A BESSSEL FUNCTION MULTIPLIER

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Abstract. We obtain nearly sharp estimates for the $L^p(\mathbb{R}^2)$ norms of certain convolution operators.

For $n \geq 1$, let $\lambda_n$ be the measure on $\mathbb{R}^2$ obtained by multiplying normalized arclength measure on $\{|x|=1\}$ by the oscillating factor $e^{in \arg(x)}$. For $1 \leq p \leq \infty$, let $C(p,n)$ denote the norm of the operator $T_n f = \lambda_n * f$ on $L^p(\mathbb{R}^2)$. The purpose of this note is to estimate the rate of decay of $C(p,n)$ as $n \to \infty$. By duality, it is enough to consider $p \geq 2$. Examples below will show that

$$C(p,n) \geq C(p)n^{-\frac{1}{p}-\frac{1}{p^*}} \quad \text{if} \quad 2 \leq p \leq 4,$$

and

$$C(p,n) \geq C(p)n^{-\frac{1}{p}} \quad \text{if} \quad 4 \leq p \leq \infty.$$ 

On the other hand, we will observe that

$$C(2,n) \leq Cn^{-\frac{1}{2}},$$

$$C(\infty,n) \leq C$$

and then prove the following result.

Theorem. There is a positive number $a$ such that

$$C(4,n) \leq Cn^{-\frac{1}{p}}(\log(n))^a.$$ 

Interpolating (3) and (4) gives upper bounds for $C(p,n)$ which differ only by a power of $\log(n)$ from the lower bounds of (1) and (2), thus providing nearly sharp estimates for $C(p,n)$.

The above question naturally arises when considering the $L^p(\mathbb{R}^3)$ mapping properties of the operator $T$ given by convolution with respect to a compact piece of arclength measure on the helix

$$t \to (\cos t, \sin t, t).$$

$T$ is an example of a folding Fourier integral operator in dimension 3, whose singular set is of dimension 1. The sharp $L^p \to L^2$ mapping properties of $T$ were established by the first author in [O]. The operator $T_n$ arises when considering the

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$L^p$ smoothing properties of $T$; that is, for which values of $\alpha_p$ is $|D|^\alpha T$ bounded on $L^p(\mathbb{R}^3)$. Since
\[
T(e^{-inx}f(x_1, x_2)) = e^{-inx}(T_n f)(x_1, x_2),
\]
the exponents in (1) and (2) give upper bounds on $\alpha_p$. In particular, the smoothing exponent for $T$ is less than that of averaging in $\mathbb{R}^2$ over the cubic $t \to (t, t^3)$, where the corresponding value of $\alpha_p$ is
\[
\alpha_p = \begin{cases} 
\frac{1}{3} & \text{if } 2 \leq p < 3, \\
\frac{1}{p} & \text{if } 3 < p < \infty . 
\end{cases}
\]
See, for example, [SW] or [SS]. The authors would like to thank Chris Sogge for discussions which led to consideration of this question.

To see (2), apply the operator $T_n f = \lambda_n * f$ to $f(x) = e^{-in\arg(x)}\chi_A(x)$ where $A$ is the annulus $|x| \leq 1 + 1/n$. One observes that there is a constant $C$ such that $|T_n f(x)| \geq C$ if $|x| \leq C/n$ and (2) follows (for all $p$, but (1) is better for $p \leq 4$).

The example for (1) is a little more complicated: for fixed $n$, and $1 \leq j \leq n^{1/3}$, let $\theta_j = jn^{-1/3}$, $\omega_j = (\cos(\theta_j), \sin(\theta_j))$, and $\omega'_j = (-\sin(\theta_j), \cos(\theta_j))$. Let $B_j$ be the disk $\{|x-\omega_j| \leq \varepsilon n^{-\frac{2}{3}}\}$ where $\varepsilon$ is a positive number independent of $n$ and small enough to insure that, for any $n$, the disks $B_j$ are pairwise disjoint. Let
\[
f_j(x) = e^{in(x \cdot \omega'_j)}\chi_{B_j}(x).
\]
One can check that
\[
|T_n f_j(x)| \geq cn^{-\frac{1}{3}} \quad \text{if } |x| \leq cn^{-\frac{1}{3}}
\]
for some small positive $c$ independent of $n$ and $j$. Let $r_j$ be the $j$th Rademacher function on $[0,1]$ and put
\[
f(t,x) = \sum_{j=1}^{n^\frac{1}{3}} r_j(t) f_j(x).
\]
Then
\[
\|f(t, \cdot)\|_p \leq Cn^{-\frac{1}{2}}.
\]
Also
\[
\int_0^1 \|T_n f(t, \cdot)\|_p^p \, dt \geq \int_{|x| \leq cn^{-1/3}} \left(\sum_j |T_n f_j(x)|^2\right)^{p/2} \, dx \geq c^2 p n^{-\frac{2}{3} - \frac{p}{2}},
\]
where the third inequality uses (5). With (6) this yields (1).

