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DISTRIBUTIVITY AND STATIONARY REFLECTIONS

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ABSTRACT. In this paper, we present some relations between generalized distributivity of quotient algebras and Mahlo operations, and show that the distributivity implies some variants of stationary relections.

The notion of generalized distributivity appeared in [3] for the first time. In that paper, we began investigating the relations between the notion and other fundamental principles in set theory, especially in the area called large cardinal axioms. For example, we showed in [3] that:

Let I be any non-trivial κ-complete ideal on a regular uncountable cardinal κ and λ any cardinal with $2 \leq \lambda < \kappa$. Then *$\langle \kappa, (2^\kappa; \lambda)\rangle$-distributivity of the quotient $\wp(\kappa)/I$ is equivalent to saying that there exists a non-trivial κ-complete prime ideal extending I* (cf. Theorem 5).

Then, it is interesting to discover the strength of distributivity in other forms, say, $\langle \kappa, (\mu; \lambda)\rangle$-distributivity with $\lambda \geq \kappa$.

In this paper, we shall show that some versions imply some variants of stationary reflections.

We shall organize our paper as follows:

In §1, we shall introduce notations and terminologies, and several definitions of distributivity will be given. In §2, we shall investigate distributive conditions which imply some closedness of the Mahlo operation. Our main result in this section is the following:

Theorem A. *Let σ be any function of S into $\wp(T)$ such that for a, b in S, $\mu_a = |\wp(\sigma(a))| < \kappa$ and if $a \prec_S b$, then $\sigma(a) \subseteq \sigma(b)$. Assume that I is a $\prec_S$-fine κ-complete $\prec_S$-normal $\langle 3, I_0^+, (H; \mu)\rangle$-distributive ideal on S, where H is the set $(\{0\} \times S) \cup (\{1\} \times T)$ and $\mu = sup(\{|T|\} \cup \{\mu_a : a \in S\})$.*

Moreover, we assume that $R = \{a \in S : cf_{\prec_T}(\sigma(a)) > \aleph_0\}$ has positive I-measure, $\{a \in S : t \in \sigma(a)\}$ has I-measure one for each $t \in T$ and if g is a function on $A \in I^+$ with $g(a) \in \sigma(a)$, then there exists a subset B of A of positive I-measure such that $g\lceil B$ is constant. Then if X is a $\prec_T$-stationary subset of T, $R - M_\sigma(X)$ has I-measure zero.

This theorem yields several corollaries. As applications of distributivity, we shall introduce those corollaries in §3. That is, the following will be shown.

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Theorem B. *Let $S = P_{<\eta}(\lambda)$ and $T = P_{<\mu}(\lambda)$, where $\aleph_0 < \mu < \eta \leq \kappa \leq \lambda$ and $2^{(\nu^{<\mu})} < \kappa$ for any $\nu < \eta$. Assume that there exists a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, J, (S;\tau)\rangle$-distributive ideal I on S, where $J = \{X \subseteq S : |X| \geq 2\}$ and $\tau = max.\{\lambda^{<\mu}, 2^{(\eta^{<\mu})}\}$. Then, if X is a $\prec_T$-stationary subset of T, $M_{\sigma_1}(X) = \{a \in S : cf_{\prec_T}(P_{<\mu}(a)) > \aleph_0$ and $X \cap P_{<\mu}(a)$ is $\prec_T$-stationary in $P_{<\mu}(a)\}$ has I-measure one.*

Theorem C. *Assume that κ is inaccessible and there exists a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, J, (S;\lambda)\rangle$-distributive ideal I on $S = P_{<\kappa}(\lambda)$ with $|X| \geq 2$ for $X \in J$. Then for every stationary set X in λ, the set $\{a \in S : cf.(sup.a) > \aleph_0$ and $X \cap sup.a$ is stationary in $sup.a\}$ has I-measure one. And so, for any $\nu < \lambda$, $E^{\nu}_{\lambda}(< \kappa)$ fails.*

Theorem D. *Assume that $\aleph_1^{\aleph_0} < \aleph_2 \leq \lambda$ and for any θ with $\lambda^{\aleph_0} < \theta$ there exists a $\prec$-fine $\aleph_2$-complete $\prec$-normal $\langle \aleph_1, (S;\lambda^{\aleph_0})\rangle$-distributive ideal I_θ on $S = P_{<\aleph_2}(H_\theta)$. Then every $\prec$-stationary set X in $T = [\lambda]^{\aleph_0}$ strongly reflects.*

§1. Notations and terminologies

We use standard set theoretic notation. For example, $|X|$ denotes the cardinality of the set X and small Greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinals, cardinals are initial ordinals and the Greek letters $\kappa, \lambda, \mu, \ldots$ are reserved for denoting cardinals. We use $(x;y)$ to denote the constant function of x with the unique value y.

