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ON POINTWISE CONVERGENCE OF FOURIER SERIES OF RADIAL FUNCTIONS IN SEVERAL VARIABLES

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Abstract. We prove the pointwise convergence of the Fourier series for radial functions in several variables, which in the case $n = 1$ is the Dirichlet-Jordan theorem itself. In our proof the method for the case of the indicator function of the ball is very useful.

1. Introduction

Z^n denotes the n dimensional integer lattice, whose points are written $m = (m_1, \cdots, m_n)$, where m_k are any integers. R^n denotes the n dimensional Euclidean spaces, whose points are written $x = (x_1, \cdots, x_n)$, and for x, $y \in R^n$, its inner product is $xy = \sum_{k=1}^n x_k y_k$. Q^n denotes the set whose points are $x = (x_1, \cdots, x_n)$, where x_k are any rational numbers.

We aim to consider the pointwise behavior of the Fourier series of some sort of bounded variation function in several variables.

The Fourier transform and its spherical partial sum of a function f on R^n are defined by

$$\hat{f}(\xi) := \int_{R^n} f(x) e^{-2\pi i \xi x} dx \qquad (\xi \in R^n)$$

and

$$f_\lambda(x) := \int_{|\xi|<\lambda} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

Next, the Fourier series and its spherical sum of a function F on T^n, where $T^n = [-\frac{1}{2}, \frac{1}{2}]^n$, are defined by

$$\hat{F}(m) := \int_{T^n} F(x) e^{-2\pi i m x} dx \qquad (m \in Z^n)$$

and

$$S_\lambda(F, x) := \sum_{|m|<\lambda} \hat{F}(m) e^{2\pi i m x}.$$

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M. A. Pinsky and others [7], [8] considered the pointwise convergence of the Fourier inversion of radial functions in several variables and the convergence at $x = 0$ of the Fourier series of the indicator function of the ball.

The indicator function of ball with radius $a > 0$ and center 0 is defined by $\chi_a(x) := 1$ for $|x| \leq a$ and $\chi_a(x) := 0$ for $|x| > a$. Also we defined $\overline{\chi_a}(x) := 1$ for $|x| < a$, $\overline{\chi_a}(x) := \frac{1}{2}$ for $|x| = a$, and $\overline{\chi_a}(x) := 0$ for $|x| > a$. Then we have

$$\hat{\chi_a}(\xi) = \begin{cases} a^{\frac{n}{2}} \frac{J_{\frac{n}{2}}(2\pi|\xi|a)}{|\xi|^{\frac{n}{2}}} & \text{for } \xi \neq 0, \\ \frac{\pi^{\frac{n}{2}} a^n}{\Gamma(\frac{n}{2}+1)} & \text{for } \xi = 0, \end{cases}$$

and

$$\chi_{a,\lambda}(x) = \begin{cases} (\frac{a}{|x|})^{\frac{n}{2}-1} \int_0^{2\pi a\lambda} J_{\frac{n}{2}}(s) J_{\frac{n}{2}-1}(\frac{|x|}{a}s)ds, & \text{for } x \neq 0, \\ \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} \int_0^{2\pi a\lambda} s^{\frac{n}{2}-1} J_{\frac{n}{2}}(s)ds, & \text{for } x = 0, \end{cases}$$

where J_α is the Bessel function of the first kind of order α.

Lemma 1. *Suppose $a > 0$ and φ is a function of bounded variation, and*

$$f_\varphi(x) := \begin{cases} \varphi(t) & \text{for } |x| = t < a, \\ 0 & \text{for } |x| > a. \end{cases}$$

Then we have

$$\hat{f_\varphi}(\xi) = \varphi(a)\hat{\chi_a}(\xi) - \int_0^a \hat{\chi_t}(\xi) d\varphi(t),$$

where $d\varphi$ is the Lebesgue-Stieltjes measure generated by φ.

