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TIMELIKE PERIODIC TRAJECTORIES IN SPATIALLY COMPACT LORENTZ MANIFOLDS

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ABSTRACT. A result on the existence of timelike periodic trajectories in a general class of Lorentzian manifolds $\mathbb{R} \times M$, with compact M, is obtained. The proof is based on arguments concerning closed geodesics and causality theory.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this article we show the existence of timelike periodic trajectories in certain general classes of Lorentzian manifolds, obtaining also accurate information on the universal periods and on the spatial part. We will focus our attention on stationary spacetimes, even though the technique can be applied to the case in which the metric has a periodic dependence on the time. Our basic assumption is that the spatial component is compact.

We will call spacetime any time-orientable connected Lorentzian (smooth) manifold $(-, +, ..., +)$. A (standard) *stationary* spacetime is a manifold $\mathcal{M}$ which can be written as a product $\mathcal{M} = \mathbb{R} \times M$, with M any connected manifold, endowed with a metric g of the form:

$$g = -(\beta \circ \pi_M)\pi_{\mathbb{R}}^*(dt^2) + \pi_M^* g_M + \pi_M^*\omega \otimes \pi_{\mathbb{R}}^* dt + \pi_{\mathbb{R}}^* dt \otimes \pi_M^*\omega \tag{1.1}$$

where $\pi_{\mathbb{R}}, \pi_M$ are the natural projections of $\mathcal{M}$ onto $\mathbb{R}$, M, respectively, dt^2 denotes the usual metric on $\mathbb{R}$, and g_M, β and ω are, respectively, a Riemannian metric, a positive function and a 1-form, all of them on M. If δ is the vector field g_M-associated to ω, the metric (1.1) can also be written as:

$$g((t,\xi),(t',\xi')) = -\beta(x_0)\cdot tt' + g_M(\xi,\xi') + t'g_M(\xi,\delta) + tg_M(\xi',\delta) \tag{1.2}$$

for all $(t,\xi),(t',\xi')$ in the tangent space $T_{z_0}\mathcal{M}, z_0 = (t_0,x_0) \in \mathcal{M}$. When $\delta \equiv 0$, the spacetime is called (standard) *static*. Observe that, in any case, the metric g is independent of the "time-variable" t; for the physical interest of stationary and static spacetimes see, for instance, [23, Ch. 7]; a review of many of their relevant properties from a mathematical point of view can be found in [22].

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A *periodic trajectory of universal period* $T \geq 0$, or T-periodic trajectory, is a non-constant geodesic, $\gamma : [0,1] \to \mathbb{R} \times M$, $\gamma(s) = (t(s), \gamma_M(s)), \forall s \in [0,1]$, satisfying:

$$\begin{aligned} t(1) &= t(0) + T, \quad t'(1) = t'(0), \\ \gamma_M(0) &= \gamma_M(1), \quad \gamma_M'(0) = \gamma_M'(1). \end{aligned} \tag{1.3}$$

These trajectories are called timelike, lightlike or spacelike according to the causal character of their velocity. A (necessarily timelike) periodic trajectory is called trivial if γ_M is constant, and non-trivial otherwise. Such trivial trajectories occur at each point $x_0 \in M$ such that $\nabla^M \beta(x_0) = 0$ (∇^M denotes the g_M–Levi-Civita gradient or connection); so, they always appear if M is compact.

Timelike periodic trajectories were introduced by Benci and Fortunato [3] for static spacetimes with $M = \mathbb{R}^3$. In this reference and in [4], the existence of certain timelike periodic trajectories is shown in the static case and with $M = \mathbb{R}^n$. In [20] these trajectories for static manifolds are shown to be the relativistic analog to the closed trajectories in Lagrangian mechanics, and their existence is proved under general hypotheses like those in this classical topic. For compact M, the following results have been obtained:

1. *Static case.* Greco [12] showed the existence of a $T_0 > 0$, and a timelike T-periodic trajectory for each $T > T_0$, if either the fundamental group $\pi(M)$ is not trivial or the maximum of β is an isolated critical level. In [20] the existence of an infinite number of timelike periodic trajectories for each non-trivial free homotopy class of M or conjugacy class of $\pi(M)$ is shown (if the trivial class exists only, it is shown in this too).
2. *General stationary case.* Masiello [14, Theorem 1.6] showed that if $\pi(M)$ is finite or it has an infinite number of conjugacy classes, then for every $m \in \mathbb{N}$ there exists a $T_0 > 0$ such that for every $T > T_0$ there are at least m timelike T-periodic trajectories.

The problem of finding timelike periodic trajectories also makes sense when the metric g has a periodic dependence on the time variable, that is, when, under natural identifications, β, δ and g_M in (1.2) must be regarded as functions not only of the point in M but also of the time coordinate in $\mathbb{R}$, and there exists $\tau > 0$ such that these three functions coincide in each couple $(t,x), (t+\tau, x), \forall (t,x) \in \mathcal{M}$. We will call this metric τ-periodic. In this case, it is natural to look for the existence of timelike periodic trajectories with a universal period multiple of τ. Note that generically, trivial trajectories do not occur now, so they are not taken into account. Greco's technique, in 1 above, for compact M still works in this case. For $M = \mathbb{R}^n$ and $\delta \equiv 0$, the existence of timelike periodic trajectories has been studied in [13], [15], [17]. For more results on periodic trajectories, not necessarily timelike, see [4], [5], [6], [7], [14], [16], [20].