For any function F with $D = dom(F) \neq \emptyset$ and $|F(a)| \geq 1$ $(a \in D)$.

$$\Pi F = \{f : f \text{ is a function on } D \text{ such that for any } a \in D,\ f(a) \in F(a)\}.$$

For any sets A and X, we set: $P_{<|A|}(X) = [X]^{<|A|} = \{Y \subseteq X : |Y| < |A|\}$.

We assume a familiarity with the definitions and basic properties of Boolean algebras.

Let I be any fixed ideal in B. A subset X of B is said to be *I-disjoint* if $X \subseteq I^+ = B - I$ and $x \wedge y \in I$ for any distinct members x and y of X. If X is $I_{\mathbf{0}}$-disjoint, it is simply said to be *disjoint*, where $I_{\mathbf{0}}$ is the trivial ideal $\{\mathbf{0}\}$ in B. Moreover, if the sum of a disjoint subset X of B is b, X is called *a partition of* b.

We usually write $[a]_I$ (or simply $[a]$) to denote the coset corresponding to a in the quotient algebra B/I. Then, if the family $\{[a]_I : a \in X\}$ is a partition of $[b]_I$ in B/I, X is said to be *an I-partition of* b. B is *λ-saturated* if there is no disjoint subset of cardinality λ, and I is *λ-saturated* if B/I is λ-saturated.

Now, we define several kinds of distributivity and introduce elementary properties of those notions. Hereafter, B always indicates a fixed Boolean algebra and I denotes an ideal in B.

Definition 1. Let D be any non-empty set and let λ and μ be any cardinals with $\mu \geq 1$. Then, we define: B is *$\langle \lambda, (D;\mu)\rangle$-distributive* if $\forall F \in {}^{D}\wp(B)\, \forall b \in B\, (\forall a \in D\, (1 \leq |F(a)| \leq \mu$ and $\mathbf{0} < b \leq \bigvee F(a)) \Longrightarrow \exists v \in \Pi F\ \ \forall t \in [D]^{<\lambda}\, (b \wedge \bigwedge_{a \in t} v(a) > \mathbf{0}))$.

Clearly the usual $(\lambda;\mu)$-distributivity is equivalent to $\langle \lambda^+, (\lambda;\mu)\rangle$-distributivity.

Just like the saturation, we can extend the notion of distributivity to that of ideals in Boolean algebras. Moreover, we can introduce some variants of the notion in this case.

Definition 2. Let D be any non-empty set and let λ and μ be any cardinals with $\mu \geq 1$.

(1) An ideal I in B is said to be $\langle\lambda,(D;\mu)\rangle$-*distributive* if the corresponding quotient algebra B/I is $\langle\lambda,(D;\mu)\rangle$-distributive.
(2) Let J be any subset of B, and assume that I is λ-complete.

Then, I is said to be $\langle\lambda,J,(D;\mu)\rangle$-*distributive* if

$$\forall F \in {}^{D}\wp(B)\,\forall b \in B\,(\,\forall a \in D\,(\,1 \le |F(a)| \le \mu \text{ and } \mathbf{0} < [b]_I \le \bigvee_{x\in F(a)} [x]_I)$$
$$\Longrightarrow \exists\, v \in \Pi F\ \ \forall t \in [D]^{<\lambda}\,(\,b \wedge \bigwedge_{a\in t} v(a) \in J\,)).$$

Clearly, I is $\langle\lambda,(D;\mu)\rangle$-distributive if and only if I is $\langle\lambda,I^+,(D;\mu)\rangle$-distributive.

Now, consider ideals in power set algebras. So, let $\langle S,\preceq\rangle$ be a partially ordered set and assume that I is an ideal in $\wp(S)$ (i.e., *an ideal on* S). Several combinatorial notions are also defined in this context, for example : Let A be any subset of S. For any subset X of A,

(1) X is $\prec$-*unbounded* in A if $\forall x \in A\,\exists a \in X\; x \prec a$.
(2) X is $\prec$-*closed* in A if whenever $\langle a_\xi : \xi < \mu\rangle$ is any $\prec$-increasing sequence in X with $b = \sup_{\xi<\mu} a_\xi$ ($b \in A$), b is also in X.
(3) X is $\prec$-*club* in A if X is both $\prec$-unbounded and $\prec$-closed in A.
(4) X is $\prec$-*stationary* in A if for every $\prec$-club C in A, $X \cap C \neq \emptyset$.