Proof.

$$\hat{f_\varphi}(\xi) = \int_0^a \varphi(t) t^{n-1} (\int_{\Sigma_{n-1}} e^{2\pi i t|\xi| x'\xi'} d\sigma_{x'}) dt$$

$$= 2\pi \int_0^a \varphi(t) t^{n-1} \frac{J_{\frac{n}{2}-1}(2\pi t|\xi|)}{(t|\xi|)^{\frac{n}{2}-1}} dt$$

$$= \frac{2\pi}{|\xi|^{\frac{n}{2}-1}} \int_0^a \varphi(t) t^{\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi t|\xi|) dt = \frac{1}{|\xi|^{\frac{n}{2}}} \int_0^a \varphi(t) \frac{d}{dt}\left(t^{\frac{n}{2}} J_{\frac{n}{2}}(2\pi t|\xi|)\right) dt$$

$$= \varphi(a) a^{\frac{n}{2}} \frac{J_{\frac{n}{2}}(2\pi a|\xi|)}{|\xi|^{\frac{n}{2}}} - \frac{1}{|\xi|^{\frac{n}{2}}} \int_0^a t^{\frac{n}{2}} J_{\frac{n}{2}}(2\pi t|\xi|) d\varphi(t)$$

$$= \varphi(a)\hat{\chi_a}(\xi) - \int_0^a \hat{\chi_t}(\xi) d\varphi(t).$$

This completes the proof of our lemma.

Next, we introduce two periodic functions F_t and F_φ with respect to each coordinate with *period* 1:

$$F_t(x) := \sum_{m \in Z^n} \chi_t(x + m)$$

and

$$F_\varphi(x) := \sum_{m\in Z^n} f_\varphi(x+m).$$

In particular, in the case of $0 < t < \frac{1}{2}$, F_t is equal on T^n to the indicator function of the ball with radius t and center 0.

Then we have the following lemma.

Lemma 2.

$$S_\lambda(F_\varphi : x) = \varphi(a)S_\lambda(F_a : x) - \int_0^a S_\lambda(F_t : x)d\varphi(t).$$

Proof. Since $\hat{F_t}(m) = \hat{\chi_t}(m)$ and $\hat{F_\varphi}(m) = \hat{f_\varphi}(m)$ by the Poisson Summation Formula, we have by Lemma 1

$$\hat{F_\varphi}(m) = \varphi(a)\hat{\chi_a}(m) - \int_0^a \hat{\chi_t}(m)d\varphi(t).$$

Therefore we have

$$S_\lambda(F_\varphi : x) = \varphi(a)\sum_{|m|<\lambda}\hat{\chi_a}(m)e^{2\pi imx} - \int_0^a\Big(\sum_{|m|<\lambda}\hat{\chi_t}(m)e^{2\pi imx}\Big)d\varphi(t)$$

$$= \varphi(a)S_\lambda(F_a : x) - \int_0^a S_\lambda(F_t : x)d\varphi(t).$$

This completes the proof of Lemma 2.

The aim of this paper is to consider the behavior of $S_\lambda(F_\varphi : x)$. Some related papers are Hardy-Landau [3], Pinsky, Stanton and Trapa [8], Pinsky [7], Kuratsubo [4], and L. Brandolini and L. Colzani [1].

Theorem (Hardy-Landau [3])**.** *If $n = 2$ and $a > 0$, then*

$$\lim_{\lambda\to\infty}\{\pi a + a\sum_{0<|m|<\lambda}\frac{J_1(2\pi a|m|)}{|m|}\} = \sum_{|m|<a} 1 + \frac{1}{2}\sum_{|m|=a} 1.$$

Theorem (Pinsky, Stanton and Trapa [8])**.** *If $0 < a < \frac{1}{2}$ and $n \geq 3$, then the Fourier series of the indicator function of the ball $|x| \leq a$ diverges at center $x = 0$.*

The results of Hardy-Landau [3] show that in the case of $n = 2$ the Fourier series of the indicator function of the ball $|x| \leq a$ converges to $\overline{\chi_a}(0)$ at center $x = 0$.

Pinsky and others [8] noted the equality:

$$\sum_{|m|\leq\lambda}\hat{\chi_a}(m) = \omega_{n-1}\int_0^\lambda \Lambda(s)s^{n-1}ds$$

$$+\Lambda(\lambda)(\sum_{|m|\leq\lambda} 1 - v_n\lambda^n) - \int_0^\lambda \Lambda'(s)(\sum_{|m|\leq s} 1 - v_n s^n)ds,$$

where ω_{n-1}, v_n are the area of the n dimensional sphere and the volume of the n dimensional ball and

$$\Lambda(s) = a^{\frac{n}{2}}\frac{J_{\frac{n}{2}}(2\pi as)}{s^{\frac{n}{2}}}.$$

Then, the first term is the leading one and the estimation of $\sum_{|m|\le\lambda} 1 - v_n\lambda^n$ is the lattice-points problem itself. The method of Kuratsubo [4] originates from the idea of Hardy-Landau.

