

## LEVEL SETS OF A TYPICAL $C^n$ FUNCTION

UDAYAN B. DARJI AND MICHAŁ MORAYNE

(Communicated by Frederick W. Gehring)

ABSTRACT. We determine the level set structure of a typical  $C^n$  function.

### INTRODUCTION

Assume that  $\mathcal{F}$  is a function space complete with respect to some norm. We say that a *typical function in  $\mathcal{F}$  satisfies property  $\mathcal{P}$*  if the subfamily of  $\mathcal{F}$  consisting of those functions which satisfy  $\mathcal{P}$  is residual (in the sense of Baire category) in  $\mathcal{F}$ . If  $y$  is in the range of  $f$ , we call  $f^{-1}(\{y\})$  a *level set* of  $f$ . In [2] Bruckner and Garg gave a full description of level sets of a typical function from  $C[0, 1]$ , the space of real-valued, continuous functions defined on  $[0, 1]$  and equipped with the sup norm. Namely, they proved:

**Theorem 1** (Bruckner and Garg). *For a typical function  $f \in C[0, 1]$  there exists a countable set  $S_f \subseteq (\min f, \max f)$  such that the level set  $f^{-1}(\{y\})$  is:*

1. a nowhere dense perfect set if  $y \notin S_f \cup \{\min f, \max f\}$ ,
2. a single point if  $y = \min f$  or  $\max f$ , and
3. the union of a nowhere dense perfect set and an isolated point of  $f^{-1}(\{y\})$  if  $y \in S_f$ .

In this paper we give the description of level sets of typical functions in  $C^n[0, 1]$ . Namely, we prove that a typical  $f \in C^1[0, 1]$  is either strictly monotone or  $f$  has uncountably many level sets having exactly one accumulation point and all other level sets of  $f$  are finite. For a typical function in  $C^n[0, 1]$  with  $n \geq 2$  the situation is simple. All level sets are finite.

### LEVEL SETS OF A TYPICAL $C^n$ FUNCTION

Let us now introduce some notation and definitions.

The symbols  $\mathbf{N}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  will denote the sets of all positive integers, rationals and reals, respectively. We use  $|A|$  to denote the cardinality of set  $A$ . The restriction of a function  $f$  to a set  $A$  will be denoted by  $f|_A$ . We use  $\lambda(A)$  to denote the Lebesgue measure of  $A$  for a Lebesgue measurable subset  $A$  of  $\mathbf{R}$ .

---

Received by the editors May 14, 1997 and, in revised form, December 23, 1997.

1991 *Mathematics Subject Classification*. Primary 26A21, 26A16.

*Key words and phrases*. Level sets,  $C^n$  functions, typical function.

The second author was supported in part by KBN Grant 2P 301 04 307.

This paper was written when the second author was visiting the Department of Mathematics of the University of Louisville, Kentucky, USA.

Let  $X$  be a metric space with a norm  $|\cdot|$ . We use  $B_{|\cdot|}(x, r)$  to denote the open ball in  $X$  with center  $x$  and radius  $r$ .

We shall consider the space  $C^n[0, 1]$  ( $n \geq 0$ ) endowed with Sobolev's norm, i.e.  $\|f\|_n = \sum_{i=0}^n \|f^{(i)}\|$ , where  $\|\cdot\|$  stands for the sup norm and  $f^{(i)}$  stands for  $i$ th derivative of  $f$ .

We prove a series of lemmas which lead us to our main results.

**Lemma 1.** *Fix  $y_0 \in \mathbf{R}$ . For a typical  $f \in C[0, 1]$ ,  $f^{-1}(\{y_0\})$  is either empty or perfect and nowhere dense.*

