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# AN EXTENSION OF THE WORK OF V. GUILLEMIN ON COMPLEX POWERS AND ZETA FUNCTIONS OF ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

BOGDAN BUCICOVSCHI

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Abstract. The purpose of this note is to extend the methods and results of Guillemin on elliptic self-adjoint pseudodifferential operators of order one, from operators defined on smooth functions on a closed manifold to operators defined on smooth sections in a vector bundle. The case of bundles of Hilbert modules of finite type over a finite von Neumann algebra will also be treated.

## 0. Introduction

Let $M$ be a closed Riemannian manifold of dimension $m$ and $E$ a vector bundle over $M$ endowed with a hermitian metric. The fibers of $E$ are finite dimensional vector spaces over $\mathbb{C}$ or, more general, finite type Hilbert modules over a von Neumann algebra $\mathcal{A}$. The first situation corresponds to the case $\mathcal{A} = \mathbb{C}$. Throughout this paper we will denote by $\Psi(E)$ or simply by $\Psi$ the algebra of classical pseudodifferential operators acting on smooth sections in $E$ (for the case when $\mathcal{A}$ is an arbitrary von Neumann algebra, see [BFKM] for definitions and properties). We will also denote by $\Psi^s(E)$ the subspace of pseudodifferential operators of complex order $s$. The total symbol $\sigma_{\mathrm{total}}(x,\xi)$ of such an operator $A \in \Psi^s$ has locally an asymptotic expansion of the form:

$$\sigma_{\mathrm{total}}(x,\xi) \sim \sum_{k=0}^{\infty} \sigma_{s-k}(x,\xi)$$

where $\sigma_{s-k}(x,\xi)$ are sections of the endomorphism bundle of the pull-back of $E$ with respect to the projection map $T^*(M) \setminus \{0\} \to M$. Each section $\sigma_{s-k}(x,\xi)$ is a homogeneous function in the variable $\xi$ of degree of homogeneity $s-k \in \mathbb{C}$, $\sigma_{s-k}(x,\lambda\xi) = \lambda^{s-k}\sigma_{s-k}(x,\xi)$ for any $\lambda > 0$.

The space $C^\infty(E)$ of smooth sections of $E$ over $M$ has a canonical metric

$$\langle f,g\rangle = \int_M \langle f(x),g(x)\rangle_x \, d\,vol$$

where $\langle\cdot\,,\cdot\rangle_x$ is the hermitian metric in the fiber above $x \in M$. The $L^2$ completion of $C^\infty(E)$ with respect to $\langle\cdot\,,\cdot\rangle$ will be denoted by $L^2(E)$. A pseudodifferential operator becomes an unbounded operator on $L^2(E)$.

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We will consider now an elliptic pseudodifferential operator of order one $A \in \Psi^1$ which is self-adjoint and positive with respect to $\langle \cdot\,, \cdot \rangle$. Suppose that the spectrum of $A$ is included in the interval $(\epsilon, \infty)$ for a sufficiently small $\epsilon > 0$. Then one can define the complex powers $A^s$, $s \in \mathbb{C}$ in the following way

$$A^s = \frac{1}{2\pi i} \int_\gamma \lambda^s (\lambda - A)^{-1}\, d\lambda \qquad \text{when } Re(s) < 0 \tag{0.1}$$

(where $\gamma$ is a contour in the complex plane obtained by joining two lines parallel to the negative real axis by a circle around the origin) and

$$A^s = A^{s-k} A^k \qquad \text{for } Re(s) \geq 0 \tag{0.2}$$

for large enough $k \in \mathbb{Z}$ so that $s - k$ is negative.

One of the goals of our paper is to show that $A^s$ is a pseudodifferential operator of complex order $s$. We remind the reader that this fact has been proven first by Seeley [S] in the case of finite dimensional hermitian bundle $E$ and extended to the case of von Neumann bundles in [BFKM]. Guillemin has provided an alternative approach in [G]; his treatment covers essentially the case of pseudodifferential operators acting on smooth functions on $M$. We will show how one can adjust his strategy to the case of operators acting on sections in a vector bundle $E$. The main difficulty arises from the fact that the algebra of endomorphisms of $E$ is noncommutative (fiberwise it is equal to the algebra of the $\mathcal{A}-$invariant endomorphisms of the fiber, as compared to Guillemin's case where the fiber is isomorphic to $\mathbb{C}$). In the same spirit of Guillemin, we will show that the zeta function of $A$ defined as:

$$\zeta_A(s) = \mathrm{Trace}_N(A^s) \qquad \text{for } Re(s) < -m$$

has a meromorphic extension over the complex plane $\mathbb{C}$ with at most simple poles at $-m$, $-m+1$, $\ldots$ . The residue of $\zeta_A$ at $-m$ will be equal to a quantity that depends only on the principal symbol $\sigma_1$ of the operator $A$.

Throughout the paper $A$ will be a classical pseudodifferential operator of order 1. The case of an operator of any other positive order can be reduced to the case in which the order is equal to 1.