Theorem (Kuratsubo [4]). *For $x \neq 0$ we have the following:*

(1) *If $2 \le n \le 4$, $\lim_{\lambda\to\infty} S_\lambda(F_a : x) = \overline{F_a}(x)$.*

(2) *If $n \ge 5$ and $x \notin Q^n$, $\limsup_{\lambda\to\infty}\left(\lambda^{\frac{5-n}{2}}|S_\lambda(F_a : x) - \overline{F_a}(x)|\right) = 0$.*

(3) *If $n \ge 5$ and $x \in Q^n$, $\limsup_{\lambda\to\infty}\left(\lambda^{\frac{7-n}{2}}|S_\lambda(F_a : x) - \overline{F_a}(x)|\right) < \infty$.*

(4) *If $n \ge 6$, $\lim_{\lambda\to\infty} S_\lambda(F_a : x) = \overline{F_a}(x)$ for almost all x.*

The main results of our paper are the following Theorems 1, 2 and 3. These theorems are extensions of Kuratsubo [4] from the case of the indicator function of a ball to the more general case of radial functions of bounded variation.

Theorem 1. *If $1 \le n \le 4$, then for $x \neq 0$, we have*

$$\lim_{\lambda\to\infty} S_\lambda(F_\varphi : x) = \varphi(a)\overline{F_a}(x) - \int_0^a \overline{F_t}(x)d\varphi(t)$$

$$(*) \qquad = \sum_{|x+m|<a} \frac{\varphi(|x+m|+) + \varphi(|x+m|-)}{2} + \frac{\varphi(a) + \varphi(a-)}{2} \sum_{|x+m|=a} 1.$$

Theorem 2. *If $n = 5$, then for any $x \notin Q^n$, we have $(*)$.*

Theorem 3. *If $n \ge 6$, for almost everywhere x, we have $(*)$.*

2. Some results on lattice point problems

A key to our methods is in lattice point problems. On these problems Fricker [2] is very thorough. Our interest is to estimate the following:

$$P_\alpha(t : x) := K_\alpha(t : x) - \frac{\pi^{\frac{n}{2}} t^{\frac{n}{2}+\alpha}}{\Gamma(\frac{n}{2} + \alpha + 1)}\delta(x),$$

where $\alpha \ge 0$,

$$K_\alpha(t : x) := \frac{1}{\Gamma(\alpha + 1)} \sum_{|m|^2<t} (t - |m|^2)^\alpha e^{2\pi i m x},$$

and $\delta(x) := 1$ if $x \in Z^n$ and $\delta(x) := 0$ otherwise. These problems have been studied by Landau, Jarník, Szegö, Novák and others. The important results are the following:

Theorem (Landau [5]). *If $n \ge 2$ and $x \in R^n$, then*

$$P_\alpha(t : x) = \begin{cases} O(t^{\frac{n-1}{4}+\frac{\alpha}{2}}) & \text{if } \alpha > \frac{n-1}{2}, \\ O(t^{\frac{n-1}{4}+\frac{\alpha}{2}} \log t) & \text{if } \alpha = \frac{n-1}{2}, \\ O(t^{\frac{n}{2}+\alpha-\frac{n}{n+1-2\alpha}}) & \text{if } 0 \le \alpha < \frac{n-1}{2} \end{cases}.$$

Theorem (Novák [6]). *If $0 \le \alpha < \frac{n}{2} - 2$, then*

$$P_\alpha(t:x) = \begin{cases} O(t^{\frac{n}{2}-1}) \ , \ \Omega(t^{\frac{n}{2}-1}) & \text{for } x \in Q^n, \\ o(t^{\frac{n}{2}-1}) & \text{for } x \notin Q^n, \\ O(t^{\frac{n}{4}+\frac{\alpha}{2}} \log^\tau t) & \text{for almost all } x, \end{cases}$$

where $\tau = 3n - 1$ if $\alpha > 0$ and $\tau = 3n$ for $\alpha = 0$.

