SOME LIE SUPERALGEBRAS ASSOCIATED TO THE WEYL ALGEBRAS

IAN M. MUSSON

(Communicated by Ken Goodearl)

Abstract. Let \( \mathfrak{g} \) be the Lie superalgebra \( \text{osp}(1,2r) \). We show that there is a surjective homomorphism from \( U(\mathfrak{g}) \) to the \( r \)th Weyl algebra \( A_r \), and we use this to construct an analog of the Joseph ideal. We also obtain a decomposition of the adjoint representation of \( \mathfrak{g} \) on \( A_r \) and use this to show that if \( A_r \) is made into a Lie superalgebra using its natural \( \mathbb{Z}_2 \)-grading, then \( A_r = \mathbb{k} \oplus [A_r,A_r] \). In addition, we show that if \([A_r,A_r]\) and \([A_s,A_s]\) are isomorphic as Lie superalgebras, then \( r = s \). This answers a question of S. Montgomery.

We work throughout over an algebraically closed field \( \mathbb{k} \) of characteristic zero.

If \( \mathfrak{g} \) is a simple Lie algebra different from \( \mathfrak{s\ell}(n) \), Joseph shows in [J2], that there is a unique completely prime ideal, \( J_0 \) whose associated variety is the closure of the minimal nilpotent orbit in \( \mathfrak{g}^* \). When \( \mathfrak{g} \) is the symplectic algebra \( \mathfrak{g} = \text{sp}(2r) \), this ideal may be constructed as follows. It is well known that the symmetric elements of degree two in the \( r \)th Weyl algebra \( A_r \) form a Lie algebra isomorphic to \( \text{sp}(2r) \) [D, Lemma 4.6.9]. Hence there is an algebra map \( \phi : U(\mathfrak{g}) \to A_r \) whose kernel is clearly completely prime and primitive. Since the image of \( \phi \) has Gel’fand Kirillov dimension \( 2r \), and this is the dimension of the minimal nilpotent orbit in \( \mathfrak{g}^* \) by [CM, Lemma 4.3.5], we have \( \ker \phi = J_0 \).

Now if \( \mathfrak{g} \) is a classical simple Lie superalgebra, and \( U(\mathfrak{g}) \) contains a completely prime primitive ideal different from the augmentation ideal, then \( \mathfrak{g} \) is isomorphic to an orthosymplectic algebra \( \text{osp}(1,2r) \) (Lemma 1). We observe that if \( \mathfrak{g} = \text{osp}(1,2r) \), then there is a surjective homomorphism \( U(\mathfrak{g}) \to A_r \) whose kernel \( J \) satisfies \( J \cap U(\mathfrak{g}_0) = J_0 \). The existence of this homomorphism has previously been shown in the Physics literature; see, for example, [F, pages 55 and 170]. It follows that \( \mathfrak{g} \) acts via the adjoint representation on \( A_r \), and we determine the decomposition of this representation explicitly.

This turns out to be a useful setting in which to study the Lie structure of certain associative algebras. A result of Herstein [He] states that if \( \mathfrak{A} \) is a simple algebra with center \( Z \), then \( [\mathfrak{A},\mathfrak{A}]/[\mathfrak{A},\mathfrak{A}] \cap Z \) is a simple Lie algebra, unless \( [\mathfrak{A}:Z] = 4 \), and \( Z \) has characteristic two. Additional results have been obtained for various generalized Lie structures in [BFM] and [Mo].
Let $A_r$ be the $r$th Weyl algebra over $k$ with generators $x_1, \ldots, x_r, \partial_1, \ldots, \partial_r$ such that $\partial_i x_j - x_j \partial_i = \delta_{ij}$.

If $A$ is any $\mathbb{Z}_2$-graded associative algebra, we can regard $A$ as a Lie superalgebra by setting

$$[a, b] = ab - (-1)^{\alpha \beta} ba$$

where $a, b$ are elements of $A$ of degree $\alpha, \beta$, respectively. We note that $A_r$ can be made into a $\mathbb{Z}_2$-graded algebra by setting $\text{deg } x_i = \text{deg } \partial_i = 1$.

In [Mo] Montgomery shows that if we consider the $r$th Weyl algebra $A_r$ as a $\mathbb{Z}_2$-graded algebra, then $[A_r, A_r]/([A_r, A_r] \cap k)$ is a simple Lie superalgebra, and that when $r = 1, A_1 = k \oplus [A_1, A_1]$.