Let h be any function of S into $\wp(S)$.

(5) X is the $\prec$-*diagonal union of* h, $X = \bigtriangledown^{\prec} h$ (or simply $X = \bigtriangledown h$), if $X = \{x \in S : \exists y \prec x\; x \in h(y)\,\}$, and then I is said to be $\prec$-*normal* if for any function h of S into I, $\bigtriangledown h \in I$.

In the above terminologies, "*in* S" is usually omitted, and in the notations introduced so far, the prefix $\prec$ is usually omitted if it is the relation $\in$.

Now, we define distributive ideals with "*normalizability*".

Definition 3. Let D be any non-empty set and let λ and μ be any cardinals with $\mu \ge 1$. Let J be any subfamily of $\wp(S)$. Then I is said to be $\langle\lambda,J,(D;\mu)\rangle$-*normal-distributive* (or simply $\langle\lambda,J,(D;\mu)\rangle$-*N-distributive*) if $\forall F \in {}^{D}\wp(B)\,\forall A \subseteq S\,(\forall a \in D(1 \le |F(a)| \le \mu$ and $\mathbf{0} < [A]_I \le \bigvee_{X\in F(a)}[X]_I) \Longrightarrow \exists\, v \in \Pi F\ \ \forall t \in [D]^{<\lambda}\,\forall g \in {}^{S}D\,([A]_I \wedge \bigwedge_{a\in t}[v(a)]_I \in I^+$ and $\triangle v \circ g \in J\,))$.

Each $\langle\lambda,I^+,(D;\mu)\rangle$-normal-distributive ideal I is also said to be $\langle\lambda,(D;\mu)\rangle$-*normal-distributive.*

Lemma 4. (1) *Assume that* $\aleph_0 \le \kappa \le \mu$, $\kappa < \lambda$ *and* B *is* κ^+*-complete. Then* B *is* $\langle\lambda,(\mu;\kappa)\rangle$*-distributive if and only if it is* $\langle\lambda,(\mu;2)\rangle$*-distributive.*

(2) *Assume that* $\lambda \le \kappa$, $J \subseteq \wp(S) - I$ *and* I *is* $\prec$*-fine and* κ*-complete, and* X *is any set. Then if* I *is* $\langle\lambda,J,(X\times S;2)\rangle$*-N-distributive,* I *is also* $\langle\lambda,J,(X;S)\rangle$*-N-distributive.*

Proof. The proof follows by an easy modification of the standard argument on the usual distributivity.

On the generalized distributivity, we have already:

Theorem 5 ([4])**.** *Let* κ *be any uncountable cardinal. Then we have:*

(1) κ *is regular if and only if* BD_κ *is non-trivial and* κ*-complete.*
(2) κ *is inaccessible if and only if* BD_κ *is non-trivial,* κ*-complete and* $(\lambda;2)$*-distributive for every* $\lambda < \kappa$*.*

(3) *κ is weakly compact if and only if BD_κ is non-trivial, κ-complete and $\langle\kappa,(\kappa;2)\rangle$-distributive.*
(4) *κ is measurable if and only if BD_κ is non-trivial, κ-complete and $\langle\kappa,(2^\kappa;2)\rangle$-distributive.*
(5) *κ is strongly compact if and only if for each regular $\lambda\ge\kappa$, BD_λ is $\langle\kappa,(2^\lambda;2)\rangle$-distributive.*
(6) *κ is strongly compact if and only if for each $\lambda\ge\kappa$, $BD^{\prec}_{\lambda^{<\kappa}}$ is $\prec$-fine, κ-complete and $\langle\kappa,(2^{\lambda^{<\kappa}};2)\rangle$-distributive.*
(7) *κ is supercompact if and only if for each $\lambda\ge\kappa$, there is a $\prec$-fine, κ-complete, $\prec$-normal and $\langle\kappa,(2^{\lambda^{<\kappa}};2)\rangle$-$N$-distributive ideal on $P_{<\kappa}(\lambda)$.*

In the above, $\prec$ denotes the relation on $P_{<\kappa}(\lambda)$ defined by $a\prec b$ iff $a\subseteq b$ and $|a|<|b|$.

Hereafter, κ denotes a regular uncountable cardinal and λ an infinite cardinal with $\kappa\le\lambda$. Then, we let $\langle S,\prec_S\rangle$ and $\langle T,\prec_T\rangle$ denote fixed partially ordered sets, and reserve $\prec$ to denote the above orderings on any suitable sets.

§2. Mahlo operations

In this section, we shall introduce some relations between distributivity and Mahlo operations.