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## 1. Complex powers of pseudodifferential operators

The goal of the section is proving the following:

**Theorem 1.1.** *Let $A$ be an elliptic, positive, self-adjoint pseudodifferential operator of order one. Suppose that $Spec(A) \in (\epsilon, \infty)$ for a sufficiently small $\epsilon > 0$. Then its complex powers $A^s$, defined as in (0.1) and (0.2), are pseudodifferential operators of order $s \in \mathbb{C}$.*

To show this we will need the following:

**Proposition 1.2.** *There exists a holomorphic family of pseudodifferential operators $A_s$ for $s \in \mathbb{C}$ such that $A_0 = Id$, $A_s A_t = A_{s+t}$ and the difference $A_1 - A$ is a smoothing operator.*

$(A_s)_{s\in\mathbb{C}}$ can be thought of as an approximation of the powers of $A$ that lie inside $\Psi$. We will show that $A_s - A^s$ are smoothing operators. Then Theorem 1.1 becomes a straightforward corollary of Proposition 1.2.

To construct the family $(A_s)_{s\in\mathbb{C}}$ it will be convenient to consider the cohomology of the group $(\mathbb{C},+)$ with coefficients in the representation of $(\mathbb{C},+)$ on the space of sections $C^\infty(\mathrm{End}(\tilde{E}))$. Here $\tilde{E}$ is the pull-back of the initial vector bundle $E$ over $M$ with respect to the projection map of the cosphere bundle $S^*(M) \to M$. This construction generalizes the cohomology considered by Guillemin in [G] for the trivial representation of $(\mathbb{C},+)$ on the space of smooth functions on $S^*(M)$.

Let $\sigma$ be a fixed section $\sigma : S^*(M) \to \mathrm{End}(\tilde{E})$ so that $\sigma(x,\xi) : E_x \to E_x$ is an invertible positive self-adjoint endomorphism for any $(x,\xi) \in S^*(M)$ ($\sigma$ will be the restriction of the principal symbol of $A$ to $S^*(M)$). The representation of $(\mathbb{C},+)$ on $C^\infty(\mathrm{End}(\tilde{E}))$ we consider is the following one: any $s \in \mathbb{C}$ acts on a section $g : S^*(M) \to \mathrm{End}(\tilde{E})$ by $s \cdot g = \sigma^{-s} g \sigma^s$.

Let $\mathcal{C}^r = \mathcal{C}^r(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E}))$ be the space of functions

$$f : \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{r \text{ times}} \to C^\infty(\mathrm{End}(\tilde{E}))$$

that are smooth, $f(\cdot)(x,\xi) : \mathbb{C} \times \mathbb{C} \times \ldots \mathbb{C} \to \mathrm{End}(E_x)$ are holomorphic for any fixed $(x,\xi) \in S^*(M)$ and $f(s_1,\ldots,s_r) = 0$ if at least one $s_i$ is equal to zero.

Let $\delta^r : \mathcal{C}^r \to \mathcal{C}^{r+1}$ defined as:

$$(\delta^r f)(s_0,s_1,\ldots,s_r) = s_0 \cdot f(s_1,\ldots,s_r) + \sum_{i=1}^{r} (-1)^i f(s_0,\ldots,s_{i-1}+s_i,\ldots,s_r) + (-1)^{r+1} f(s_0,\ldots,s_{r-1}).$$

Let $\mathcal{H}^r(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E}))) = \mathrm{Ker}\,\delta^r / \mathrm{Im}\,\delta^{r-1}$.

**Proposition 1.3.** $\mathcal{H}^2(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E}))) = 0$.

*Moreover, for each 2-cocycle $f$ there exists a unique 1-cochain $h$ such that $\delta h = f$ and $h$ has a prescribed value at* 1, $h(1)$.

*Proof.* Let $f : \mathbb{C} \times \mathbb{C} \to C^\infty(\mathrm{End}(\tilde{E}))$ so that for all $a,b,c \in \mathbb{C}$

$$\begin{cases} f(0,b) = f(a,0) = 0, \\ (\delta^2 f)(a,b,c) = a \cdot f(b,c) - f(a+b,c) + f(a,b+c) - f(a,b) = 0. \end{cases}$$

We will try to find $h : \mathbb{C} \to C^\infty(\mathrm{End}(\tilde{E}))$ such that

$$(\delta^1 h)(a,b) = \sigma^{-a} h(b) \sigma^a - h(a+b) + h(a) = f(a,b).$$

The existence of an $h$ as above implies:

$$h'(a) = \sigma^{-a} h'(0) \sigma^a - \frac{\partial f}{\partial b}(a,0). \tag{1.1}$$

Consider $h$ to be the unique solution of the previous equation with $h(0) = 0$ and with a fixed prescribed value at 1, $h(1)$. $h$ can be found in the following way:

Let $\Phi(t)$ be the automorphism of $C^\infty(\mathrm{End}(\tilde{E}))$ given by $A \to \sigma^{-t} A \sigma^t$. Then

$$h(a) = -\int_0^a \frac{\partial f}{\partial b}(t,0)\,dt + \int_0^a \Phi(t)(h'(0))\,dt.$$

If $T(a)A = \displaystyle\int_0^a \Phi(t)A\,dt$, then, in order to get any prescribed value for $h(1)$, we need to show that $T(1)$ is surjective. Indeed, we have:

$$\begin{aligned} T(1)A &= \int_0^{\frac{1}{2}} \sigma^{-t} A \sigma^t \, dt + \int_{\frac{1}{2}}^1 \sigma^{-t} A \sigma^t \, dt \\ &= T(\tfrac{1}{2})A + \Phi(\tfrac{1}{2})T(\tfrac{1}{2})A = (\mathrm{Id} + \Phi(\tfrac{1}{2}))T(\tfrac{1}{2})A \end{aligned}$$

and by induction

$$T(1)A = (\mathrm{Id} + \Phi(\tfrac{1}{2}))(\mathrm{Id} + \Phi(\tfrac{1}{4}))\dots(\mathrm{Id} + \Phi(\tfrac{1}{2^n}))T(\tfrac{1}{2^n})A.$$

