{ K \times P : K \in \mathcal{K}_X, P \in \mathcal{K}_Y \}$;
(2) if the restriction of $p : X \to Y$ on $K \subseteq X$ is continuous for every $K \in \mathcal{K}$, where $\mathcal{K}$ determines the topology of X, then p is continuous.

§2. Example

Let us define the following operations on arbitrary subsets K, K_0, and K_1 of $\mathbb{F}$:

(3) $(K_0, K_1) \mapsto K_0 + K_1$;
(4) $(K_0, K_1) \mapsto K_0 \cdot K_1$;
(5) $K \mapsto \{q\}$, where $q \in \mathbb{F}$;
(6) $K \mapsto (K \cap O)^{-1}$, where $O \in \mathcal{B}^+$.

Let $\{T_i\}_{i\in\omega}$ list all $T : P(\mathbb{F})^{n_T} \to P(\mathbb{F})$ obtained as compositions of (3)–(6), where n_T is the number of arguments of T and $P(\mathbb{F})$ is the power set of $\mathbb{F}$. Abusing this notation we will write $T_i(K)$ instead of $T_i(K, \dots, K)$ meaning that all the arguments of T_i are the same. It is obvious that T_i's preserve both $\mathcal{C}$ and $[\mathbb{F}]^{<\omega}$. Now let K be a compact subset of $\mathbb{F}$. Denote

$$C(K) = \{ T_i(K) : i \in \omega \}.$$

Then $C(K)$ consists of compact subsets of $\mathbb{F}$, contains all singletons, and is closed under (3)–(6). Let us introduce a new topology on $\mathbb{F}$ as follows: $H \subseteq \mathbb{F}$ is closed if $H \cap C$ is closed for each $C \in C(K)$. Then in the new topology each member of $C(K)$ has the original topology induced by $\mathbb{F}$. Denote $\mathbb{F}$ with this topology as $F(K)$. Now $C(K)$ determines the topology of $F(K)$.

The idea of the short proof of the following lemma presented here is borrowed from [MMO]:

Lemma 2.1. *The topology of $F(K)$ is the maximal field topology on $\mathbb{F}$ inducing the original topology on K.*

Proof. To show that $F(K)$ is a topological field it is enough to prove that the mappings $p_1 : F(K) \times (F(K) \setminus \{0\}) \to F(K)$ and $p_2 : F(K) \times F(K) \to F(K)$, where $p_1(x_1, x_2) = x_1 \cdot x_2^{-1}$ and $p_2(x_1, x_2) = x_1 - x_2$ are continuous. Put

$$C^+(K) = \{ T_i(K) \cap O : i \in \omega, O \in \mathcal{B}^+ \}.$$

Then, since $\mathcal{B}^+$ is closed under taking $^{-1}$, $C^+(K) \subseteq C(K)$ and it is immediate that $C^+(K)$ determines the topology of $F(K) \setminus \{0\}$. Now both $C(K)$ and $C^+(K)$ are countable families of compact subsets of $\mathbb{F}$ so it follows from (1) and (2) that to prove the continuity of p_1 it is enough to show that the restriction of p_1 on $T_i(K) \times (T_j(K) \cap O)$ is continuous for every $i, j \in \omega$, $O \in \mathcal{B}^+$. Now

$$p_1(T_i(K) \times (T_j(K) \cap O)) = T_i(K) \cdot (T_j(K) \cap O)^{-1} \in C(K)$$

(since $(T_j(K) \cap O)^{-1} \in C(K)$ as a result of application of one of the operations (6) to a member of $C(K)$). Thus both $T_i(K) \times (T_j(K) \cap O)$ and $p_1(T_i(K) \times (T_j(K) \cap O))$

have the topology induced by $\mathbb{F}$. Since p_1 is continuous in the original topology of $\mathbb{F}$, the argument for p_1 is complete.

Similarly, for p_2 one observes that $p_2(T_i(K)\times T_j(K)) = T_i(K)-T_j(K) = T_i(K)+\{-1\}\cdot T_j(K) \in C(K)$ since $\{-1\} \in C(K)$ by (5).

Now if τ is a field topology on $\mathbb{F}$ inducing the original topology on K, then for every $i \in \omega$ the set $T_i(K)$ is compact. So if $H \subseteq \mathbb{F}$ is closed in τ, then for every $i \in \omega$ the set $H \cap T_i(K)$ is closed in the usual topology of $\mathbb{F}$. Thus H is closed in the topology of $F(K)$. □

Lemma 2.2. *For any $i \in \omega$ the operation T_i satisfies the following properties (below n stands for the number of arguments of T_i):*