To state our result, let us review the Mahlo operation in a general setting.

Let σ be any function of S into $\wp(T)$ such that $a\prec b$ implies $\sigma(a)\subseteq\sigma(b)$. Then, *Mahlo operation, M_σ, with respect to σ* is defined by

$$M_\sigma(X)=\{a\in S: cf_{\prec_T}(\sigma(a))>\aleph_0 \textit{ and } X\cap\sigma(a) \textit{ is } \prec_T\textit{-stationary in } \sigma(a)\,\}$$

for each subset X of T, where $cf_{\prec_T}(\sigma(a))$ is the *$\prec_T$-cofinality of* $\sigma(a)$, i.e. the least ordinal α such that there exists an $\prec_T$-increasing sequence in $\sigma(a)$ of length α with no $\prec_T$-upper bound in $\sigma(a)$.

If the function σ is the identity map or defined by $\sigma(a)=pr_T(a)$, the corresponding Mahlo operation is simply denoted by M, where $pr_T(a)=\{b\in T: b\prec_T a\,\}$.

Now, let us introduce our result.

Theorem 6. *Let σ be any function of S into $\wp(T)$ such that for a,b in S, $\mu_a=|\wp(\sigma(a))|<\kappa$ and if $a\prec_S b$, then $\sigma(a)\subseteq\sigma(b)$. Assume that I is a $\prec_S$-fine κ-complete $\prec_S$-normal $\langle 3,I_0^+,(H;\mu)\rangle$-distributive ideal on S, where H is the set $(\{0\}\times S)\cup(\{1\}\times T)$ and $\mu=sup(\{|T|\}\cup\{\mu_a: a\in S\,\})$.*

Moreover, we assume that $R=\{a\in S: cf_{\prec_T}(\sigma(a))>\aleph_0\,\}$ has positive I-measure, $\{a\in S: t\in\sigma(a)\,\}$ has I-measure one for each $t\in T$ and if g is a function on $A\in I^+$ with $g(a)\in\sigma(a)$, then there exists a subset B of A of positive I-measure such that $g\lceil B$ is constant. Then, if X is a $\prec_T$-stationary subset of T, $R-M_\sigma(X)$ has I-measure zero.

To prove this theorem, we need the following lemma.

Lemma 7 (cf. [3])**.** *Let D be any non-empty set and let σ be any function of S into $\wp(T)$ such that for any $a.b\in S$, $\mu_a=|\wp(\sigma(a))|<\kappa$ and if $a\prec_S b$, then $\sigma(a)\subseteq\sigma(b)$. Assume that there exists a $\prec_S$-fine κ-complete $\langle 3,I_{\mathbf{0}}^+,(H;\mu)\rangle$-distributive ideal I on S, where H is the set $(\{0\}\times S)\cup(\{1\}\times T)$ and μ is a cardinal such that for all $a\in S$, $\mu_a\le\mu$.*

Then, whenever a subset A of S has positive I-measure, $\langle t_a: a\in A\rangle$ is a sequence with $t_a\subseteq\sigma(a)$ $(a\in A)$ and $\langle W_d: d\in D\,\rangle$ is a sequence of subsets of

$\wp(S)$ with $|W_d| \leq \mu$ and $[A]_I \leq \bigvee_{X \in W_d}[X]_I$ $(d \in D)$, there exist a set E and a function h on H such that for any $d \in D$, $h(1,d) \in W_d$ and for any $a \in S$, the set $h(0,a) \subseteq \{b \in A : a \prec_S b$ and $E \cap \sigma(a) = t_b \cap \sigma(a)\}$ has positive I-measure, and moreover for any $p \in [H]^{<3}$, $\bigcap_{\langle i,b\rangle \in p} h(i,b)$ is non-empty.

Although the proof of this lemma which we will give below is almost the same as that of Lemma 5 in [3], we think it convinient for the reader to give a complete proof.