The proofs of our main theorems need the following lemmas. Let us define by k_0 the smallest of integers k such that it satisfies the inequality: $\frac{n-1}{2} < k$.

Lemma 3. *If $n \ge 2$,*

$$P_\alpha(t:x) = O(t^{\psi_1(\alpha)}) \quad \text{for } x \in R^n \text{ and } 0 \le \alpha \le k_0,$$

where $\psi_1(\alpha)$ is the linear function tying two points: $(0, \frac{n}{2} - \frac{n}{n+1})$, $(k_0, \frac{n-1}{4} + \frac{k_0}{2})$.

Lemma 4. *If $n \ge 5$,*

$$P_\alpha(t:x) = O(t^{\psi_2(\alpha)}) \quad \text{for } x \in R^n \text{ and } 0 \le \alpha \le k_0,$$

where $0 < \varepsilon < \frac{1}{2}$ and $\psi_2(\alpha)$ is the linear curve tying three points: $(0, \frac{n}{2} - 1)$, $(\frac{n}{2} - 2 - \varepsilon, \frac{n}{2} - 1)$, and $(k_0, \frac{n-1}{4} + \frac{k_0}{2})$.

Lemma 5. *If $n \ge 5$,*

$$P_\alpha(t:x) = O(t^{\psi_3(\alpha)}) \quad \text{for almost all } x \text{ and } 0 \le \alpha \le k_0,$$

where $\varepsilon > 0$ and $\psi_3(\alpha)$ is the linear function tying two points: $(0, \frac{n}{4} + \varepsilon)$ and $(k_0, \frac{n-1}{4} + \frac{k_0}{2})$.

The proofs of these lemmas are derived from the above two theorems and Riesz's convexity theorem:

Riesz's convexity theorem. *For any numerical sequence $\{c_k\}_{k=0}^\infty$, consider its Riesz means:*

$$\sigma_t^\alpha = \sum_{0 \le k < t} (1 - \frac{k}{t})^\alpha c_k \ , \ \alpha \ge 0.$$

Suppose that $\sigma_t^{\alpha_j} = O(t^{a_j})$ as $t \to \infty$, for $j = 0, 1$. Then,

$$\text{if } 0 < \theta < 1 \ , \ \alpha = \alpha_0(1-\theta) + \alpha_1\theta \ , \text{ and } a = a_0(1-\theta) + a_1\theta \ ,$$

we can conclude that $\sigma_t^\alpha = O(t^a)$ as $t \to \infty$.

(See Stein-Weiss [9], p. 285.)

3. Proofs of main theorems

The starting point of our proofs is the following theorem.

Theorem (Kuratsubo [4]). *Suppose k_0 is the smallest of integers k which satisfies $k > \frac{n-1}{2}$; we have*

$$S_\lambda(F_t:x) = \overline{F_t}(x) + (\chi_{t,\lambda}(0) - 1)\delta(x) + t^{\frac{n}{2}} \sum_{l=0}^{k_0} (\pi t)^l P_l(\lambda^2 : x) \Lambda_l(t : \lambda^2)$$

$$+ \sum_{m-x \ne 0} \left(\frac{t}{|m-x|} \right)^{\frac{n}{2}+k_0} \int_{2\pi t\lambda}^\infty J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t} s) J_{\frac{n}{2}+k_0+1}(s) ds,$$

where $\delta(x) := 1$ *if* $x \in Z^n$ *and* $\delta(x) := 0$ *if* $x \notin Z^n$, *and*

$$\overline{F_t}(x) := \sum_m \overline{\chi_t}(m+x) \text{ and } \Lambda_\alpha(t:\lambda) := \frac{J_{\frac{n}{2}+\alpha}(2\pi t\sqrt{\lambda})}{\lambda^{\frac{n}{4}+\frac{\alpha}{2}}}.$$

Furthermore, the last term is absolutely convergent with respect to m *and* $o(1)$ *as* $\lambda \to \infty$.

(See Kuratsubo [4].)