In most of the references on this topic, the geodesics are seen as critical points of the action functional for curves joining two given points. For Lorentzian manifolds, this functional is rather complicated because it does not satisfy the Palais-Smale condition. So, some variational methods have been developed to reduce the problem to one in which the Palais-Smale condition holds. From a more geometrical point of view, in the references [12], [13] and [16], some variants of a shortening procedure introduced to study closed geodesics in Riemannian manifolds have been developed. Our point of view in this article is quite different. Note that timelike T-periodic trajectories can be regarded as closed timelike geodesics of the quotient obtained by identifying each (t,x) and $(t+T,x)$; then, some ideas which have been used already to study such closed geodesics are also used by us (regarding this topic, see [8], [9], [24]). So, the properties of causality theory and especially of the timelike distance function on Lorentzian manifolds, are taken into account. For some comments on the relation of these techniques and the variational ones, see [21].

Recall that it makes sense to speak of the conjugacy class in $\pi(M)$ of any closed curve in M. Our main result is the following one:

Theorem 1.1. *Consider a stationary spacetime with compact M, and choose a non-zero class of conjugacy $\mathcal{C}$ of $\pi(M)$. Then there is a $T_{\mathcal{C}} > 0$ such that:*

(i) for each $T > T_{\mathcal{C}}$ there exists a non-trivial timelike periodic trajectory $\gamma(s) = (t(s), \gamma_M(s))$ with $\gamma_M \in \mathcal{C}$ and universal period T, and

(ii) if $0 < T \leq T_{\mathcal{C}}$ there exists no such timelike T-periodic trajectory.

Remarks. (1) With this technique, if $\pi(M)$ is trivial we cannot obtain the existence of timelike periodic trajectories; this case is also an exception in Greco's result [12] commented on in 1 above. In fact, one can find examples in which the conclusions of Theorem 1.1 still hold (simply, take a product manifold), however, it is also easy to find examples in which non-trivial T-periodic trajectories with arbitrarily small T appear [20]. Anyway, the existence of timelike periodic trajectories in this case can also be stated by using [20, Corollaries 3.5 and 3.10] and Masiello's result 2 above [14, Theorem 1.6]. If $\pi(M) \neq 0$, the results in [12] are refined and generalized to stationary manifolds.

(2) From the proof it will be clear that, when $\pi(M)$ has an infinite number of conjugacy classes, then the result 2 above is also reobtained (compare with [21, Proposition 4.5]).

(3) Part (ii) only depends on causal considerations and holds for general timelike curves (not necessarily geodesics). In fact, we will prove the following stronger result:

(ii′) There exists no piecewise smooth future-directed timelike curve $\gamma : [0, 1] \to \mathcal{M}, \gamma(s) = (t(s), \gamma_M(s))$ with $t(1) - t(0) \leq T_{\mathcal{C}}$ and $\gamma_M \in \mathcal{C}$.

This dependence on just causal facts is also true in [4, Theorem 1.1 (i)] and [11, Theorem 1.6 (ii)]. In these references, the result on the non-existence of timelike geodesics obtained by variational methods can be strengthened in a result on the non-existence of timelike curves obtained for causal considerations, as we will do here. Moreover, our result is accurate in the sense that the same critical value of $T_{\mathcal{C}}$ is valid for (i) and (ii). On the other hand, the critical value $T_{\mathcal{C}}$ is obtained from causal considerations and, thus, it is a conformal invariant.

(4) Consider now lightlike periodic trajectories. Under the compactness assumption for M, Candela [7] showed the existence of one such T-periodic trajectory in stationary spacetimes; for the static case, the existence of at least two is showed in [20]. From the proof of Theorem 1.1 it is not difficult to show the existence of a lightlike periodic trajectory for each non-trivial conjugacy class $\mathcal{C}$ with universal period $T_{\mathcal{C}}$.

(5) Even though at the beginning it was assumed that the objects were smooth, in fact, the order of differentiability which is necessary can be reduced. So, the manifold can be assumed as C^2 and the metric C^1. Due to technical reasons concerning Nash's embedding theorem, the order of differentiability typically used to develop variational methods is C^3 (see, for example, [14]).

(6) A straightforward consequence of Theorem 1.1 (putting $\delta \equiv 0$ and $\beta \equiv 1$) is the well-known Riemannian result which ensures the existence of closed geodesics

for each (non-trivial) conjugacy class of $\pi(M)$ (see, for example, [1] or references therein). Moreover, given the relation between geodesics in $\mathcal{M}$ and the solutions to certain variational problems on the Riemannian manifold (M, g_M) (see [20]), an indirect proof of the existence of solutions to these problems is also deduced.

