

A BESSEL FUNCTION MULTIPLIER

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ABSTRACT. We obtain nearly sharp estimates for the $L^p(\mathbb{R}^2)$ norms of certain convolution operators.

For $n \geq 1$ let λ_n be the measure on \mathbb{R}^2 obtained by multiplying normalized arclength measure on $\{|x| = 1\}$ by the oscillating factor $e^{in \arg(x)}$. For $1 \leq p \leq \infty$, let $C(p, n)$ denote the norm of the operator $T_n f \doteq \lambda_n * f$ on $L^p(\mathbb{R}^2)$. The purpose of this note is to estimate the rate of decay of $C(p, n)$ as $n \rightarrow \infty$. By duality, it is enough to consider $p \geq 2$. Examples below will show that

$$(1) \quad C(p, n) \geq C(p)n^{-\frac{1}{6} - \frac{1}{3p}} \quad \text{if } 2 \leq p \leq 4,$$

and

$$(2) \quad C(p, n) \geq C(p)n^{-\frac{1}{p}} \quad \text{if } 4 \leq p \leq \infty.$$

On the other hand, we will observe that

$$(3) \quad \begin{aligned} C(2, n) &\leq Cn^{-\frac{1}{3}}, \\ C(\infty, n) &\leq C \end{aligned}$$

and then prove the following result.

Theorem. *There is a positive number a such that*

$$(4) \quad C(4, n) \leq Cn^{-\frac{1}{4}}(\log(n))^a.$$

Interpolating (3) and (4) gives upper bounds for $C(p, n)$ which differ only by a power of $\log(n)$ from the lower bounds of (1) and (2), thus providing nearly sharp estimates for $C(p, n)$.

The above question naturally arises when considering the $L^p(\mathbb{R}^3)$ mapping properties of the operator T given by convolution with respect to a compact piece of arclength measure on the helix

$$t \rightarrow (\cos t, \sin t, t).$$

T is an example of a folding Fourier integral operator in dimension 3, whose singular set is of dimension 1. The sharp $L^p \rightarrow L^2$ mapping properties of T were established by the first author in [O]. The operator T_n arises when considering the

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L^p smoothing properties of T ; that is, for which values of α_p is $|D|^{\alpha_p} T$ bounded on $L^p(\mathbb{R}^3)$. Since

$$T(e^{-inx_3} f(x_1, x_2)) = e^{-inx_3} (T_n f)(x_1, x_2),$$

the exponents in (1) and (2) give upper bounds on α_p . In particular, the smoothing exponent for T is less than that of averaging in \mathbb{R}^2 over the cubic $t \rightarrow (t, t^3)$, where the corresponding value of α_p is

$$\alpha_p = \begin{cases} \frac{1}{3} & \text{if } 2 \leq p < 3, \\ \frac{1}{p} & \text{if } 3 < p < \infty. \end{cases}$$

See, for example, [SW] or [SS]. The authors would like to thank Chris Sogge for discussions which led to consideration of this question.

To see (2), apply the operator $T_n f \doteq \lambda_n * f$ to $f(x) = e^{-in \arg(x)} \chi_A(x)$ where A is the annulus $\{1 \leq |x| \leq 1 + 1/n\}$. One observes that there is a constant C such that $|T_n f(x)| \geq C$ if $|x| \leq C/n$ and (2) follows (for all p , but (1) is better for $p \leq 4$).

The example for (1) is a little more complicated: for fixed n , and $1 \leq j \leq n^{1/3}$, let $\theta_j = jn^{-1/3}$, $\omega_j = (\cos(\theta_j), \sin(\theta_j))$, and $\omega'_j = (-\sin(\theta_j), \cos(\theta_j))$. Let B_j be the disk $\{|x - \omega_j| \leq \varepsilon n^{-1/3}\}$ where ε is a positive number independent of n and small enough to insure that, for any n , the disks B_j are pairwise disjoint. Let

$$f_j(x) = e^{in(x \cdot \omega'_j)} \chi_{B_j}(x).$$

One can check that

$$(5) \quad |T_n f_j(x)| \geq cn^{-\frac{1}{3}} \quad \text{if } |x| \leq cn^{-\frac{1}{3}}$$

for some small positive c independent of n and j . Let r_j be the j th Rademacher function on $[0, 1]$ and put

$$f(t, x) = \sum_{j=1}^{n^{\frac{1}{3}}} r_j(t) f_j(x).$$

Then

$$(6) \quad \|f(t, \cdot)\|_p \leq Cn^{-\frac{1}{3p}}.$$

Also

$$\int_0^1 \|T_n f(t, \cdot)\|_p^p dt \geq \int_{|x| \leq cn^{-1/3}} \left(\sum_j |T_n f_j(x)|^2 \right)^{p/2} dx \geq c^{2+p} n^{-\frac{2}{3} - \frac{p}{6}},$$

where the third inequality uses (5). With (6) this yields (1).