Lemma 6. *For any* x, *we have*

$$S_\lambda(F_\varphi : x) = \Bigg[\{\varphi(a)\overline{F_a}(x) - \int_0^a \overline{F_t}(x)d\varphi(t)\} + \{\varphi(a)(\chi_{a,\lambda}(0)-1)$$

$$-\int_0^a (\chi_{t,\lambda}(0)-1)d\varphi(t)\}\delta(x) + \varphi(a)a^{\frac{n}{2}}\sum_{l=0}^{k_0}(\pi a)^l P_l(\lambda^2 : x)\Lambda_l(a:\lambda^2)$$

$$-\sum_{l=0}^{k_0} P_l(\lambda^2 : x)\int_0^a t^{\frac{n}{2}}(\pi t)^l \Lambda_l(t:\lambda^2)d\varphi(t)\Bigg] + o(1) \qquad \text{as } \lambda \to \infty.$$

Proof. Since

$$\int_0^a S_\lambda(F_t : x)d\varphi(t) = \int_0^a \overline{F_t}(x)d\varphi(t)$$

$$+\left(\int_0^a (\chi_{t,\lambda}(0)-1)d\varphi(t)\right)\delta(x) + \sum_{l=0}^{k_0} P_l(\lambda^2 : x)\int_0^a t^{\frac{n}{2}}(\pi t)^l\Lambda_l(t:\lambda^2)d\varphi(t)$$

$$+\sum_{m-x\neq 0}\int_0^a \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0}\{\int_{2\pi t\lambda}^\infty J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t}s)J_{\frac{n}{2}+k_0+1}(s)ds\}d\varphi(t),$$

we conclude the proof of Lemma 6 from the following lemma.

Throughout this section, the notation $A \ll B$ implies $|A| \le CB$ for some absolute constant C.

Lemma 7. *The infinite series*

$$\Sigma(\lambda) := \sum_{m-x\neq 0}\int_0^a \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0}\{\int_{2\pi t\lambda}^\infty J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t}s)J_{\frac{n}{2}+k_0+1}(s)ds\}d\varphi(t)$$

is absolutely convergent and

$$\Sigma(\lambda) = o(1) \qquad \text{as } \lambda \to \infty.$$

Proof. Firstly in the case of $|m-x| \ge 2a$,

$$|\int_0^a \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0}\{\int_{2\pi t\lambda}^\infty J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t}s)J_{\frac{n}{2}+k_0+1}(s)ds\}d\varphi(t)|$$

$$\ll \int_0^a \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0}\int_{2\pi t\lambda}^\infty |J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t}s)J_{\frac{n}{2}+k_0+1}(s)ds||d\varphi|(t)$$

$$\ll \int_0^a \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0} \left(\frac{1}{\left(\frac{|m-x|}{t}\right)^{\frac{1}{2}}} | \int_{||m-x|-t|2\pi\lambda}^{\infty} \frac{\sin s}{s} ds| \right.$$

$$\left. + \frac{1}{\left(\frac{|m-x|}{t}\right)^{\frac{1}{2}}} \frac{1}{t\lambda}(1+\frac{1}{t\lambda}) \right) |d\varphi|(t)$$

$$\ll \frac{1}{|m-x|^{\frac{n+1}{2}+k_0}} \frac{1}{\lambda} \int_0^a \left(t^{\frac{n+1}{2}+k_0} + t^{\frac{n-1}{2}+k_0}(1+\frac{1}{t\lambda}) \right) |d\varphi|(t)$$

$$\ll \frac{1}{\lambda} \frac{1}{|m-x|^{\frac{n+1}{2}+k_0}}.$$

From this estimation the series $\Sigma(\lambda)$ is absolutely convergent.

Secondly in the case of $|m-x| < 2a$,

$$|\left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0} \left\{ \int_{2\pi t\lambda}^{\infty} J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t}s) J_{\frac{n}{2}+k_0+1}(s) ds \right\}|$$

$$\ll \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0} \left(\frac{1}{\left(\frac{|m-x|}{t}\right)^{\frac{1}{2}}} + \left(\frac{|m-x|}{t}\right)^{\frac{n}{2}+k_0} \right)$$

$$\ll \left(\frac{t}{|m-x|}\right)^{\frac{n+1}{2}+k_0} + 1 \ll \left(\frac{a}{|m-x|}\right)^{\frac{n+1}{2}+k_0} + 1 \qquad \text{for } 0 \le t \le a$$

and

$$\lim_{\lambda\to\infty} \left(\frac{t}{|m-x|}\right)^{\frac{n}{2}+k_0} \int_{2\pi t\lambda}^{\infty} J_{\frac{n}{2}+k_0}(\frac{|m-x|}{t}s) J_{\frac{n}{2}+k_0+1}(s) ds = 0 \qquad \text{for } 0 \le t \le a.$$

Therefore by Lebesgue's convergence theorem we conclude the proof of Lemma 7. □

In the proof of Lemma 7 we used some properties of the Bessel function.