A computation shows that $\hat{T_n}(\xi) = e^{in\arg(\xi)}J_n(|\xi|)$ (whence the name of this note). Thus (3) follows from the estimate, uniform in $n$,
\[
|J_n(r)| \leq C r^{-\frac{1}{2}} \quad \text{if } r \geq 1
\]
(see p.357 in [S]) combined with the observation
\[
|J_n(r)| \leq \frac{C}{n} \quad \text{if } 0 \leq r \leq \frac{3n}{4}.
\]
To begin the proof of (4), let \( \rho \) be a smooth cutoff function which is equal to 1 on the annulus \( \{ \frac{1}{4} \leq |\xi| \leq \frac{5}{4} \} \) and is supported in the annulus \( \{ \frac{1}{2} \leq |\xi| \leq \frac{3}{2} \} \). Let \( S_n \) be the operator defined by \( \hat{S}_n(\xi) = \hat{T}_n(\xi)\rho(|n^{-1}\xi|) \). The easy estimate

\[
|J_n(r)| \leq Cn^{-\frac{1}{2}} \quad \text{if} \quad r \geq \frac{5n}{4}
\]

combines with (7) to show that the \( L^2(\mathbb{R}^2) \) operator norm \( \|T_n - S_n\|_{2,2} \) is \( O(n^{-\frac{1}{2}}) \). Interpolating this with \( \|T_n - S_n\|_{\infty,\infty} = O(1) \) yields \( \|T_n - S_n\|_{4,4} = O(n^{-\frac{1}{2}}) \). Thus (4) will follow from

\[
\|S_n\|_{4,4} \leq Cn^{-\frac{1}{2}}(\log(n))^a,
\]

which is our principal result. The Fourier transform \( \hat{S}_n(\xi) \) is supported in the annulus \( A_n = \{ \frac{n}{2} \leq |\xi| \leq \frac{3n}{2} \} \). Having fixed \( n \), we will decompose \( S_n \) by decomposing \( A_n \) into a union of annuli \( A_n^j \) as follows:

for \( j \geq 1 \), set \( A_n^j = \{ n + 2^j n^{\frac{1}{2}} \leq |\xi| \leq n + 2^{j+1} n^{\frac{1}{2}} \} \);

set \( A_n^0 = \{ n - 2n^{\frac{1}{2}} \leq |\xi| \leq n + 2n^{\frac{1}{2}} \} \);

for \( j \leq -1 \), set \( A_n^j = \{ n - 2|j+1| n^{\frac{1}{2}} \leq |\xi| \leq n - 2|j| n^{\frac{1}{2}} \} \).

Introducing a suitable partition of unity on the Fourier transform side leads to the decomposition

\[ S_n = \sum_j S_n^j. \]

For fixed \( n \), the number of terms \( S_n^j \) is \( O(\log(n)) \). Thus (8) will follow from

\[
\|S_n^j\|_{4,4} \leq Cn^{-\frac{1}{2}}(\log(n))^b
\]

for all \( j \) and \( n \) and some \( b > 0 \). At this point we make a further decomposition of \( A_n^j \) into sectors \( A_n^{jl} \) of opening angle \( \delta = 2^{|j|/2}n^{-\frac{1}{4}} \). This leads to a decomposition

\[ S_n^j = \sum_{l=1}^{\delta^{-1}} S_n^{jl}. \]

The function \( \hat{S}_n^{jl} \) is supported in a set \( R_n^{jl} \) obtained from the intersection of the annulus \( n + \frac{1}{2}n\delta^2 \leq |\xi| \leq n + 3n\delta^2 \) with a sector of angle \( \delta \); thus, \( R_n^{jl} \) is essentially a rectangle of dimensions \( n\delta \) by \( n\delta^2 \), with major dimension \( n\delta \) normal to the vector through the center of \( R_n^{jl} \).

**Lemma.**

\[
\|S_n^{jl}\|_{4,4} \leq Cn^{-\frac{1}{4}}\delta^{\frac{1}{2}}.
\]

**Proof.** We will obtain the lemma by interpolating the following estimates:

\[
\|S_n^{jl}\|_{2,2} \leq C(n\delta)^{-\frac{1}{2}},
\]

(10) \[
\|S_n^{jl}\|_{\infty,\infty} \leq C\delta.
\]

The first estimate in (10) is a bound on \( J_n(r) \) over the annulus \( A_n^j \). The desired estimates are well known, but we provide the simple argument here for completeness.
For \( j = 0 \), the desired bounds follow from the uniform bound \( |J_n(r)| \leq C n^{-\frac{1}{2}} \).

For \( j \neq 0 \), it suffices to show that
\[
\left| \int_0^\pi e^{int - in(1 \pm \delta^2) \sin t} dt \right| \leq C (n\delta)^{-\frac{1}{2}},
\]
where \( C \) is uniform over \( n \in \mathbb{Z} \) and \( \delta^2 \leq 1/2 \).