(7) *for any finite $f_1 \subseteq \mathbb{F}, \ldots, f_n \subseteq \mathbb{F}$ the set $T_i(f_1, \ldots, f_n)$ is finite;*
(8) *for any $q \in T_i(K_1, \ldots, K_n)$ there are finite $f_1 \subseteq K_1, \ldots, f_n \subseteq K_n$ such that $q \in T_i(f_1, \ldots, f_n)$;*
(9) *if $K'_1 \subseteq K_1, \ldots, K'_n \subseteq K_n$, then $T_i(K'_1, \ldots, K'_n) \subseteq T_i(K_1, \ldots, K_n)$;*
(10) *for any open $U \subseteq \mathbb{F}$ such that $T_i(K_1, \ldots, K_n) \subseteq U$ there are open $V_1, \ldots, V_n$ such that $K_1 \subseteq V_1, \ldots, K_n \subseteq V_n$ and $T(V_1, \ldots, V_n) \subseteq U$.*

Proof. It is easy to show that taking compositions preserves properties (7)–(10). So to prove the lemma it is enough to show that operations (3)–(6) have properties (7)–(10). The only nontrivial case is (10). For operations (3) and (4) the property immediately follows from the well known property of multiplication (resp. addition): whenever $K_0 \cdot K_1 \subseteq U$ (resp. $K_0 + K_1 \subseteq U$) for compact K_0, K_1, and an open U, then there exist open sets $V_i \supseteq K_i$ such that $V_0 \cdot V_1 \subseteq U$ (resp. $V_0 \cdot V_1 \subseteq U$). The case of (5) is trivial. To show (10) for operations (6) suppose that $(P \cap O)^{-1} \subseteq U$ for some $O \in \mathcal{B}^+$, compact P, and open U. Since $0 \notin O^{-1}$, we may assume without loss of generality that $0 \notin U$. Put $V = \mathbb{F} \setminus (O \setminus U^{-1})$. Obviously V is an open subset of $\mathbb{F}$ and a routine check shows that $(V \cap O)^{-1} \subseteq U$ and $P \subseteq V$. □

Lemma 2.3. *Let $S \to 0$ in the usual topology of $\mathbb{F}$. Then there are infinite $K \subseteq S$ and $D \subseteq S$ such that D is a closed discrete subset of $G(K \cup \{0\})$.*

Proof. Let us construct for every $i \in \omega$ points $x_i, z_i \in S$ such that

(11) $x_i \neq x_j$ and $z_i \neq z_j$ if $i \neq j$;
(12) if $n \geq k$ and $l \geq k$, then $z_l \notin T_k(\bigcup_{j \leq n}\{x_j\} \cup \{0\})$.

Let x_0 be an arbitrary point of $\mathbb{F}$ and choose z_0 so that $z_0 \notin T_0(\{x_0\} \cup \{0\})$. Suppose x_i, z_i have been constructed for $i < n$ so that they satisfy (11)–(12). It follows from (7) that the set $\bigcup_{i \leq n} T_i(\bigcup_{j<n}\{x_j\} \cup \{0\})$ is finite. So there exists $z_n \in S$ such that

$$z_n \notin \bigcup_{i \leq n} T_i(\bigcup_{j<n}\{x_j\} \cup \{0\}) \cup \{\, z_i : i < n \,\}.$$

Now by (10) for every $l, k \in \omega$ such that $n \geq l \geq k$ there exists an open $U_{l,k} \ni 0$ such that

$$z_l \notin T_k(\bigcup_{j<n}\{x_j\} \cup (U_{l,k} \cap S) \cup \{0\}).$$

Put $U = \bigcap_{n \geq l \geq k} U_{l,k}$. Now if $x_n \in (S \cap U) \setminus \{\, x_i : i < n \,\}$, then (11)–(12) are easy to check. Put $K = \{\, x_i : i \in \omega \,\}$. Then $K \to 0$ and (12) and imply that for any $i \in \omega$ and $j \in \omega$ such that $j > i$

$$z_j \notin T_i(K \cup \{0\}).$$

So $D = \{ z_i : i \in \omega \}$ intersects each element of $C(K \cup \{0\})$ in a finite subset and therefore is a closed discrete subset of $F(K \cup \{0\})$. $\square$

Proof of Theorem 1.2. Let $\mathbb{F}$ be an infinite countable field. By [W, Theorem 6, p. 23] there exists a nondiscrete topology on $\mathbb{F}$ which can be made metrizable using the main result of [S1]. Using Zorn's lemma, construct a maximal nondiscrete field topology (i.e. a field topology such that any finer field topology is discrete) τ on $\mathbb{F}$ which is finer than the metrizable one.

Suppose $S \to 0$ in τ for some infinite $S \subseteq \mathbb{F}$. By Lemma 2.1 τ is the topology of $F(S \cup \{0\})$. Lemma 2.3 implies that there is $K \subseteq S$ such that the topology of $F(K \cup \{0\})$ is finer than the topology of $F(S \cup \{0\})$, a contradiction. $\square$

The author is thankful to Professor G. Gruenhage for his valuable remarks and suggestions, to Professor D. Shakhmatov for helpful communication about the problem, and to the referee for suggesting various improvements to the proof.

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