But the map $A \to 2^n \displaystyle\int_0^{\frac{1}{2^n}} \sigma^{-t} A \sigma^t \, dt$ is close to the identity for a sufficiently large $n$ so $T(\frac{1}{2^n})$ is invertible. $(\mathrm{Id} + \Phi(\frac{1}{2^i}))$ is invertible as well, because $\Phi(t)$ is positive self-adjoint in any fiber above $(x, \xi) \in S^*(M)$, for any real $t$.

Thus we obtain a continuous map $h : \mathbb{C} \to C^\infty(\mathrm{End}(\tilde{E}))$ that is holomorphic in all fibers $E_{(x,\xi)}$ and $h \in \mathcal{C}^1$. We will show that $\delta\, h = f$ so $f$ is a coboundary. To see this, let

$$g(a, b) = f(a, b) - (\sigma^{-a} h(b) \sigma^a - h(a + b) + h(a)).$$

Clearly $\delta\, h = f$ if and only if $g \equiv 0$. Denote by $\dfrac{\partial}{\partial b}$ the partial derivative with respect to the second variable. Then:

$$\frac{\partial g}{\partial b}(a, b) = \frac{\partial f}{\partial b}(a, b) - \sigma^{-a} h'(b) \sigma^a + h'(a + b). \tag{1.2}$$

From (1.1) we get:

$$\begin{aligned} h'(b) &= \sigma^{-b} h(0) \sigma^b - \frac{\partial f}{\partial b}(b, 0) \quad \text{and} \\ h'(a + b) &= \sigma^{-(a+b)} h'(0) \sigma^{(a+b)} - \frac{\partial f}{\partial b}(a + b, 0). \end{aligned}$$

These two equalities and (1.2) imply

$$\begin{aligned} \frac{\partial g}{\partial b}(a, b) &= \frac{\partial f}{\partial b}(a, b) - \sigma^{-a} \left( \sigma^{-b} h'(0) \sigma^b - \frac{\partial f}{\partial b}(b, 0) \right) \sigma^a + \sigma^{-(a+b)} h'(0) \sigma^{(a+b)} \\ &\quad - \frac{\partial f}{\partial b}(a + b, 0) \\ &= \sigma^{-a} \frac{\partial f}{\partial b}(b, 0) \sigma^a - \frac{\partial f}{\partial b}(a + b, 0) + \frac{\partial f}{\partial b}(a, b) \\ &= \frac{\partial}{\partial c} \left[ (\delta^2 f)(a, b, c) \right]_{|c=0}. \end{aligned}$$

So $\dfrac{\partial g}{\partial b} = 0$; hence $g(a, b)$ is constant in $b$. When $b = 0$ we have

$$g(a, 0) = f(a, 0) - \left( \sigma^{-a} h(0) \sigma^a - h(a) + h(a) \right) = 0.$$

So $g \equiv 0$. Because $f$ was chosen arbitrarily we conclude $\mathcal{H}^2(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E}))) = 0$. ∎

We now proceed with the proof of Proposition 1.2. To show the existence of a family $(A_s)_{s\in\mathbb{C}}$ as stated in the proposition, we will show that there exists a family $(A_{(s)})_{s\in\mathbb{C}}$ of pseudodifferential operators that satisfies the conditions of Proposition 1.2 only up to smoothing operators. More precisely:

**Proposition 1.4.** *There exists a holomorphic family of pseudodifferential operators* $(A_{(s)})_{s\in\mathbb{C}}$ *with principal symbols* $\sigma_{pr}(A_{(s)}) = (\sigma_{pr}(A))^s$ *such that* $A_{(0)} = Id$, $A_{(1)} \equiv A$ *and* $A_{(s)}A_{(t)} \equiv A_{(s+t)}$ *modulo smoothing operators. This family is unique up to smoothing operators.*

*Proof.* The statement of the Theorem is equivalent to:

$$\begin{cases} A_{(s)}A_{(t)}A^{-1}_{(s+t)} \equiv Id & (\text{mod } \Psi^{-\infty}), \\ A_{(1)}A^{-1} \equiv Id & (\text{mod } \Psi^{-\infty}), \\ A_{(0)} = Id \end{cases} \tag{1.3}$$

(we denoted the space of smoothing operators by $\Psi^{-\infty}$).

To prove Proposition 1.4, we will construct $A_{(s)}$ inductively in $k \in \mathbb{N}$, such that

$$\begin{cases} A_{(s)}A_{(t)}A^{-1}_{(s+t)} \equiv Id & (\text{mod } \Psi^{-k}), \\ A_{(1)}A^{-1} \equiv Id & (\text{mod } \Psi^{-k}), \\ A_{(0)} = Id. \end{cases} \tag{1.4}$$

For $k = 1$ we can choose $(A_{(s)})_{s\in\mathbb{C}}$ to be a holomorphic family of pseudodifferential operators of order $s$ with the principal symbol equal to $\sigma^s$ where $\sigma$ is the principal symbol of $A$. We can construct such a family using a partition of unity. Moreover $A_{(0)}$ can be chosen to be the identity. The operators $A_{(s)}A_{(t)}A^{-1}_{(s+t)}$ and $A_{(1)}A^{-1}$ are operators of order 0 with the principal symbol equal to the principal symbol of the identity. The relations (1.4) are satisfied modulo $\Psi^{-1}$.