Proof of Lemma 7. Let $\langle t_{\xi,a} : \xi < \mu_a \rangle$ be an enumeration of $\wp(\sigma(a))$ for each $a \in S$. Define $A_{\xi,a} = \{b \in A : a \prec_S b \textit{ and } t_b \cap \sigma(a) = t_{\xi,a}\}$ for each $a \in S$ and $\xi < \mu_a$. Then the family $P_a = \{A_{\xi,a} : \xi < \mu_a \textit{ and } A_{\xi,a} \in I^+\}$ is clearly an I-partition of A for each $a \in S$. By the $\langle 3, I_{\mathbf{0}}, (H;\mu)\rangle$-distributivity of I, there exists a function h of H into $(\bigcup_{a\in S} P_a) \cup (\bigcup_{d \in D} W_d)$ such that for any $a \in S$ $h(0,a) \in P_a$ and for any $d \in D$ $h(1,d) \in W_d$ and moreover for any $p \in [H]^{<3}$, $\bigcap_{\langle i,b\rangle \in p} h(i,b)$ is non-empty. Let k be the function on S defined by $h(0,a) = A_{k(a),a}$. Let a and b be any distinct elements of S. Then if we pick a $c \in h(0,a) \cap h(0,b)$, then $t_c \cap \sigma(a) = t_{k(a),a}$ and $t_c \cap \sigma(b) = t_{k(b),b}$. Thus $t_{k(a),a} \cap \sigma(b) = t_{k(b),b} \cap \sigma(a)$. Putting $E = \bigcup_{a \in S} t_{k(a),a}$, we can check that for any $a \in S$, if $b \in h(0,a)$, then $E \cap \sigma(a) = t_b \cap \sigma(a)$. For, since $\sigma(a) \subseteq \sigma(b)$, for every $c \in S$, $t_{k(c),c} \cap \sigma(a) = t_{k(b),b} \cap \sigma(a) \cap \sigma(c)$ holds. Hence the following chain of equalities holds:

$$\begin{aligned} E \cap \sigma(a) &= \bigcup_{c \in S} t_{k(a),a} \cap \sigma(a) = t_{k(b),b} \cap \bigcup_{c\in S} \sigma(c) \cap \sigma(a) \\ &= t_{k(b),b} \cap \sigma(a) = t_{k(a),a} = t_b \cap \sigma(a). \end{aligned}$$

This completes the proof.

Note that in the above lemma if we assume that I is $\prec_S$-normal, for each $\xi < \kappa$, $\{\xi\}$ is a minimal element of $S - \{\emptyset\}$ with respect to $\prec_S$ and $\{\{\xi\} : \xi < \kappa\}$ has I-measure zero, then we can replace the condition $\mu_a < \kappa$ by $\mu_a \leq \kappa$. To see this, we have only to prove that P_a is an I-partition of A. Otherwise, there is a subset B of A of positive I-measure such that $B \cap A_{\xi,a} \in I$ for all $\xi < \mu_a \leq \kappa$. Notice that $B \subseteq \bigcup_{\xi < \mu_a} A_{\xi,a}$. Then the function g of B into S defined by $g(x) = \{\xi\}$ if x is in $A_{\xi,a}$, is $\prec_S$-regressive on $B - \{\{\xi\} : \xi < \kappa\}$, and so for some subset E of B of positive I-measure, g is constant. This is absurd.

This remark can be also applied to Theorem 6.

Proof of Theorem 6. On the contrary, we assume that $A = R - M_\sigma(X)$ has positive I-measure and will produce a contradiction.

For each $b \in A$, we can find a $\prec_T$-club C_b in $\sigma(b)$ with $X \cap C_b = \emptyset$. Let t be any element of T and f_t a choice function of $\{C_b : t \in \sigma(b) \textit{ and } b \in A\}$ such that $t \prec_T f_t(b) \in C_b$. Then by the $\prec_S$-normality of I, there exists a maximal family $\{\langle D_{t,\xi}, e_{t,\xi}\rangle : \xi < \eta_t\}$ with $\eta_t \leq |T|$ such that i) $\{D_{t,\xi} : \xi < \eta_t\}$ is an I-partition of A and ii) $t \prec_T e_{t,\xi}$ and $e_{t,\xi} \in C_b$ for every $b \in D_{t,\xi}$. Then by Lemma 7, there exist a set C and a function h on H such that for any $a \in S$, $h(0,a) \subseteq \{b \in A : a \prec_S b$ *and* $C \cap \sigma(a) = C_b \cap \sigma(a)\}$ and for any $p \in [H]^{<3}$, $\bigcap_{i \in p} h(i)$ is non-empty. Let b be any element in A and t be any element of T with $h(1,t) = D_{t,\xi_t}$ and $e_{t,\xi_t} \in \sigma(b)$. Such b exists, since A has positive I-measure. Then, picking $d \in h(0,b) \cap h(1,t)$, we can easily notice that $e_{t,\xi_t} \in C_d$ and $t \prec_T e_{t,\xi_t} \in \sigma(b)$. And so, e_{t,ξ_t} is in $C_d \cap \sigma(b) = C \cap \sigma(b)$. Thus C is $\prec_T$-unbounded. For $\prec_T$-closedness of C, let

$\langle a_\xi : \xi < \mu \rangle$ be any $\prec_T$-increasing sequence in C with $c = sup_{\xi<\mu} a_\xi$. Then, since A has positive I-measure, there exists an element a_0 of A with $c \in \sigma(a_0)$. So, for any $b \in h(0, a_0)$, $\langle a_\xi : \xi < \mu \rangle$ is a sequence in C_b. Since C_b is $\prec_T$-closed, c is in $C_b \cap \sigma(a_0)$ and in C. Thus C is $\prec_T$-closed. Hence C is $\prec_T$-club in T and has a non-empty intersection with X. This is a contradiction.