*Proof.* Let  $p, q \in [0, 1]$  and  $p < q$ . Let  $\mathcal{F}_{p,q} = \{f \in C[0, 1] : |f^{-1}(\{y_0\}) \cap [p, q]| = 1\}$ . We first want to show that  $\mathcal{F}_{p,q}$  is nowhere dense in  $C[0, 1]$ . We will accomplish this by showing that for each  $h \in \mathcal{F}_{p,q}$  and  $\epsilon > 0$ , there exists a nonempty open set which misses  $\mathcal{F}_{p,q}$  and is contained in  $B_{\|\cdot\|}(h, \epsilon)$ . To this end, let  $h \in \mathcal{F}_{p,q}$  and  $\epsilon > 0$ . Let  $(u, v) \subset [p, q]$  be such that  $h((u, v)) \subset (y_0 - \epsilon/4, y_0 + \epsilon/4)$ . Let  $a < b < c$  be any points of  $(u, v)$ . Let us define a new function  $\phi$  as follows:  $\phi(x) = h(x)$  for  $x \notin (u, v)$ ,  $\phi(a) = \phi(c) = y_0 - \epsilon/4$ ,  $\phi(b) = y_0 + \epsilon/4$  and make  $\phi$  linear on intervals  $(u, a)$ ,  $(a, b)$ ,  $(b, c)$  and  $(c, v)$ . Every function from  $B_{\|\cdot\|}(\phi, \epsilon/4)$  now takes the value  $y_0$  in at least two points on the interval  $(u, v) \subset [p, q]$  whence  $B_{\|\cdot\|}(\phi, \epsilon/4) \cap \mathcal{F}_{p,q} = \emptyset$ . As  $\|h - \phi\| \leq \epsilon/2$ ,  $B_{\|\cdot\|}(\phi, \epsilon/4) \subseteq B_{\|\cdot\|}(h, \epsilon)$ . Thus,  $\mathcal{F}_{p,q}$  is nowhere dense.

Let us notice now that for every function  $f$  in the residual set

$$C[0, 1] \setminus \bigcup \{\mathcal{F}_{p,q} : p, q \in \mathbf{Q} \cap [0, 1], p < q\}$$

there is no isolated point in the level set  $f^{-1}(\{y_0\})$ . So  $f^{-1}(\{y_0\})$  must be either empty or perfect.

It is a well-known fact that a typical function in  $C[0, 1]$  is nowhere constant. (For example, this follows from the aforementioned theorem of Bruckner and Garg.) This together with what we just showed finishes the proof of the lemma.

The following is a  $C^1$  counterpart of Lemma 1.

**Lemma 2.** *Fix  $y_0 \in \mathbf{R}$ . For a typical  $f \in C^1[0, 1]$ ,  $f^{-1}(\{y_0\})$  is finite.*

*Proof.* Of course, it is enough to show the lemma for  $y_0 = 0$ . It is easy to see that a function  $f$  that has infinite level set  $f^{-1}(\{0\})$  must satisfy at some point the equation  $f(x) = f'(x) = 0$ . We shall show that the family of functions

$$\mathcal{F} = \{f \in C^1[0, 1] : \text{there exists } x \text{ such that } f(x) = f'(x) = 0\}$$

is of first category. One can readily observe that it is closed, so it is enough to show that its interior is empty.

Let  $f \in \mathcal{F}$ . From Sard's theorem (see for instance Theorem 6.3, p. 226, [3]) we know that  $\lambda(f(\{x : f'(x) = 0\})) = 0$ . Thus there exists as small a positive  $\epsilon$  as we want such that  $f(x) \neq \epsilon$  at any point  $x$  where  $f'(x) = 0$ . Let us now consider the function  $h = f - \epsilon$ . If  $h(x) = h'(x) = 0$  at some  $x \in [0, 1]$ , then  $\epsilon \in f(\{x : f'(x) = 0\})$  which is impossible. Obviously,  $h \in B_{\|\cdot\|_1}(f, 2\epsilon)$ . Thus the interior of  $\mathcal{F}$  is empty. This finishes the proof.

Let us now recall the following definition. We say that a function  $f : [0, 1] \rightarrow \mathbf{R}$  is *monotone at a point  $x$*  if there exists an open (relative to  $[0, 1]$ ) neighborhood  $V$

of  $x$  such that one of the following conditions is satisfied:

- $f(s) \leq f(x)$  whenever  $s < x$  and  $s \in V$ , and  $f(s) \geq f(x)$  whenever  $s > x$  and  $s \in V$ ; or
- $f(s) \geq f(x)$  whenever  $s < x$  and  $s \in V$ , and  $f(s) \leq f(x)$  whenever  $s > x$  and  $s \in V$ .

**Lemma 3.** *A typical function  $f$  in  $C^1[0, 1]$  has the property that if there is an  $x$  such that  $f'(x) = 0$ , then  $f$  is not monotone at  $x$ .*

*Proof.* Let

$$\mathcal{F} = \{f \in C^1[0, 1] : \text{there exists } x \text{ such that } f'(x) = 0 \text{ and } f \text{ is monotone at } x\}.$$