A computation shows that $\widehat{T}_n(\xi) = e^{in \arg(\xi)} J_n(|\xi|)$ (whence the name of this note). Thus (3) follows from the estimate, uniform in n ,

$$|J_n(r)| \leq Cr^{-\frac{1}{3}} \quad \text{if } r \geq 1$$

(see p.357 in [S]) combined with the observation

$$(7) \quad |J_n(r)| \leq \frac{C}{n} \quad \text{if } 0 \leq r \leq \frac{3n}{4}.$$

To begin the proof of (4), let ρ be a smooth cutoff function which is equal to 1 on the annulus $\{\frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}$ and is supported in the annulus $\{\frac{1}{2} \leq |\xi| \leq \frac{3}{2}\}$. Let S_n be the operator defined by $\widehat{S}_n(\xi) = \widehat{T}_n(\xi)\rho(|n^{-1}\xi|)$. The easy estimate

$$|J_n(r)| \leq Cn^{-\frac{1}{2}} \quad \text{if } r \geq \frac{5n}{4}$$

combines with (7) to show that the $L^2(\mathbb{R}^2)$ operator norm $\|T_n - S_n\|_{2,2}$ is $O(n^{-\frac{1}{2}})$. Interpolating this with $\|T_n - S_n\|_{\infty,\infty} = O(1)$ yields $\|T_n - S_n\|_{4,4} = O(n^{-\frac{1}{4}})$. Thus (4) will follow from

$$(8) \quad \|S_n\|_{4,4} \leq Cn^{-\frac{1}{4}}(\log(n))^a,$$

which is our principal result. The Fourier transform $\widehat{S}_n(\xi)$ is supported in the annulus $A_n = \{\frac{n}{2} \leq |\xi| \leq \frac{3n}{2}\}$. Having fixed n , we will decompose S_n by decomposing A_n into a union of annuli A_n^j as follows:

$$\text{for } j \geq 1, \text{ set } A_n^j = \{n + 2^j n^{\frac{1}{3}} \leq |\xi| \leq n + 2^{j+1} n^{\frac{1}{3}}\};$$

$$\text{set } A_n^0 = \{n - 2n^{\frac{1}{3}} \leq |\xi| \leq n + 2n^{\frac{1}{3}}\};$$

$$\text{for } j \leq -1, \text{ set } A_n^j = \{n - 2^{|j|+1} n^{\frac{1}{3}} \leq |\xi| \leq n - 2^{|j|} n^{\frac{1}{3}}\}.$$

Introducing a suitable partition of unity on the Fourier transform side leads to the decomposition

$$S_n = \sum_j S_n^j.$$

For fixed n , the number of terms S_n^j is $O(\log(n))$. Thus (8) will follow from

$$(9) \quad \|S_n^j\|_{4,4} \leq Cn^{-\frac{1}{4}}(\log(n))^b$$

for all j and n and some $b > 0$. At this point we make a further decomposition of A_n^j into sectors A_n^{jl} of opening angle $\delta \doteq 2^{|j|/2} n^{-\frac{1}{3}}$. This leads to a decomposition

$$S_n^j = \sum_{l=1}^{\delta^{-1}} S_n^{jl}.$$

The function \widehat{S}_n^{jl} is supported in a set R^{jl} obtained from the intersection of the annulus $n + \frac{1}{2}n\delta^2 \leq |\xi| \leq n + 3n\delta^2$ with a sector of angle δ ; thus, R^{jl} is essentially a rectangle of dimensions $n\delta$ by $n\delta^2$, with major dimension $n\delta$ normal to the vector through the center of R^{jl} .

Lemma.

$$\|S_n^{jl}\|_{4,4} \leq Cn^{-\frac{1}{4}}\delta^{\frac{1}{4}}.$$

Proof. We will obtain the lemma by interpolating the following estimates:

$$(10) \quad \|S_n^{jl}\|_{2,2} \leq C(n\delta)^{-\frac{1}{2}},$$

$$\|S_n^{jl}\|_{\infty,\infty} \leq C\delta.$$

The first estimate in (10) is a bound on $J_n(r)$ over the annulus A_n^j . The desired estimates are well known, but we provide the simple argument here for completeness.

For $j = 0$, the desired bounds follow from the uniform bound $|J_n(r)| \leq C n^{-\frac{1}{3}}$. For $j \neq 0$, it suffices to show that

$$\left| \int_0^\pi e^{int - in(1 \pm \delta^2) \sin t} dt \right| \leq C (n\delta)^{-\frac{1}{2}},$$

where C is uniform over $n \in \mathbb{Z}$ and $\delta^2 \leq 1/2$.