Now suppose that the relations (1.4) hold for a certain $k \in \mathbb{N}$. We will construct a new family $(\tilde{A}_{(s)})_{s\in\mathbb{C}}$ that satisfies (1.4) for $k+1$ that is of the following form:

$$\tilde{A}_{(s)} = A_{(s)}(Id - H_{(s)}), \qquad H_{(s)} \in \Psi^{-k}. \tag{1.5}$$

In this way $\tilde{A}_{(s)} - A_{(s)} \in \Psi^{s-k}$. We have:

$$\begin{aligned} \tilde{A}_{(s)}\tilde{A}_{(t)}\tilde{A}^{-1}_{(s+t)} &\equiv A_{(s)}(Id - H_{(s)})A_{(t)}(Id - H_{(t)})(Id + H_{(s+t)})A^{-1}_{(s+t)} \\ &\equiv A_{(s)}A_{(t)}A^{-1}_{(s+t)} - A_{(s)}H_{(s)}A_{(t)}A^{-1}_{(s+t)} - A_{(s)}A_{(t)}H_{(t)}A^{-1}_{(s+t)} \\ &\quad + A_{(s)}A_{(t)}H_{(s+t)}A^{-1}_{(s+t)} \\ &\equiv Id + F_{(s,t)} - A_{(s)}H_{(s)}A_{(t)}A^{-1}_{(s+t)} - A_{(s)}A_{(t)}H_{(t)}A^{-1}_{(s+t)} \\ &\quad + A_{(s)}A_{(t)}H_{(s+t)}A^{-1}_{(s+t)} \qquad (\text{mod } \Psi^{-k-1}) \end{aligned} \tag{1.6}$$

where $F_{(s,t)} = A_{(s)}A_{(t)}A^{-1}_{(s+t)} - Id$ , $F_{(s,t)} \in \Psi^{-k}$ by the induction step. To proceed with the induction we have to find a family $(H_{(s)})_{s\in\mathbb{C}}$ that makes the right hand side of the equivalence (1.6) equal to the identity modulo $\Psi^{-k-1}$. If $\sigma_{pr}(F(s,t))$ and $h(s) = \sigma_{pr}(H(s))$ are the principal symbols, then the condition on $H(s)$ is equivalent to:

$$\begin{aligned} &\sigma_{pr}(F(s,t)) = \sigma^s h(s)\sigma^{-s} + \sigma^{s+t}h(t)\sigma^{-(s+t)} - \sigma^{s+t}h(s+t)\sigma^{-(s+t)} \quad \text{or} \\ &\sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t} = \sigma^{-t}h(s)\sigma^t - h(s+t) + h(t). \end{aligned} \tag{1.7}$$

Because both sides are sections in the bundle $\mathrm{End}(\tilde{E})$ over $T^*(M)\setminus\{0\}$ of degree of homogeneity $-k$, then the above equality is satisfied if it holds when both sections are restricted to the cosphere bundle $S^*(M)$. Let:

$$f(t,s) = \sigma^{-(s+t)}\sigma_{pr}(F(s,t))\sigma^{s+t} \qquad \text{restricted to } S^*(M). \tag{1.8}$$

We will show that $f \in \mathcal{C}^2(\mathbb{C}; C^\infty(\mathrm{End}(\tilde{E})))$ and $\delta^2 f = 0$. Then $h$ as in (1.7) will be a 1-cochain so that $\delta h = f$.

We would also want the second condition of (1.4) to be satisfied so:

$$\begin{aligned} A^{-1}\tilde{A}_{(1)} &\equiv A^{-1}A_{(1)}(Id - H_{(1)}) \\ &\equiv Id + (A^{-1}A_{(1)} - Id) - A^{-1}A_{(1)}H_{(1)} \\ &\equiv Id \qquad (\mathrm{mod}\ \Psi^{-k-1}) \end{aligned}$$

and this holds if

$$h(1) = \sigma_{pr}(A^{-1}A_{(1)} - Id) \tag{1.9}$$

(we already know that $(A^{-1}A_{(1)} - Id) \in \Psi^{-k}$ from the induction step).