Property. *Suppose $\beta > 0$. If $A \ge \beta$ and $\omega > 0$, then*

$$|\int_{\omega}^{\infty} J_{\alpha-1}(As) J_{\alpha}(s) ds + \frac{sign(A-1)}{\pi\sqrt{A}} \int_{|A-1|\omega}^{\infty} \frac{\sin s}{s} ds| \le \frac{C}{\sqrt{A}\omega}\left(1+\frac{1}{\omega}\right),$$

where C is a constant number depending upon β and α only.

In particular

$$\lim_{\omega\to\infty} \int_{\omega}^{\infty} J_{\alpha-1}(As) J_{\alpha}(s) ds = 0$$

and

$$|\int_{\omega}^{\infty} J_{\alpha-1}(As) J_{\alpha}(s) ds| \le C(\frac{1}{\sqrt{A}} + A^{\alpha-1}).$$

(See Hardy-Landau [3] and Kuratsubo [4].)

Proof of Theorem 1. Since if $2 \le n \le 4$, then $\psi_1(\alpha) < \frac{n+1}{4} + \frac{\alpha}{2}$, by Lemma 3 and $\Lambda_l(t:\lambda^2) = O(\frac{1}{\lambda^{\frac{n+1}{2}+\alpha}})$. Therefore $P_l(\lambda^2:x)\Lambda_l(a:\lambda^2)$ and

$$P_l(\lambda^2:x)\int_0^a t^{\frac{n}{2}}(\pi t)^l \Lambda_l(t:\lambda^2)d\varphi(t) = o(1)$$

for $0 \le l \le k_0$. This completes the proof of Theorem 1. □

Proof of Theorem 2. We have the following by Lemma 4:

$$\psi_2(\alpha) < \frac{n+1}{4} + \frac{\alpha}{2} \quad \text{for} \quad \frac{n}{2} - 2 \le \alpha \le k_0.$$

Therefore we have

$$P_\alpha(\lambda^2:x)\Lambda_\alpha(a:\lambda^2) = \begin{cases} o(\lambda^{n-2})O(\frac{1}{\lambda^{\frac{n+1}{2}}}) & \text{for } \alpha = 0, \\ o(\lambda^{n-2})O(\frac{1}{\lambda^{\frac{n+1}{2}+\alpha}}) & \text{for } 0 < \alpha < \frac{n}{2} - 2, \\ O(\lambda^{2\psi_2(\alpha)})O(\frac{1}{\lambda^{\frac{n+1}{2}+\alpha}}) & \text{for } \frac{n}{2} - 2 \le \alpha \le k_0 \end{cases}$$

$$= o(1) \qquad \text{for } 0 \le \alpha \le k_0$$

and

$$P_\alpha(\lambda^2:x)\int_0^a t^{\frac{n}{2}}(\pi t)^l \Lambda_l(t:\lambda^2)d\varphi(t) = o(1) \qquad \text{for } 0 \le \alpha \le k_0.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. If $n \ge 6$, by Lemma 5, we have the following for almost all x: $\psi_3(\alpha) < \frac{n+1}{4} + \frac{\alpha}{2}$, for $\alpha \ge 0$, and $\Lambda_\alpha(t:\lambda^2) = O(\frac{1}{\lambda^{\frac{n+1}{2}+\alpha}})$. Therefore $P_l(\lambda^2:x)\Lambda_l(\lambda^2)$ and $P_\alpha(\lambda^2:x)\int_0^a t^{\frac{n}{2}}(\pi t)^l\Lambda_l(t:\lambda^2)d\varphi(t) = o(1)$ for $0 \le l \le k_0$. This completes the proof of Theorem 3. □

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