We let $\phi(t) = t - (1 \pm \delta^2) \sin t$. On the interval $0 \leq t \leq \delta$, we have $|\phi'(t)| \geq c\delta^2$, and $\phi'(t)$ is monotonic, so Proposition 2 of [S], page 332, implies that

$$\left| \int_0^\delta e^{int - in(1 \pm \delta^2) \sin t} dt \right| \leq C (n\delta^2)^{-1} \leq C (n\delta)^{-\frac{1}{2}}.$$

On the interval $\delta \leq t \leq \pi - \delta$, it follows that $|\phi''(t)| \geq c\delta$, and the same proposition implies that

$$\left| \int_\delta^{\pi - \delta} e^{int - in(1 \pm \delta^2) \sin t} dt \right| \leq C (n\delta)^{-\frac{1}{2}}.$$

On the interval $\pi - \delta \leq t \leq \pi$, $|\phi'(t)| \geq 1$, and the integral is bounded by n^{-1} .

For the second estimate of (10), it suffices to consider the term S_n^{j0} , associated to the rectangle R_n^{j0} with center on the positive ξ_2 axis. The partition of unity element associated to this rectangle is of the form $\widehat{\psi}((n\delta)^{-1}\xi_1, (n\delta^2)^{-1}(\xi_2 - n))$, where ψ is a Schwartz function, whose seminorms are bounded by constants independent of n, j, l . Thus, the convolution kernel associated to S_n^{j0} is of the form

$$K_n^{j0}(x) = n^2 \delta^3 \int_{-\pi}^\pi e^{in(x_2 - \sin t) + int} \psi(n\delta(x_1 - \cos t), n\delta^2(x_2 - \sin t)) dt.$$

We need to show that

$$(11) \quad \int |K_n^{j0}(x)| dx \leq C \delta.$$

The contribution from the integral over $|t| \leq \delta$ trivially satisfies (11), so it suffices to consider the following term:

$$\widetilde{K}(x) = n^2 \delta^3 \int e^{in(t - \sin t)} \chi(\delta^{-1}t) \psi(n\delta(x_1 - \cos t), n\delta^2(x_2 - \sin t)) dt,$$

where $\chi(s) = 1$ for $|s| \geq 2$, and $\chi(s) = 0$ for $|s| \leq 1$. Integration by parts yields

$$\widetilde{K}(x) = in\delta^3 \int e^{in(t - \sin t)} \frac{\partial}{\partial t} \left[\frac{\chi(\delta^{-1}t)}{1 - \cos t} \psi(n\delta(x_1 - \cos t), n\delta^2(x_2 - \sin t)) \right] dt.$$

The term where the derivative falls on the term in front of ψ satisfies (11), since

$$\int \left| \frac{\partial}{\partial t} \left(\frac{\chi(\delta^{-1}t)}{1 - \cos t} \right) \right| dt \leq C\delta^{-2} \leq Cn\delta.$$

The term where the derivative falls on the x_2 place of ψ also satisfies (11), since

$$\int \left| \frac{\chi(\delta^{-1}t) \cos t}{1 - \cos t} \right| dt \leq C\delta^{-1}.$$

The term where the derivative falls on the x_1 place of ψ would appear to lead to bounds comparable to $\delta \log(\delta^{-1})$; however, one further integration by parts shows that this term too satisfies (11). \square

We now prove (9) by noting that the angle δ was chosen so that the sets $R^{jl} + R^{j'l'}$ have bounded overlap for R^{jl} and $R^{j'l'}$ in the same quadrant, i.e., so that the orthogonality argument of [F] applies. This argument yields

$$\left\| \sum_l S_n^{jl} f \right\|_4 \leq C \left\| \left(\sum_l |S_n^{jl} f|^2 \right)^{\frac{1}{2}} \right\|_4.$$

The number of indices l is $O(\delta^{-1})$, so

$$\sum_l |S_n^{jl} f(x)|^2 \leq C \delta^{-\frac{1}{2}} \left(\sum_l |S_n^{jl} f(x)|^4 \right)^{\frac{1}{2}}.$$

With \widehat{f}_{jl} representing the localisation of \widehat{f} to an appropriate sector, we thus have

$$\begin{aligned} \left\| \sum_l S_n^{jl} f \right\|_4 &\leq C \delta^{-\frac{1}{4}} \left\| \left(\sum_l |S_n^{jl} f|^4 \right)^{\frac{1}{4}} \right\|_4 \\ &\leq C n^{-\frac{1}{4}} \left\| \left(\sum_l |f_{jl}|^4 \right)^{\frac{1}{4}} \right\|_4 \\ &\leq C n^{-\frac{1}{4}} \left\| \left(\sum_l |f_{jl}|^2 \right)^{\frac{1}{2}} \right\|_4. \end{aligned}$$

A result of Córdoba [C] gives

$$\left\| \left(\sum_l |f_{jl}|^2 \right)^{\frac{1}{2}} \right\|_4 \leq C (\log(n))^b \|f\|_4$$

for some positive b , which completes the proof of (9).

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