We will have to show that $f$ is a cocycle in $\mathcal{C}^2$. Obviously, $f(0,t) = f(s,0) = 0$. We have:

$$\begin{aligned} (\delta^2 f)(s,t,r) &= \sigma^{-s} f(t,r)\sigma^s - f(s+t,r) + f(s,t+r) - f(s,t) \\ &= \sigma^{-s}\left[\sigma^{-(t+r)}\sigma_{pr}(F(r,t))\sigma^{t+r}\right]\sigma^s - \sigma^{-(s+t+r)}\sigma_{pr}(F(r,s+t))\sigma^{s+t+r} \\ &\quad + \sigma^{-(s+t+r)}\sigma_{pr}(F(t+r,s))\sigma^{s+t+r} - \sigma^{-(s+t)}\sigma_{pr}(F(t,s))\sigma^{s+t} = 0 \end{aligned}$$

is equivalent to

$$\sigma_{pr}(F(r,t)) - \sigma_{pr}(F(r,s+t)) + \sigma_{pr}(F(t+r,s)) - \sigma^r\sigma_{pr}(F(t,s))\sigma^{-r} = 0. \tag{1.10}$$

To see this, consider the following equivalences modulo $\Psi^{-k}$:

$$\begin{aligned} &(Id + F(r,t))(Id + F(t+r,s))(Id - F(r,s+t))A_{(r)}(Id - F(t,s))A_{(r)}^{-1} \\ &\equiv A_{(r)}A_{(t)}A_{(t+r)}^{-1}A_{(t+r)}A_{(s)}A_{(s+t+r)}^{-1}A_{(s+t+r)}A_{(s+t)}^{-1}A_{(r)}^{-1}A_{(r)}A_{(s+t)}A_{(s)}^{-1}A_{(t)}^{-1}A_{(r)}^{-1} \\ &\equiv Id \end{aligned}$$

and the first term is also equivalent to

$$Id + F(r,t) - F(r,s+t) + F(t+r,s) - A_{(r)}F(t,s)A_{(r)}^{-1}$$

which proves (1.10). So $f(s,t) = \sigma^{-(s+t)}\sigma_{pr}(F(t,s))\sigma^{s+t}$ is a cocycle.

Proposition 1.3 provides us with a family $h(s)$ such that $\delta h = f$. We can choose this family so that (1.9) holds as well. This determines $h$ in a unique way. If $(H_{(s)})_{s\in\mathbb{C}}$ is a holomorphic family of pseudodifferential operators of fixed order $-k$ with principal symbol $h(s)$ and $H_{(1)} = Id$, then $\tilde{A}_{(s)} = A_{(s)}(Id - H_{(s)})$ satisfies the equivalences (1.4) modulo $\Psi^{-k-1}$.

In this way we obtain a sequence of families of operators $(A_{(s)}^{(k)})_{s\in\mathbb{C}}$ that satisfy the relations (1.4) for each $k \in \mathbb{N}$. Moreover, $A_{(s)}^{(k+1)} - A_{(s)}^{(k)} \in \Psi^{s-k}$. Then, using a standard procedure as in Lemma 1.2.8 in [Gi], we can construct a family $(A_{(s)})_{s\in\mathbb{C}}$ whose asymptotic expansion of the total symbol will be equal to:

$$\sigma_{\mathrm{total}}(A_{(s)}) \sim \sigma_{\mathrm{total}}(A_{(s)}^{(1)}) + \sum_{k\geq 0}\sigma_{\mathrm{total}}(A_{(s)}^{(k+1)} - A_{(s)}^{(k)}).$$

The family $(A_{(s)})_{s\in\mathbb{C}}$ will satisfy the conditions of Proposition 1.4.

$(A_{(s)})_{s\in\mathbb{C}}$ is unique up to smoothing operators because it must satisfy the relations (1.4) for all $k \in \mathbb{N}$ and so it must be equal to $(A^{(k)}_{(s)})_{s\in\mathbb{C}}$ modulo $\Psi^{-k}$. □

*Proof of Theorem 1.1 and Proposition 1.2.* Once we obtained the family of pseudodifferential operators $(A_{(s)})_{s\in\mathbb{C}}$, the proofs of Thm. 1.1 and Prop. 1.2 are identical to the proof of Theorem 5.1 in [G]. We can construct the one parameter group of operators as in Prop. 1.2 using the differential equation:

$$\dot{A}_s = PA_s \qquad \text{with} \quad A_0 = Id$$

where $P = \dot{A}_{(0)}$. If $A_{(s)}$ is made a self-adjoint family in $s$ (i.e., $A^*_{(s)} = A_{(\overline{s})}$) by replacing it with $\frac{1}{2}(A_{(s)} + A^*_{(\overline{s})})$, $P$ becomes a self-adjoint operator. By construction $A_s \in \Psi^s$. Then, using a theorem of Stone (Thms. VIII.7 and VIII.8 [RS]), it can be shown that $A_s = (A_1)^s$ with $P$ the infinitesimal generator of this one parameter group. In this case:

$$(A_1)^s - A^s = \frac{1}{2\pi i}\int_\gamma \lambda^s(\lambda - A)^{-1}(A - A_1)(\lambda - A_1)^{-1}\, d\lambda$$

and this is a smoothing operator. Because $(A_1)^s = A_s \in \Psi^s$ we obtain $A^s \in \Psi^s$. □

## 2. Zeta function of an elliptic pseudodifferential operator

Let $(A_{(s)})_{s\in\mathbb{C}}$ be a family of pseudodifferential operators depending holomorphically on the complex parameter $s$, $A_{(s)} \in \Psi^s$. For $Re(s) < -dim(M)$, $A_{(s)}$ is a trace-class operator.

**Definition 2.1.** The trace function of the family$A_{(s)}$ is the holomorphic function $\mathrm{Trace}_N\left(A_{(s)}\right)$ defined on the half-plane $Re(s) < -dim(M)$.