We first show that the complement of  $\mathcal{F}$  is dense in  $C^1[0, 1]$ . Let  $f \in C^1[0, 1]$  and  $\epsilon > 0$ . Let us define  $g \in B_{\|\cdot\|_1}(f, \epsilon/3)$ . If  $f'(0) = 0$  or  $f'(1) = 0$ , let us put  $g = f + \alpha x$  with  $\alpha < \epsilon/6$  such that  $g'(0) \neq 0$  and  $g'(1) \neq 0$ . If  $f'(0) \neq 0$  and  $f'(1) \neq 0$ , then we put  $g = f$ . One can easily see that  $g \in B_{\|\cdot\|_1}(f, \epsilon/3)$ . As the set  $M = \{x \in [0, 1] : g'(x) = 0\}$  is compact and the set  $\{x \in [0, 1] : |g'(x)| < \epsilon/6\}$  is open in  $[0, 1]$  and contains  $M$ , there exists a finite sequence of points  $0 \leq a_1 < b_1 < \dots < a_n < b_n \leq 1$  such that  $|g'(t)| < \epsilon/6$ , for each  $t \in (a_i, b_i)$ ,  $i \leq n$ , and  $M \subset \bigcup\{(a_i, b_i) : i \leq n\}$ . Let  $\phi \in C[0, 1]$  be defined as follows:  $\phi(x) = g'(x)$  if  $x \notin \bigcup\{(a_i, b_i) : i \leq n\}$  and  $\phi$  is linear on every interval  $(a_i, b_i)$ ,  $i \leq n$ . Finally, let  $h \in C^1[0, 1]$  be defined as

$$h(x) = g(0) + \int_0^x \phi(t) dt.$$

It is easy to observe that  $h$  has strict extremum at every point  $x$  where  $h'(x) = 0$  and thus is not monotone at any such point. We also have  $\|f - h\|_1 \leq \|f - g\|_1 + \|g - h\|_1 < \epsilon$ . Thus, the complement of  $\mathcal{F}$  is dense in  $C^1[0, 1]$ .

To complete the proof of the lemma, we now show that  $\mathcal{F}$  is  $F_\sigma$ . Let us first observe that

$$\mathcal{F} = \bigcup_{n=1}^{\infty} (\mathcal{F}_n^+ \cup \mathcal{F}_n^-),$$

where

$$\mathcal{F}_n^+ = \{f \in C^1[0, 1] : \text{there exists } p \in [0, 1] \text{ such that } f'(p) = 0 \text{ and}$$

$$f|(p - 1/n, p] \cap [0, 1] \leq f(p) \text{ and } f|[p, p + 1/n) \cap [0, 1] \geq f(p)\}$$

and

$$\mathcal{F}_n^- = \{f \in C^1[0, 1] : \text{there exists } p \in [0, 1] \text{ such that } f'(p) = 0 \text{ and}$$

$$f|(p - 1/n, p] \cap [0, 1] \geq f(p) \text{ and } f|[p, p + 1/n) \cap [0, 1] \leq f(p)\}.$$

It is easy to see that  $\mathcal{F}_n^+$  and  $\mathcal{F}_n^-$  are closed in  $C^1[0, 1]$ . Thus  $\mathcal{F}$  is of first category.

**Proposition 1.** *The family*

$$\mathcal{S} = \{f \in C^1[0, 1] : \text{there is } 0 \leq u < v \leq 1 \text{ such that } f(u) = f(v)$$

$$\text{and } (u = 0 \text{ or } f'(u) = 0) \text{ and } (v = 1 \text{ or } f'(v) = 0)\}$$

*is of first category in  $C^1[0, 1]$ .*

*Proof.* We shall use the fact that the space  $C^1[0, 1]$  is homeomorphic with the product space  $\mathbf{R} \times C[0, 1]$  where the homeomorphism  $\Psi : \mathbf{R} \times C[0, 1] \rightarrow C^1[0, 1]$  is given by the formula

$$(1) \quad \Psi(a, f)(x) = a + \int_0^x f(t) dt.$$

We have  $\Psi^{-1}(\mathcal{S}) = \mathbf{R} \times \mathcal{G}$ , where

$$\mathcal{G} = \left\{ \varphi \in C[0, 1] : \text{there exist } 0 \leq u < v \leq 1 \text{ such that} \right.$$

$$\left. (u = 0 \text{ or } \varphi(u) = 0) \text{ and } (v = 1 \text{ or } \varphi(v) = 0) \text{ and } \int_u^v \varphi(t) dt = 0 \right\}.$$

Thus to show that  $\mathcal{S}$  is of first category in  $C^1[0, 1]$  or, equivalently, that  $\Psi^{-1}(\mathcal{S})$  is of first category in  $\mathbf{R} \times C[0, 1]$  it is enough to show that  $\mathcal{G}$  is of first category in  $C[0, 1]$ .