§3. Applications

Theorem 6 is fertile in producing results on stationary reflection.

Theorem 8. *Assume that κ is inaccessible and I is a non-trivial κ-complete normal $\langle \aleph_1, J, (\kappa; \kappa) \rangle$-distributive ideal on κ, where $J = \{ X \subseteq \kappa : |X| \geq 2 \}$. Then I is an M-ideal, i.e. whenever X has I-measure one, so does the set $M(X) = \{ \alpha < \kappa : cf.(\alpha) > \aleph_0$ and $X \cap \alpha$ is stationary in $\alpha \}$. And so, κ is greatly Mahlo.*

Proof. The $\langle \aleph_1, J, (\kappa; \kappa) \rangle$-distributivity of I implies that $R = \{ \alpha < \kappa : cf.(\alpha) > \aleph_0 \}$ has I-measure one. If X has positive I-measure, X is stationary in κ. So $R - M(X)$ has I-measure zero by the theorem, that is, $M(X)$ has I-measure one. Thus I is an M-ideal. □

Notice that an ideal I is a WC-ideal on κ is equivalent to saying that it is $\langle \kappa, (\kappa; \kappa) \rangle$-distributive (cf. [2]). By the result in [2] that *if κ is weakly compact, then there is a normal WC-ideal on κ*, the above corollary also holds for any weakly compact cardinal κ. That is, we get the well-known result:

Corollary (Baumgartner, Taylor and Wagon, or Kakuda)**.** *If κ is weakly compact, κ is greatly Mahlo.*

Set $S = P_{<\eta}(\lambda)$ and $T = P_{<\mu}(\lambda)$ with $\aleph_0 < \mu < \eta \leq \kappa$, and let $\prec_T$ denote the inclusion relation on T. Recall that $\prec$ denotes the ordering defined by $a \prec b$ iff $a \subseteq b$ and $|a| < |b|$.

Assume that κ is inaccessible and for any $\nu < \eta$, $2^{(\nu^{<\mu})} < \kappa$, and I is a $\prec$-fine $\prec$-normal κ-complete ideal on S. Consider the map σ_1 on S defined by $\sigma_1(a) = P_{<\mu}(a)$. Then clearly it holds that if $a \prec_S b$, then $\sigma_1(a) \subseteq \sigma_1(b)$, and for any $t \in T$, the set $\{ a \in S : t \in \sigma_1(a)$ *and* $|a| \geq \mu \}$ has I-measure one. Moreover, if g is a function on $A \in I^+$ with $g(a) \in \sigma_1(a)$, then $g \lceil B$ is constant for some $B \in I^+ \cap \wp(A)$. Therefore, the following is established.

Theorem 9. *Let $S = P_{<\eta}(\lambda)$ and $T = P_{<\mu}(\lambda)$, where $\aleph_0 < \mu < \eta \leq \kappa \leq \lambda$ and $2^{(\nu^{<\mu})} < \kappa$ for any $\nu < \eta$. Assume that there exists a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, J, (S; \tau) \rangle$-distributive ideal I on S, where $J = \{ X \subseteq S : |X| \geq 2 \}$ and $\tau = max.\{ \lambda^{<\mu}, 2^{(\eta^{<\mu})} \}$. Then, if X is a $\prec_T$-stationary subset of T, $M_{\sigma_1}(X) = \{ a \in S : cf_{\prec_T}(P_{<\mu}(a)) > \aleph_0$ and $X \cap P_{<\mu}(a)$ is $\prec_T$-stationary in $P_{<\mu}(a) \}$ has I-measure one.*

By Theorem 5, if κ is λ-supercompact, there is a $\prec$-fine κ-complete $\prec$-normal $\langle \kappa, (S^S; 2) \rangle$-$N$-distributive ideal I on $S = P_{<\kappa}(\lambda)$. By Lemma 4, I is also $\langle \aleph_1, (S; S) \rangle$-$N$-distributive. Hence, using Lemma 1.2 in [1], we get:

Corollary 1. *Let κ be λ-supercompact. Then for every uncountable regular $\mu < \kappa$, for every $\prec_T$-stationary $X \subseteq T = P_{<\mu}(\lambda)$ and for every tight and $\prec$-unbounded $A \subseteq S = P_{<\kappa}(\lambda)$, $\{ a \in A : X \cap P_{<\mu}(a)$ is $\prec_T$-stationary in $P_{<\mu}(a) \}$ is contained in some $\prec$-fine κ-complete $\prec$-normal ultrafilter on S.*

In the above, "tight" means that if $D \subseteq A$ is ω-directed, i.e. D satisfies that for any sequence $\langle a_n : n < \omega \rangle$ in D there is an $a \in D$ with $\bigcup_{n<\omega} a_n \subseteq a$ and if $\omega < cf.(|D|) \leq |D| < \kappa$, then $\bigcup D \in A$ holds.

So, we have another proof of the following:

Corollary 2 (Feng and Magidor [1]). *Assume that κ is λ-supercompact with $\lambda \geq \kappa$ regular. Then for every stationary $S \subseteq P_{<\omega_1}(\lambda)$ and for every tight and unbounded $A \subseteq P_{<\kappa}(\lambda)$, there is an $X \in A$ such that $S \cap P_{<\omega_1}(X)$ is stationary in $P_{<\omega_1}(X)$.*

Next, assume that κ is inaccessible and I is a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, J, (S;\lambda)\rangle$-distributive ideal on $S = P_{<\kappa}(\lambda)$ with $|X| \geq 2$ for $X \in J$. Consider the map σ_2 on S defined by $\sigma_2(a) = sup.a < \lambda$. Then the following facts are verified:

(1) if $a \prec b$, then $sup.a \leq sup.b$, i.e. $\sigma_2(a) \leq \sigma_2(b)$,
(2) for any $\alpha < \lambda$, the set $\{a \in S : \alpha < \sigma_2(a)\}$ has I-measure one,
(3) the set $\{a \in S : cf.(sup.a) > \aleph_0\}$ has I-measure one and
(4) if g is a function on $A \in I^+$ with $g(a) < sup.a$, then there is a $B \in I^+ \cap \wp(A)$ such that $g \lceil B$ is constant.

By these facts, we can get:

Theorem 10. *Assume that κ is inaccessible and there exists a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, J, (S;\lambda)\rangle$-distributive ideal I on $S = P_{<\kappa}(\lambda)$ with $|X| \geq 2$ for $X \in J$. Then for every stationary set X in λ, the set $\{a \in S : cf.(sup.a) > \aleph_0$ and $X \cap sup.a$ is stationary in $sup.a\}$ has I-measure one. And so, for any $\nu < \lambda$, $E^{\nu}_{\lambda}(<\kappa)$ fails.*

Recall that $E^{\nu}_{\lambda}(<\kappa)$ denotes the assertion that there is a stationary set $S \subseteq \{\alpha < \lambda : cf.(\alpha) = \nu\}$ such that for each limit ordinal $\xi < \lambda$ with $cf.(\xi) < \kappa$, $S \cap \xi$ is not stationary in ξ.

As a corollary, we shall give a result of cardinal arithmetic. Borrowing the argument of Theorem 2 in Matsubara [4], we can verify that some distributivity implies the same conclusion that he did.

Corollary 1. *Assume that κ is inaccessible and there exists a $\prec$-fine κ-complete $\prec$-normal $\langle \aleph_1, J, (S;\lambda)\rangle$-distributive ideal on $S = P_{<\kappa}(\lambda)$ with $|X| \geq 2$ for $X \in J$. Then it holds that $\lambda^{<\kappa} = \lambda$.*

Proof. We follow Matsubara's argument of Theorem 2 in [4]. By the well-known theorem of Solovay, there is a disjoint family $\{A_\xi \subseteq \lambda : \xi < \lambda\}$ of stationary sets in λ. For each $\xi < \lambda$, we set $t_\xi = \{\alpha < \lambda : A_\alpha \cap \xi$ *is stationary in* $\xi\}$. Clearly, the set $D = \{a \in S : |t_{sup.a}| < \kappa\}$ has I-measure one and for each $\alpha < \lambda$, $E_\alpha = \{a \in S : A_\alpha \cap sup.a$ *is stationary in* $sup.a\}$ also has I-measure one by Theorem 10. Let x be any element in $[\lambda]^{<\kappa}(= S)$, and pick an element $a \in D \cap \bigcap_{\alpha \in x} E_\alpha$. Then it can be checked that $x \subseteq t_{sup.a}$ and $|t_{sup.a}| < \kappa$. Hence, putting $E = \{\xi < \lambda : |t_\xi| < \kappa\}$, we can get $[\lambda]^{<\kappa} \subseteq \bigcup_{\xi \in E} \wp(t_\xi)$ and so $\lambda^{<\kappa} \leq |E| \cdot 2^{<\kappa} = \lambda$. This completes the proof. □