The von Neumann trace of $A_{(s)}$ is obtained by integrating the von Neumann trace of the Schwartz kernel on $M$ for $Re(s) < -dim(M)$. If $A$ is an elliptic positive self-adjoint pseudodifferential operator of order 1 with $Spec(A) \in (\epsilon, \infty)$, then its zeta function $\zeta_A$ is equal to the trace function associated with the family of its complex powers $A^s$.

In this section of our paper we will show that $\mathrm{Trace}_N\left(A_{(s)}\right)$ has a meromorphic continuation to the whole complex plane with at most simple poles at $-m$, $-m+1$, ..., where $m = dim(M)$. This fact has been proved by Seeley [S]. Guillemin has a different proof in [G] that applies only for scalar pseudodifferential operators. We will adapt his proof for the case of operators that act on sections in a vector bundle $E$ over the base space $M$.

We start by recalling some definitions and constructions in [G].

Let $\omega$ be the canonical symplectic form on the cotangent space $Y = T^*(M) \setminus \{0\}$. The multiplicative group $(\mathbb{R}^+, \cdot)$ acts on $Y$ by multiplication along the fiber $(t, (x, \xi)) \xrightarrow{\rho} (x, t\xi)$. By identifying the groups $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$ via $\ln : \mathbb{R}^+ \to \mathbb{R}$, $\rho$ can be seen as a 1-parameter group of isomorphisms. Let $\Xi$ be the vector field on $Y$ associated with this 1-parameter group and $\alpha = \iota_\Xi\, \omega$ be the contraction of $\omega$ along $\Xi$. Then the $(2m-1)$-form on $Y$, $\mu = \alpha \wedge \omega^{m-1}$, is homogeneous of degree $m$, $\rho^*_t\, \mu = t^m \mu$, and it is horizontal with respect to the fibration $Y = T^*(M) \setminus \{0\} \xrightarrow{\pi} S^*(M)$.

Let $\mathcal{B}$ be a von Neumann algebra. In our case $\mathcal{B}$ will be the field of complex numbers $\mathbb{C}$, our initial von Neumann algebra $\mathcal{A}$, or the von Neumann algebra of $\mathcal{A}$

linear endomorphisms $\mathrm{End}_{\mathcal{A}}(V)$, where $V$ is a Hilbert module of finite type over $\mathcal{A}$. Let $\overline{\mathcal{P}}_s$ be the space of smooth homogeneous $\mathcal{B}$-valued functions defined on $Y$ of degree of homogeneity $s \in \mathbb{C}$ and $\mathcal{P}_s$ the space of smooth scalar functions on $Y$ of degree of homogeneity $s$. If $f \in \overline{\mathcal{P}}_{-m}$, then the $\mathcal{B}$-valued $(2m-1)$ form $f\mu$ is horizontal and invariant under the action of $(\mathbb{R}^+, \cdot)$ so it is of the form $\pi^*\mu_f$ where $\mu_f$ is a $(2m-1)$-form on $S^*(M)$.

**Definition 2.2.** The residue of $f \in \overline{\mathcal{P}}_{-m}$ is equal to the integral

$$\overline{\mathrm{Res}}\, f = \int_{S^*(M)} \mu_f \quad \in \mathcal{B}.$$

For $f \in \overline{\mathcal{P}}_s$, $s \neq -m$, we define $\overline{\mathrm{Res}}\, f = 0$. If $\mathcal{B} = \mathbb{C}$, we will denote the residue simply by $\mathrm{Res}\, f$.

Consider the Poisson bracket $\{\ ,\ \}$ on $T^*(M)$ associated with the canonical symplectic form $\omega$. Let $\{\mathcal{P}_s, \overline{\mathcal{P}}_t\}$ be the space of functions spanned by $\{f, g\}$ with $f \in \mathcal{P}_s$ and $g \in \overline{\mathcal{P}}_t$. Then $\{\mathcal{P}_s, \overline{\mathcal{P}}_t\} \subset \overline{\mathcal{P}}_{s+t-1}$. Following the same method as in [G] (Theorem 6.2), it can be shown that:

a) If $s \neq -m$, then $\{\mathcal{P}_1, \overline{\mathcal{P}}_s\} = \overline{\mathcal{P}}_s$.

b) If $s = -m$, then $\{\mathcal{P}_1, \overline{\mathcal{P}}_s\}$ consists of all functions $f$ for which $\overline{\mathrm{Res}}\, f = 0$.

Moreover, one can construct a family of functions $(g_i)_{i \in I}$, $g_i \in \mathcal{P}_1$ such that for any analytic family with parameter $s$, $f_s \in \overline{\mathcal{P}}_s$, defined on a strip $a - \epsilon \leq Im(s) \leq a + \epsilon$, $c \leq Re(s) \leq d$ for which $\overline{\mathrm{Res}}\, f_{-m} = 0$, one can find $\delta \leq \epsilon$ and homogeneous functions $h_{i,s} \in \overline{\mathcal{P}}$ which are analytic in $s$ on a narrower strip $a - \delta \leq Im(s) \leq a + \delta$, $c \leq Re(s) \leq d$, such that

$$f_s = \sum_{i \in I} \{g_i, h_{i,s}\}$$

(cf. [G], Theorem 6.7).

Let us consider now a holomorphic family of pseudodifferential operators $(A_{(s)})_{s \in \mathbb{C}}$, $A_{(s)} \in \Psi^s$ and its associated trace function $\mathrm{Trace}_N\left(A_{(s)}\right)$. We define the residue of the family $A$ to be $\mathrm{Res}\, A = \mathrm{Res}(\mathrm{Trace}_N\, \sigma_{pr}(A_{(-m)})) \in \mathbb{C}$.