Using the adjoint representation of $\mathfrak{g}$ on $A_r$ we show that $A_r = k \oplus [A_r, A_r]$ for all $r$. In addition if $r \neq s$, then $[A_r, A_s]$ is not isomorphic to $[A_s, A_s]$ as a Lie superalgebra. This answers a question of Montgomery.

Much is known about the enveloping algebras of the Lie superalgebras $osp(1,2r)$ [F], [M1], [M2], [P]. However, we have tried to keep this paper as self-contained as possible.

**Lemma 1.** If $\mathfrak{g}$ is a classical simple Lie superalgebra which is not isomorphic to $osp(1,2r)$ for any $r$, then the only completely prime ideal of $U(\mathfrak{g})$ is the augmentation ideal.

**Proof.** It is shown in [B, pages 17-20], that if $\mathfrak{g} \neq osp(1,2r)$, then $\mathfrak{g}$ contains an odd element $x$ such that $[x,x] = 0$. Hence if $P$ is a completely prime ideal, then $x^2 = 0 \in P$ forces $x \in P$. Since $P \cap \mathfrak{g}$ is an ideal of $\mathfrak{g}$, this implies $\mathfrak{g} \subseteq P$.

**Lemma 2.** If $\mathfrak{g} = osp(1,2r)$, there is a surjective homomorphism $U(\mathfrak{g}) \twoheadrightarrow A_r$.

**Proof.** Set

$$\mathfrak{g}_1 = \sum_i kx_i + \sum_i k\partial_i$$

and

$$\mathfrak{g}_0 = \sum_{i,j} kx_i x_j + \sum_{i,j} k\partial_i \partial_j + \sum_{i,j} k(x_i \partial_j + \partial_j x_i).$$

We may identify $\mathfrak{g}_0$ with the second symmetric power $S^2 \mathfrak{g}_1$ of $\mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ becomes a Lie superalgebra under the bracket

$$[a, b] = ab - (-1)^{\alpha \beta} ba$$

where $a \in \mathfrak{g}_0$ and $b \in \mathfrak{g}_1$. It follows immediately from the description of $osp(m,n)$ given in [K, 2.1.2, supplement] that $\mathfrak{g} \cong osp(1,2r)$.

Now let $\mathfrak{a}_r$ be the $r$th Heisenberg Lie algebra with basis $X_1, \ldots, X_r, Y_1, \ldots, Y_r, Z$ and nonvanishing brackets given by $[X_i, Y_j] = \delta_{ij} Z$. Thus $U(\mathfrak{a}_r)/(Z-1)$ is isomorphic to $A_r$ via the map sending $X_i$ to $x_i$ and $Y_i$ to $y_i$. By [D, Lemma 4.6.9], $\mathfrak{g}_0 = sp(2r)$ acts by derivations on $\mathfrak{a}_r$, and hence on $U(\mathfrak{a}_r)$ and on the symmetric algebra $S(\mathfrak{a}_r)$. Therefore by [D, Proposition 2.4.9], the symmetrisation map $w : S(\mathfrak{a}_r) \rightarrow U(\mathfrak{a}_r)$ is an isomorphism of $\mathfrak{g}_0$-modules. Set $S = S(\mathfrak{a}_r)/(Z-1)$. Clearly $w$ induces an isomorphism $\overline{w} : S \rightarrow A_r$ of $\mathfrak{g}_0$-modules. Now $S$ is a polynomial algebra in $2r$ variables, and we let $S(n)$ be the subspace of homogeneous polynomials of degree $n$. Clearly $S(n)$ is a $\mathfrak{g}_0$-module.Set $A(n) = \overline{w}(S(n))$. Our
The main result is the following:

**Theorem 3.** Under the adjoint action
1) $A(n)$ is a simple $g_0$-module for all $n$.
2) $A(2n) \oplus A(2n-1)$ is a simple $g$-module for all $n$.

**Remark.** When $r = 1$ the decomposition of the adjoint representation is given in [P, Lemma 7.4.1]. Part 1) of the theorem also follows from arguments in [PS, Section 3].

In order to prove the theorem, we need some notation.