Let us first notice that  $\mathcal{G} = \bigcup_{n=1}^\infty \mathcal{G}_n$ , where

$$\mathcal{G}_n = \left\{ \varphi \in C[0, 1] : \text{there exist } 0 \leq u < v \leq 1 \text{ such that } v - u \geq 1/n, \right.$$

$$\left. (u = 0 \text{ or } \varphi(u) = 0) \text{ and } (v = 1 \text{ or } \varphi(v) = 0) \text{ and } \int_u^v \varphi(t) dt = 0 \right\}.$$

It is easy to check that  $\mathcal{G}_n$  is closed in  $C[0, 1]$ . Now to show that  $\mathcal{G}$  is of first category it is enough to show that the complement of  $\mathcal{G}$  is dense in  $C[0, 1]$ .

Let  $f \in C[0, 1]$ . Let  $\epsilon > 0$ . By the Weierstrass approximation theorem there exists a nonzero polynomial  $w$  such that  $\|f - w\| < \epsilon/2$  (of course, we consider only the restriction of  $w$  to  $[0, 1]$ ). Let  $0 = x_0 \leq x_1 < \dots < x_{n-1} \leq x_n = 1$  and the set  $\{x_1, \dots, x_{n-1}\}$  be the set of all roots of  $w$  which belong to  $[0, 1]$ . Let

$$\delta = \min \left\{ \left| \int_{x_i}^{x_j} w(t) dt \right| : 0 \leq i < j \leq n \text{ and } \int_{x_i}^{x_j} w(t) dt \neq 0 \right\}.$$

Let

$$0 < \alpha < \min \left\{ \frac{\epsilon}{2\|w^2\|}, \frac{1}{\|w\|}, \frac{\delta}{\|w^2\|} \right\}.$$

We define a new polynomial  $v(t) = w(t) + \alpha w^2(t)$ . Let us notice that

- $v$  has the same roots in the interval  $[0, 1]$  as  $w$ ,
- $\|v(t) - w(t)\| < \epsilon/2$ , and
- for  $x_i < x_j$  we have that

$$\left| \int_{x_i}^{x_j} v(t) dt \right| \geq \min \left\{ \delta - \alpha\|w^2\|, \alpha \cdot \int_{x_i}^{x_j} w^2(t) dt \right\} > 0.$$

Thus, we have found a  $v \in C[0, 1]$  such that  $\|v - f\| < \epsilon$  and  $v \notin \mathcal{G}$ . Therefore, the complement of  $\mathcal{G}$  is dense in  $C[0, 1]$  and the proof of the proposition is complete.

The next three lemmas are immediate corollaries to Proposition 1.

**Lemma 4.** *No level set of a typical function in  $C^1[0, 1]$  contains two points at which  $f$  attains a local extremum.*

Let us notice that if an extremum is attained at 0 or at 1 the derivative need not be equal to zero, and this was the reason for considering  $u = 0$  and  $v = 1$  in Proposition 1.

**Lemma 5.** *A typical function  $f$  in  $C^1[0, 1]$  has the property that if  $f$  attains a local extremum at  $x$ , then  $f^{-1}(f(x))$  is finite.*

**Lemma 6.** *No level set of a typical function in  $C^1[0, 1]$  contains two accumulation points.*

**Lemma 7.** *A typical function  $f \in C^1[0, 1]$  has the property that  $\{x : f'(x) = 0\}$  is either empty or perfect.*

*Proof.* Let us consider the family  $\mathcal{T}$  of functions  $f \in C^1[0, 1]$  for which the set  $\{x : f'(x) = 0\}$  is non-empty and is not perfect. We want to show that  $\mathcal{T}$  is of first category. We again use the fact that the space  $C^1[0, 1]$  is homeomorphic with the product space  $\mathbf{R} \times C[0, 1]$  where the homeomorphism  $\Psi : \mathbf{R} \times C[0, 1] \rightarrow C^1[0, 1]$  is given by the formula (1). Let us notice that  $\Psi^{-1}(\mathcal{T}) = \mathbf{R} \times \mathcal{H}$ , where

$$\mathcal{H} = \{\phi \in C[0, 1] : \phi^{-1}(\{0\}) \neq \emptyset \text{ and } \phi^{-1}(\{0\}) \text{ is not perfect}\}.$$

By Lemma 1  $\mathcal{H}$  is of first category in  $C[0, 1]$  and thus  $\mathcal{T}$  is of first category in  $C^1[0, 1]$ .

We are now in a position to prove our main theorem.