We have the following theorem:

**Theorem 2.3.** *The trace function of the analytic family $(A_{(s)})_{s \in \mathbb{C}}$ has a meromorphic continuation to the whole complex plane with at most simple poles at $-m$, $-m+1, \ldots$ . The residue of $\mathrm{Trace}_N\left(A_{(s)}\right)$ at $s = -m$ is equal to*

$$\mathrm{res}_{|_{s=-m}} \mathrm{Trace}_N\left(A_{(s)}\right) = \gamma_0\, \mathrm{Res}\, A$$

*where $\gamma_0$ is a constant depending only on $dim(M)$. For $A_{(s)} = A^s$ – the complex powers of an elliptic positive self-adjoint pseudodifferential operator of order one, the residue of the zeta function at $s = -m$ depends only on its principal symbol $\sigma = \sigma_{pr}(A)$ and is equal to $\gamma_0\, \mathrm{Res}(\sigma^{-m})$.*

*Proof.* Let $(U_\alpha)_\alpha$ be an open cover of $M$ with chosen trivializations of the vector bundle $E$ over each $U_\alpha$, $E_{|U_\alpha} \cong U_\alpha \times V$, with $V$ the generic fiber. Using a partition of unity argument, we can write:

$$A_{(s)} = \sum_\alpha A_{\alpha\,(s)} + K_{(s)} \tag{2.1}$$

where $A_{\alpha\,(s)}$ are pseudodifferential operators of order $s$ with support inside $U_\alpha$ and $K_{(s)}$ is a family of smoothing operators. Because the residue of the trace function of the family $A_{(s)}$ and $\operatorname{Res} A$ are both linear in $A$, it is sufficient to prove the theorem for $A_{\alpha\,(s)}$ and $K_{(s)}$. But $K_{(s)}$ is a family of smoothing operators and both the residues of their trace function and the residue $\operatorname{Res} K$ are zero. Thus we reduced the proof of the theorem to the case of one family $A_{(s)} = A_{\alpha\,(s)}$ supported in an open set $U = U_\alpha$ over which we have a trivialization of the vector bundle $\chi_\alpha : E_{|U} \to U \times V$. Moreover, because both the trace function $\operatorname{Trace}_N\left(A_{(s)}\right)$ and the residue $\operatorname{Res} A$ are obtained by integrating quantities that depend on the local expression of the total symbol of $A_{(s)}$, we can replace the bundle $E \to M$ with the trivial bundle $M \times V \to M$ and the operators $A_{\alpha\,(s)}$ with the pseudodifferential operators acting on sections of the trivial bundle $M \times V$ that are supported in the open set $U_\alpha$ and equal to $A_{\alpha\,(s)}$ via the isomorphism $\chi_\alpha$. To make things simple, we will denote this new family of operators by $A_{(s)}$ as well, and the new trivial bundle by $E$.

Following the ideas in [G], we consider the family $(s+m)A_{(s)}$. The principal symbol $(s+m)\sigma_{pr}(A_{(s)})$ can be represented by the $\mathcal{B}$-valued smooth homogeneous functions of degree $s$, $f_{(s)} : T^*(M) \setminus \{0\} \to \mathcal{B}$, with $\mathcal{B} = \operatorname{End}_{\mathcal{A}}(V)$. For $s = -m$ we have $f = 0$, so $\overline{\operatorname{Res}}\, f = 0$. Then there exist $\mathcal{B}$-valued functions $h^k_{(s)}$, $h^k_{(s)} \in \overline{\mathcal{P}}_s$ such that

$$f_{(s)} = \sum_k \{g_k, h^k_{(s)}\}$$

and $h^k_{(s)}$ are analytic on a strip $a - \epsilon \leq Im(s) \leq a + \epsilon$, $c \leq Re(s) \leq d$.

Let $G_k = G'_k \hat{\otimes} Id$ be a pseudodifferential operator acting on the space of sections $C^\infty(M)\hat{\otimes} V$ of the trivial bundle $E$ with $G'_k$ a scalar pseudodifferential operator that has the principal symbol equal to $g_k$ and $Id$ the identity operator. Let $(H^k_{(s)})$ be a holomorphic family of pseudodifferential operators with the principal symbol equal to $h^k_{(s)}$. Then the principal symbol of the commutator is equal to

$$\sigma_{pr}\left[G_k, H^k_{(s)}\right] = \{g_k, h^k_{(s)}\}$$

so

$$(s+m)A_{(s)} = \sum_k \left[G_k, H^k_{(s)}\right] + B_{(s)} \qquad \text{with } B_{(s)} \in \Psi^{s-1}.$$

For $Re(s)$ sufficiently small, $\operatorname{Trace}_N\left([G_k, H^k_{(s)}]\right) = 0$, so $\operatorname{Trace}_N\left(A_{(s)}\right) = \dfrac{1}{s+m}\operatorname{Trace}_N\left(B_{(s)}\right)$ for $Re(s) < -m$. But $\dfrac{1}{s+m}\operatorname{Trace}_N\left(B_{(s)}\right)$ is a meromorphic function on the half-plane $Re(s) < -m+1$ with a simple pole at $s = -m$. So $\operatorname{Trace}_N\left(A_{(s)}\right)$ has a meromorphic extension to $Re(s) < -m+1$. Replacing the family $A_{(s)}$ by $B_{(s)}$ and using an induction argument, we can extend $\operatorname{Trace}_N\left(A_{(s)}\right)$ to a meromorphic function on the complex plane with at most simple poles at $-m$, $-m+1, \ldots$ .