For $1 \leq i \leq r - 1$, consider the elements of $g$ given by
\[ e_i = x_{i+1} \partial_i, \quad f_i = x_i \partial_{i+1} \]
and
\[ h_i = [e_i, f_i] = x_{i+1} \partial_{i+1} - x_i \partial_i. \]

In addition, set $e_r = \partial_r$, $f_r = x_r$ and $h_r = -(e_r, f_r)/2 = -(x_r, \partial_r + \partial_r x_r)/2$. Then $h = \text{span}\{h_i|1 \leq i \leq r\}$ is a Cartan subalgebra of $g$. We let $\alpha_1, \ldots, \alpha_r \in h^*$ be the positive roots determined by $[h, e_i] = \alpha_i(h)e_i$ for all $h \in h$. The values $\alpha_i(h_j)$ are the entries in the (symmetrized) Cartan matrix
\[
\begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& & \ddots & & \ddots & -1 \\
& & & -1 & 2 & -1 \\
& & & & -1 & 1
\end{bmatrix}.
\]

Let $n$ be the subalgebra of $g$ generated by $e_1, \ldots, e_r$ and $n_0 = n \cap g_0$. If $L$ is a $g$-module (resp. $g_0$-module), we say that $v \in L$ is a highest weight vector for $g$ (resp. for $g_0$) of weight $\lambda \in h^*$ if $hv = \lambda(h)v$ for all $h \in h$ and $nv = 0$ (resp. $n_0v = 0$).

The bilinear form $(,)$ defined on $h^*$ by $(\alpha_i, \alpha_j) = \alpha_i(h_j)$ is invariant under the action of the Weyl group. For later computations involving $(,)$ it is convenient to use the following alternative description [K, 2.5.4]. Identify $h^*$ with $k^r$ with standard basis $e_1, \ldots, e_r$ and $(,)$ with the usual inner product. Then $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq r - 1$ and $\alpha_r = \epsilon_r$. Let $\rho_0$ (resp. $\rho_1$) denote the half-sum of the positive even (resp. odd) roots of $g$ and $\rho = \rho_0 - \rho_1$. Under the identification above we have $\rho_0 = \sum_{i=1}^r (r-i+1)e_i$, $\rho_1 = \frac{1}{2} \sum_{i=1}^r e_i$ and $\rho = \frac{1}{2} \sum_{i=1}^r (2r-2i+1)e_i$.

We now return to the homomorphism $\phi : U(g) \rightarrow A_r$. Set $J = \text{Ker} \phi$. Note that $R = \mathbb{C}[x_1, \ldots, x_r]$ is a simple $A_r$-module and hence a faithful simple $U(g)/J$-module. Also $1 \in R$ is a highest weight vector of weight $\lambda$ where $\lambda(h_i) = 0$ for $1 \leq i \leq r - 1$, and $\lambda(h_r) = -1/2$. An easy computation shows that $\lambda = -\frac{1}{2} \sum_{i=1}^r i \alpha_i = -\rho_1$. Thus we have shown

**Corollary 4.** $J$ is the annihilator of the simple highest weight module with weight $-\rho_1$.

We comment briefly on the geometric significance of the embedding $U(g)/J_0 \hookrightarrow U(g)/J \cong A_r$. Passing to the associated graded rings we have
\[ R(\overline{\mathcal{O}}_{\text{min}}) = \text{gr}(U(g)/J_0) \hookrightarrow \text{gr}(A_r) = R(k^{2r}) \]
where \( R(X) \) denotes the ring of regular functions on \( X \) and \( \overline{O}_{\text{min}} \) is the minimal coadjoint orbit. Let \( \mu : k^{2r} \to g_0^* \) be the moment map for the natural action of \( Sp(2r) \) on \( k^{2r} \) [CG, 1.4]. Then the image of \( \mu \) is contained in \( O_{\text{min}} \) and the above inclusion is the comorphism \( \mu^* \). All of this is quite well known. The new twist that Lie superalgebras bring to this situation is a consequence of the next result.

**Lemma 5.** Suppose \( k = \mathbb{C} \), let \( g = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be any Lie superalgebra over \( \mathbb{C} \) and define \( \pi : \mathfrak{g}_1 \to \mathfrak{g}_0 \) by \( \pi(y) = y^2 = \frac{1}{2}[y, y] \). Then \( (\exp \text{ad} x)(\pi(y)) = \pi(\exp \text{ad} x(y)) \) for all \( x \in \mathfrak{g}_0 \) and \( y \in \mathfrak{g}_1 \).

In the above situation, we can identify \( \mathfrak{g}_0^* \) with \( \mathfrak{g}_0 \) and \( \mathbb{C}^{2r} \) with \( \mathfrak{g}_1 \) in such a way that \( \mu = \pi \).