We let \( \phi(t) = t - (1 \pm \delta^2) \sin t \). On the interval \( 0 \leq t \leq \delta \), we have \( |\phi'(t)| \geq c\delta^2 \), and \( \phi'(t) \) is monotonic, so Proposition 2 of [S], page 332, implies that
\[
\left| \int_0^\delta e^{int - in(1 \pm \delta^2) \sin t} dt \right| \leq C (n\delta)^{-1} \leq C (n\delta)^{-\frac{1}{2}}.
\]

On the interval \( \delta \leq t \leq \pi - \delta \), it follows that \( |\phi''(t)| \geq c\delta \), and the same proposition implies that
\[
\left| \int_\delta^{\pi-\delta} e^{int - in(1 \pm \delta^2) \sin t} dt \right| \leq C (n\delta)^{-\frac{1}{2}}.
\]

On the interval \( \pi - \delta \leq t \leq \pi \), \( |\phi'(t)| \geq 1 \), and the integral is bounded by \( n^{-1} \).

For the second estimate of \( (10) \), it suffices to consider the term \( S_n^{j0} \), associated to the rectangle \( R_n^j \) with center on the positive \( \xi_j^0 \) axis. The partition of unity element associated to this rectangle is of the form \( \hat{\psi}(n\delta^{-1} \xi_1, (n\delta^2)^{-1} (\xi_2 - n)) \), where \( \hat{\psi} \) is a Schwartz function, whose seminorms are bounded by constants independent of \( n, j, l \). Thus, the convolution kernel associated to \( S_n^{j0} \) is of the form
\[
K_n^{j0}(x) = n^{2\delta^3} \int_{-\pi}^\pi e^{in(x_2 - \sin t) + int} \hat{\psi}(n\delta(x_1 - \cos t), n\delta^2(x_2 - \sin t)) dt.
\]

We need to show that
\[
(11) \quad \int \left| K_n^{j0}(x) \right| dx \leq C \delta.
\]

The contribution from the integral over \( |t| \leq \delta \) trivially satisfies (11), so it suffices to consider the following term:
\[
\tilde{K}(x) = n^{2\delta^3} \int e^{in(t-\sin t)} \chi(\delta^{-1} t) \psi(n\delta(x_1 - \cos t), n\delta^2(x_2 - \sin t)) dt,
\]
where \( \chi(s) = 1 \) for \( |s| \geq 2 \), and \( \chi(s) = 0 \) for \( |s| \leq 1 \). Integration by parts yields
\[
\tilde{K}(x) = in\delta^3 \int e^{in(t-\sin t)} \frac{\partial}{\partial t} \left[ \frac{\chi(\delta^{-1} t)}{1 - \cos t} \psi(n\delta(x_1 - \cos t), n\delta^2(x_2 - \sin t)) \right] dt.
\]

The term where the derivative falls on the term in front of \( \psi \) satisfies (11), since
\[
\int \left| \frac{\partial}{\partial t} \left( \frac{\chi(\delta^{-1} t)}{1 - \cos t} \right) \right| dt \leq C\delta^{-2} \leq Cn\delta.
\]

The term where the derivative falls on the \( x_2 \) place of \( \psi \) also satisfies (11), since
\[
\int \left| \frac{\chi(\delta^{-1} t) \cos t}{1 - \cos t} \right| dt \leq C\delta^{-1}.
\]

The term where the derivative falls on the \( x_1 \) place of \( \psi \) would appear to lead to bounds comparable to \( \delta \log(\delta^{-1}) \); however, one further integration by parts shows that this term too satisfies (11). \( \qed \)
We now prove (9) by noting that the angle $\delta$ was chosen so that the sets $R^{ij} + R^{ij'}$ have bounded overlap for $R^{ij}$ and $R^{ij'}$ in the same quadrant, i.e., so that the orthogonality argument of [F] applies. This argument yields
\[
\left\| \sum_l S_{jl}^i f \right\|_4 \leq C \left( \sum_l |S_{jl}^i f|^2 \right)^{1/2} \left\| \sum_l |S_{jl}^i f|^4 \right\|^{1/4}.
\]
The number of indices $l$ is $O(\delta^{-1})$, so
\[
\sum_l |S_{jl}^i f(x)|^2 \leq C\delta^{-1/2} \left( \sum_l |S_{jl}^i f(x)|^4 \right)^{1/2}.
\]
With $\widehat{f}_{jl}$ representing the localisation of $\hat{f}$ to an appropriate sector, we thus have
\[
\left\| \sum_l S_{jl}^i f \right\|_4 \leq C\delta^{-1/2} \left( \sum_l |S_{jl}^i f|^4 \right)^{1/2} \left\| \sum_l |\widehat{f}_{jl}|^4 \right\|^{1/4} \leq Cn^{-1/2} \left( \sum_l |\widehat{f}_{jl}|^2 \right)^{1/2}.
\]
A result of Córdoba [C] gives
\[
\left\| \left( \sum_l |f_{jl}|^2 \right)^{1/2} \right\|_4 \leq C(\log(n))^{b} \|f\|_4
\]
for some positive $b$, which completes the proof of (9).

REFERENCES


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