**Theorem 2.** *A typical function  $f \in C^1[0, 1]$  is either strictly monotone or there exist a perfect nowhere dense subset  $P_f$  of  $(\min f, \max f)$  and a countable dense subset  $D_f$  of  $P_f$  such that the level set  $f^{-1}(\{y\})$  is:*

1. *a set with exactly one accumulation point if  $y \in P_f \setminus D_f$ ,*
2. *a finite set if  $y \in D_f \cup ((\min f, \max f) \setminus P_f)$ ; and*
3. *a single point if  $y \in \{\min f, \max f\}$ .*

*Proof.* As the intersection of countably many residual families is residual in a complete space, we have that a typical function satisfies the properties of Lemmas 3 - 7. Let  $f$  be such a function.

Assume that  $f$  is not monotone. Then, there exists a point  $x \in (0, 1)$  where  $f'(x) = 0$ . As  $f$  satisfies the property of Lemma 7,  $M_f = \{x : f'(x) = 0\}$  is perfect. As  $f$  satisfies the property of Lemma 6, the set  $P_f = f(M_f)$  is also perfect. Let

$$D_f = \{y \in P_f : \text{there is } x \in f^{-1}(\{y\}) \text{ such that } f \text{ has a local extremum at } x\}.$$

It is well-known that  $D_f$  has to be countable [1]. Of course,  $D_f$  is dense in  $P_f$ .

Let  $y \in P_f \setminus D_f$ . As  $f$  has no extremum at any point of the level set  $f^{-1}(\{y\})$  and at some point  $x \in f^{-1}(\{y\})$  we have  $f'(x) = 0$ , then, as  $f$  satisfies the property of Lemma 3, this must be an accumulation point of the level set  $f^{-1}(\{y\})$ . But, as  $f$  satisfies the property of Lemma 6, there is no other accumulation point of the level set  $f^{-1}(\{y\})$ . Thus 1. is satisfied.

As  $f$  satisfies the property of Lemma 5 and the fact that for any  $y$  for which  $f^{-1}(\{y\})$  is infinite there must be a point  $x \in f^{-1}(\{y\})$  such that  $f'(x) = 0$ , we have that 2. is satisfied.

That 3. is satisfied follows from the fact that  $f$  satisfies the property of Lemma 4.

Let us finally state the following theorem on the level sets of a typical function in  $C^n[0, 1]$  for  $n \geq 2$ .

**Theorem 3.** *A typical function in  $C^n[0, 1]$ ,  $n \geq 2$ , has all level sets finite.*

*Proof.* Let us first prove the theorem for  $n = 2$ . Notice that the space  $C^2[0, 1]$  is homeomorphic with the product space  $\mathbf{R} \times C^1[0, 1]$  where the homeomorphism  $\Psi : \mathbf{R} \times C^1[0, 1] \rightarrow C^2[0, 1]$  is again given by the formula (1). Let

$$\mathcal{W} = \{f \in C^2[0, 1] : \text{there exists } y \text{ such that the set } f^{-1}(\{y\}) \text{ is infinite}\}.$$

Let us notice that

$$\Psi^{-1}(\mathcal{W}) \subseteq \mathbf{R} \times \mathcal{J},$$

where

$$\mathcal{J} = \{\phi \in C^1[0, 1] : \text{the set } \phi^{-1}(\{0\}) \text{ is infinite}\}.$$

By Lemma 2  $\mathcal{J}$  is of first category in  $C^1[0, 1]$  whence  $\mathcal{W}$  is of first category in  $C^2[0, 1]$ . Thus the conclusion of our theorem is proved for  $n = 2$ .

For  $n > 2$ , the theorem can be easily proved in a similar fashion with induction.

#### REFERENCES

- [1] A. M. Bruckner, *Differentiation of Real Functions*, Second edition. CRM Monograph Series, 5. American Mathematical Society. MR **94m**:26001
- [2] A. M. Bruckner and K. M. Garg, *The level structure of a residual set of continuous functions*, Trans. Amer. Math. Soc. **232** (1977), 307–321. MR **57**:16487
- [3] S. Saks, *Theory of Integral*, Monografie Matematyczne 7, Warszawa-Lwów, 1937.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISVILLE, LOUISVILLE, KENTUCKY 40292  
*E-mail address:* ubdarj01@athena.louisville.edu

MATHEMATICAL INSTITUTE, POLISH ACADEMY OF SCIENCES, UL. KOPERNIKA 18, 51-617 WROCLAW, POLAND  
*E-mail address:* morayne@im.pwn.wroc.pl