Corollary 2 (Solovay [5]). *If κ is a supercompact cardinal, for every regular cardinal $\lambda \geq \kappa$, $\lambda^{<\kappa} = \lambda$ holds.*

Finally, we shall present some conditions which imply *the strong reflection*. In [6], Veličković introduced the following notion:

A $\prec_T$-stationary subset X of $T = [\lambda]^{\aleph_0}$ *strongly reflects* if for every sufficiently large regular cardinal θ, there exists a continuous elementary chain $\langle M_\xi : \xi < \omega_1 \rangle$

of countable submodels of H_θ containing λ and X such that the set $\{\xi < \omega_1 : M_\xi \cap \lambda \in X\}$ is stationary in ω_1, where H_θ is the family of sets x with hereditary cardinality less than θ, that is, $\{x \in V_\theta : \textit{the cardinality of the transitive closure of } x \textit{ is less than } \theta\}$.

Now, we shall present the condition in terms of distributivity which implies that every stationary set in T strongly reflects.

Again, Let $T = [\lambda]^{\aleph_0}$ and let θ be any regular cardinal with $\aleph_1^{\aleph_0} < \aleph_2 \le \lambda$ and $\lambda^{\aleph_0} < \theta$. Let I be a $\prec_S$-fine $\aleph_2$-complete $\prec_S$-normal $\langle \aleph_1, J, (S; \lambda^{\aleph_0})\rangle$-distributive ideal on S, where $S = P_{<\aleph_2}(H_\theta)$ and S and T are considered as partially ordered sets with the inclusion relation. Let σ_3 be the function on S defined by $\sigma_3(N) = [N \cap \lambda]^{\aleph_0}$, and let $C = \{N \in S : |N| = \aleph_1,\ N \prec_e H_\theta \textit{ and } \lambda \in N\}$, where $N \prec_e H_\theta$ asserts that N is an elementary substructure of H_θ.

Clearly we have that:

(1) if $N_1 \prec_S N_2$, then $\sigma_3(N_1) \subseteq \sigma_3(N_2)$,
(2) the set $\{N \in S : t \in \sigma_3(N)\}$ has I-measure one for every $t \in T$,
(3) the set $\{N \in S : cf_{\prec_T}(\sigma_3(N)) > \aleph_0\}$ has I-measure one,
(4) if g is a function on a set $A \in I^+$ with $g(a) \in \sigma_3(N)$, then there exists a subset B of A of positive I-measure such that $g\lceil B$ is constant, and
(5) C is $\prec_S$-club.

Hence by Theorem 6 and the remark after the proof of Lemma 7, for every $\prec_T$-stationary set X in T, the set $M_{\sigma_3}(X) \cap C$ has I-measure one. Picking an $N \in M_{\sigma_3}(X) \cap C$ with $\lambda \in N$ and $X \in N$, we can find a continuous elementary chain $\langle M_\xi : \xi < \omega_1\rangle$ of countable submodels of N (and so, of H_θ) in S such that $N = \bigcup_{\xi<\omega_1} M_\xi$, $\lambda \in M_\xi$ and $X \in M_\xi$ for all $\xi < \omega_1$. Clearly, for every club C in ω_1, the set $\{M_\xi \cap \lambda : \xi \in C\}$ is $\prec_T$-club in $\sigma_3(N)$. Since $X \cap \sigma_3(N)$ is $\prec_T$-stationary in $\sigma_3(N)$, the set $\{\xi < \omega_1 : M_\xi \cap \lambda \textit{ is in } X\}$ is stationary in ω_1.

Thus we have established the following:

Theorem 11. *Assume that* $\aleph_1^{\aleph_0} < \aleph_2 \le \lambda$ *and for any* θ *with* $\lambda^{\aleph_0} < \theta$ *there exists a* $\prec_T$*-fine* $\aleph_2$*-complete* $\prec_T$*-normal* $\langle \aleph_1, (S; \lambda^{\aleph_0})\rangle$*-distributive ideal* I_θ *on* $S = P_{<\omega_2}(H_\theta)$. *Then every* $\prec_T$*-stationary set* X *in* $T = [\lambda]^{\aleph_0}$ *strongly reflects.*

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