**Proof.** The second claim follows from the first and the uniqueness of the moment map. The formula involving \( \pi \) is proved by formally expanding both sides. Define \( [x, y]_0 = y \) and \( [x, y]_n = (\text{ad} x)^n(y) = [x, [x, y]_{n-1}] \) for \( n > 0 \). Similarly we define \( [x, y^2]_n \). Then \( (\exp \text{ad} x)\pi(y) = \sum_{n \geq 0} [x, y^2]_n / n! \). To show that this equals \( \pi(\exp \text{ad} x(y)) \) we use the identity \( [x, y^2]_{2m} = \binom{2m}{m} [x, y]^2_m \) and a similar identity for \( [x, y^2]_{2m+1} \). The identities are easily proved by induction.

**Lemma 6.** Under the adjoint action of \( \mathfrak{g}_0 \) or \( \mathfrak{g} \) on \( A_r \),
1) \( \partial_1^n \) is a highest weight vector for \( \mathfrak{g}_0 \) of weight \( n \epsilon_1 \).
2) If \( n \) is even, \( \partial_1^n \) is a highest weight vector for \( \mathfrak{g} \).

**Proof.** A simple computation.

If \( \lambda \in \mathfrak{h}^* \), we denote the simple \( \mathfrak{g}_0 \)-module with highest weight \( \lambda \) by \( L(\lambda) \).

**Lemma 7.** We have \( \dim L(n \epsilon_1) = \binom{2r + n - 1}{n} \) for all \( n \).

**Proof.** By Weyl’s dimension formula
\[
\dim L(\lambda) = \prod_{\alpha > 0} \frac{(\lambda + \rho_0, \alpha)}{(\rho_0, \alpha)}
\]
where the product is taken over all positive even roots \( \alpha \). The even roots \( \alpha \) for which \( (\epsilon_1, \alpha) > 0 \) are listed in the first column of the table below. The other columns give the information we need.

<table>
<thead>
<tr>
<th>( \epsilon_1 - \epsilon_{i+1} ), ( 1 \leq i \leq r - 1 )</th>
<th>( \epsilon_1 + \epsilon_j ), ( 2 \leq j \leq r )</th>
<th>( 2 \epsilon_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_0, \alpha )</td>
<td>( i )</td>
<td>( 2r )</td>
</tr>
<tr>
<td>( n \epsilon_1, \alpha )</td>
<td>( n )</td>
<td>( 2n )</td>
</tr>
</tbody>
</table>
Therefore

\[ \dim L(n\epsilon_1) = \prod_{i=1}^{r} \frac{n+i}{i} \prod_{j=2}^{r} \frac{2r+n-j+1}{2r-j+1} = \left( \frac{2r+n-1}{n} \right). \]

**Proof of Theorem 3.** Set \( A = A_r \). Part 1) of the theorem follows from Lemmas 6 and 7, since \( \dim A(n) = \left( \frac{2r+n-1}{n} \right) \). Thus \( B(n) = A(2n) \oplus A(2n-1) \) is a direct sum of two nonisomorphic simple \( \mathfrak{g}_0 \)-modules. Also the highest weight vectors \( \partial_{1}^{2n} \) and \( \partial_{1}^{2n-1} \) for these \( \mathfrak{g}_0 \)-modules satisfy

\[ [x_1, \partial_{1}^{2n}] = -2n\partial_{1}^{2n-1}, \]
\[ [\partial_{1}, \partial_{1}^{2n-1}] = 2\partial_{1}^{2n}. \]

Let \( M \) be the ad\( \mathfrak{g} \)-submodule of \( A \) generated by \( \partial_{1}^{2n} \). It follows that \( B(n) \subseteq M \). Also \( M \) is a finite dimensional image of a Verma module (which has a unique simple quotient). On the other hand all finite dimensional simple \( \mathfrak{g} \)-modules are completely reducible by [DH]. It follows that \( M \) is a simple ad\( \mathfrak{g} \)-module. (cf. the argument in [Jan, Lemma 5.14]).

We do not know yet that \( B(n) \) is an ad\( \mathfrak{g} \)-module. This can be seen as follows. We define a filtration \( \{B_n\} \) on \( A \) by setting \( B_n = \bigoplus_{m \leq n} B(m) \). Note that this filtration is the image of the filtration \( \{U_n\} \) of \( U(\mathfrak{g}) \) defined by \( U_n = U_1^n \) where \( U_1 = k \oplus \mathfrak{g} \). Hence the associated graded ring \( \bigoplus_{n \geq 0} B_n/B_{n-1} \) is supercommutative. It follows that \( [\mathfrak{g}, B_n] \subseteq B_n \) and so \( M \subseteq B_n \). If \( M \) strictly contains \( B(n) \), we would have \( M \cap (B(n-1) \oplus \ldots \oplus B(1) \oplus k) \neq 0 \). By induction, the \( B(i) \) with \( i < n \) are simple ad\( \mathfrak{g} \)-modules, so \( M \) would contain \( \partial_{1}^{2i} \) for some \( i < n \). However a simple \( U(\mathfrak{g}) \)-module cannot contain more than one highest weight vector. This contradiction shows that \( M = B(n) \) and completes the proof.