We will compare the residue of $\operatorname{Trace}_N\left(A_{(s)}\right)$ at $-m$ to the residue of the family $(A_{(s)})_{s\in\mathbb{C}}$, $\operatorname{Res} A = \operatorname{Res}(\operatorname{Trace}_N \sigma_{pr}(A_{(-m)}))$. Guillemin has showed ([G], Theorem 7.5) that in the scalar case there exists a constant $\gamma_0$ that depends only on the dimension of the manifold $M$ such that

$$\operatorname{res}_{|s=-m} \operatorname{Trace} A = \gamma_0 \operatorname{Res} A. \tag{2.2}$$

We will extend this equality for the pseudodifferential operators acting on sections in the vector bundle $E$.

We will show a stronger equality:

$$\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}}\, A = \gamma_0\, \overline{\operatorname{Res}}\, A_{(-m)} \tag{2.3}$$

where $(A_{(s)})_s$ is a holomorphic family of pseudodifferential operators acting on the sections of the trivial bundle $M \times V$, $\overline{\operatorname{Trace}}\, A_{(s)} = \int_M K_s(x,x)\, dx$ with $K_s(x,y)$ the Schwartz kernel of $A_{(s)}$, and $\overline{\operatorname{Res}}\, A_{(-m)} = \overline{\operatorname{Res}}\, \sigma_{pr}(A_{(-m)})$, both sides of the equality (2.3) being in the von Neumann algebra $\mathcal{B} = \operatorname{End}_{\mathcal{A}}(V)$. The equality (2.2) will be then a direct consequence of (2.3) after passing to the von Neumann traces.

Both sides of the equality (2.3) depend only on the principal symbol of the operator $A_{(-m)}$. This is obvious for the right-hand side. If one considers another family $B_{(s)}$ with $\sigma_{pr}(B_{(-m)}) = \sigma_{pr}(A_{(-m)})$, then $(B_{(s)} - A_{(s)})$ is a family for which $\overline{\operatorname{Res}}\, \sigma_{pr}(B_{(-m)} - A_{(-m)}) = 0$, so, by a previous observation, $\overline{\operatorname{Trace}}(B_{(s)} - A_{(s)})$ has a meromorphic extension which is holomorphic at $s = -m$. So $\overline{\operatorname{Trace}}\, B_{(s)}$ and $\overline{\operatorname{Trace}}\, A_{(s)}$ will have the same residue at $s = -m$ and this shows that the left-hand side of (2.3) depends only on $\sigma_{pr}(A_{(-m)})$.

Both sides of (2.3), as functions of holomorphic families, will factor through the projection $A_{(s)} \to \sigma_{pr}(A_{(-m)}) \in \overline{\mathcal{P}}_{-m}$. It will be sufficient to show that the equality (2.3) holds on $\overline{\mathcal{P}}_{-m}$.

$\overline{\operatorname{Res}}$ vanishes exactly on $\{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\}$ and realizes a $\mathcal{B}$ isomorphism $\overline{\mathcal{P}}_{-m}/\{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\} \xrightarrow{\sim} \mathcal{B}$. For $f \in \{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\}$, $f = \sum\{g_k, h^k\}$, one can extend it to a holomorphic family of homogeneous symbols of degree of homogeneity $s \in \mathbb{C}$ by considering first the homogenous holomorphic extensions $h^k_{(s)} \in \overline{\mathcal{P}}_s$ and then taking $f_{(s)} = \sum\{g_k, h^k_{(s)}\}$. If $G_k = G'_k \hat{\otimes} Id$ is a pseudodifferential operator such that the scalar operator $G'_k$ has the principal symbol equal to $g_k$ and $(H^k_{(s)})$ is a holomorphic family of pseudodifferential operators with the principal symbol equal to $h^k_{(s)}$, then $A_{(s)}$ defined as $\sum[G_k, H^k_{(s)}]$ has the principal symbol at $s = -m$ equal to $f$ and its trace is identically zero. This shows that $\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}}\, A$ vanishes on $\{\mathcal{P}_1, \overline{\mathcal{P}}_{-m}\}$ as well. Because both $\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}}\, A$ and $\overline{\operatorname{Res}}\, A_{(-m)}$ are $\mathcal{B}$ linear, one gets $\operatorname{res}_{|s=-m} \overline{\operatorname{Trace}}\, A = \overline{\operatorname{Res}}\, A_{(-m)} \cdot C$ with $C \in \mathcal{B}$.

Guillemin already showed this equality for a holomorphic family of scalar pseudodifferential operators $(A_{(s)})$ in which case $C$ is a scalar constant $\gamma_0$. So $C = \gamma_0 \cdot \operatorname{Id}_{\mathcal{B}}$ and the equality (2.3) holds. Passing to the von Neumann trace, we get (2.2). ☐

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Department of Mathematics, Ohio State University, 231 W 18th Ave., Columbus, Ohio 43210

*E-mail address*: `bogdanb@math.ohio-state.edu`