**Theorem 8.** We have \( [A_r, A_r] = \bigoplus_{n \geq 0} A(n) \). In particular \( A_r = k \oplus [A_r, A_r] \).

**Proof.** Note that if \( a, b, c \in A \) have degrees \( \alpha, \beta \) and \( \gamma \), then as noted in [Mo, Lemma 1.4 (3)]

\[ [ab, c] = [a, bc] + (-1)^{\alpha(\beta+\gamma)}[b, ca]. \]

Therefore, since \( A_r \) is generated by the image of \( \mathfrak{g} \), we have \( [A_r, A_r] = [A_r, \mathfrak{g}] \). The result now follows from Theorem 3.

**Remark.** From [Mo, Theorem 4.1] it follows that \( [A_r, A_r] \) is a simple Lie superalgebra for all \( r \).

A question raised in [Mo] is whether, for different \( r \), the \( [A_r, A_r] \) are all nonisomorphic. We show that this is the case by finding the largest rank of a finite dimensional simple Lie subalgebra of \( [A_r, A_r] \). Note that \( sp(2r) \cong A(2) \subseteq [A_r, A_r] \). On the other hand we have

**Lemma 9.** If \( L \) is a finite dimensional simple Lie subalgebra of \( [A_r, A_r] \), then \( \text{rank}(L) \leq r \).

**Proof.** Note that under the stated hypothesis, \( L \) is a Lie subalgebra of \( A_r \) with the usual Lie bracket \([a, b] = ab - ba\). Now in [J1], Joseph investigates for each simple Lie algebra \( L \), the least integer \( n = n_A(L) \) such that \( L \) is isomorphic to a Lie subalgebra of \( A_n \). (The integer \( n_A(L) \) is determined to within one for all
classical Lie algebras.) In particular it follows from Lemma 3.1 and Table 1 of [J1] that $n_A(L) \geq \text{rank}(L)$.

**Corollary 10.** If $[A_r, A_r] \cong [A_s, A_s]$ as Lie superalgebras, then $r = s$.

For the sake of completeness, we give a proof of Corollary 10 which is independent of [J1]. It is enough to show that if $\mathfrak{g}_0 = \mathfrak{sp}(2r)$ is a Lie subalgebra of a Weyl algebra $A_n$, then $n \geq r$. The elements $x_1, x_i, x_i \partial_1$, with $2 \leq i \leq r$ and $x_i^2$ span a Heisenberg subalgebra $\mathfrak{a} = \mathfrak{a}_{r-1}$ of $\mathfrak{g}_0$ with center spanned by $x_1^2$. The inclusion $\mathfrak{g}_0 \subset A_n$ induces a homomorphism $\phi : U(\mathfrak{g}_0) \to A_n$. If $I = \ker \phi \cap U(\mathfrak{a}) = 0$, then we have $GK(U(\mathfrak{a})) = 2r - 1 \leq GK(A_n) = 2n$, where $GK(\ )$ denotes Gel’fand-Kirillov dimension, and so $r \leq n$. However if $I \neq 0$, then since the localization of $U(\mathfrak{a})$ at the nonzero elements of $k[x_1^2]$ is a simple ring, we would have $x_1^2 - \alpha \in I$ for some scalar $\alpha$. This would imply that $x_1^2$ is central in $\mathfrak{g}_0$, a contradiction.

**Remark.** It is shown in [Mo, Proposition 4.2] that if $A_r$ is isomorphic to $A_s$, then $r = s$. Corollary 10 also follows from this and Theorem 8.

Finally, we note that the proof of Theorem 8 works for certain other algebras.

**Theorem 11.** Let $\mathfrak{g}$ be a semisimple Lie algebra, and $A$ a primitive factor algebra of $U(\mathfrak{g})$, then $A = k \oplus [A,A]$.

**Proof.** As before we have $[A,A] = [A,\mathfrak{g}]$. Also $A = \bigoplus V$, a direct sum of finite dimensional simple submodules under the adjoint representation. Since $[V,\mathfrak{g}]$ is a submodule of $V$ for any such $V$, and the center of $A$ equals $k$, we obtain $[A,A] = \bigoplus_{V \neq k} V$, and the result follows.

**References**


Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201
E-mail address: musson@csd.uwm.edu
URL: www.uwm.edu/~musson/