Theorem 1.1 can be easily extended to the case in which the metric g has a periodic dependence with the time (which is the more general spacetime in which the definition of periodic trajectory makes sense). Furthermore, as now the trivial case has not to be taken into account, the corresponding result also includes the trivial conjugacy class:

Theorem 1.2. *Consider a spacetime $(\mathcal{M}, g)$ where g is a τ-periodic metric. Then for each class of conjugacy $\mathcal{C}$ of $\pi(M)$, there is a non-negative integer $n_{\mathcal{C}}$ such that:*
(i) for every $n > n_{\mathcal{C}}$ there exists a timelike T-periodic trajectory

$$\gamma(s) = (t(s), \gamma_M(s))$$

with $\gamma_M \in \mathcal{C}$ and universal period $T = n\tau$;
(ii) for $0 \leq n \leq n_{\mathcal{C}}$ no such timelike $n\tau$-periodic trajectory exists.
Moreover, when $\mathcal{C}$ is the trivial conjugacy class, then $n_{\mathcal{C}} = 0$.

In Section 2 we will see some technical properties of the causal structure of stationary spacetimes which are necessary for the proof (Propositions 2.1 and 2.2). In Section 3, Theorem 1.1 will be proved; the demonstration of Theorem 1.2 would follow analogous steps and, so, it will not be done here.

2. Set-up and causality in stationary spacetimes

We will consider the (future) time-orientation on $\mathcal{M}$ yielded by $\partial/\partial t$, and use notation which is usual in causality theory (as, for instance, [2], [18]). Following [18], a tangent vector $v \in T\mathcal{M}$ will be timelike (resp. lightlike; spacelike; causal) if $g(v, v) < 0$ (resp. $g(v, v) = 0$ and $v \neq 0$; $g(v, v) > 0$ or $v = 0$; v is timelike or lightlike). If $A \subseteq \mathcal{M}$ is any subset, $I^+(A)$ (resp. $J^+(A)$) will denote the subset containing all the points of $\mathcal{M}$ which can be joined with a future-directed timelike (resp. future-directed causal or identically constant) curve starting at any point of A. The open subset $I^-(A)$ is defined dually with past-directed curves. The transitive property,

$$p_2 \in J^+(p_1),\ p_3 \in I^+(p_2) \ \Rightarrow\ p_3 \in I^+(p_1) \tag{2.1}$$

is well-known (see, for example, [18, Corollary 14.1]). A Cauchy hypersurface for a spacetime is a topological hypersurface which is crossed exactly once by any inextendible timelike curve and, then, by any inextendible causal curve [18, Lemma 14.29]; a spacetime admitting it will be called globally hyperbolic. It is also well-known that the timelike distance function (or time-separation) d is continuous in these spacetimes. We will need some natural causal properties of our ambient spacetime.

Proposition 2.1. *Consider the stationary spacetime $(\mathcal{M}, g)$. If M is compact, then each slice $\{t_0\} \times M$ is a Cauchy hypersurface.*

Proof. General computations including τ-periodic metrics and non-compact M can be found in [21] (for some additional properties of the two dimensional case, see [19]). The proof under our hypotheses is easier and can be regarded as an exercise. □

The following proposition will be applied to the universal Lorentzian covering of $(\mathcal{M}, g)$ in Theorem 1.1. Of course, this spacetime will also be stationary and globally hyperbolic (the universal covering of a globally hyperbolic spacetime is globally hyperbolic too). In order to get a clearer notation afterwards, we will consider a new stationary spacetime $(\mathcal{M}', g), \mathcal{M}' = \mathbb{R} \times M'$, with g defined formally as in (1.2).

Proposition 2.2. *Assume that in the stationary spacetime $(\mathcal{M}', g)$ each slice $\{t_0\} \times M'$ is a Cauchy hypersurface. Then:*

(A) Given $x, x' \in M'$, there exists $0 \leq T_0 < \infty$ such that if $T > T_0$, then $(T, x') \in I^+(0, x)$, and if $T \leq T_0$, then $(T, x') \notin I^+(0, x)$; the equality $T_0 = 0$ holds if and only if $x = x'$.

(B) The function $T_0 \equiv T_0(x, x')$ depends continuously on x and x' in M'.

(C) Consider two compact subsets, $K, K^* \subset M'$ such that K is included in the interior of K^*. Then there exists $\epsilon > 0$ such that $T_0(x, x') > \epsilon, \forall x \in K, \forall x' \in M' - K^*$.

Proof. (A) The case $T_0 = 0$ is straightforward, so we will assume $x \neq x'$. From the global hyperbolicity of $\mathcal{M}'$, $J^+(0, x)$ is closed [18, p. 412], and, thus, its boundary is $\partial J^+(0, x) = J^+(0, x) - I^+(0, x)$. In fact, the inclusion $\subseteq$ is obvious, and the converse is true in any spacetime; otherwise, take $p \in (J^+(0, x) - I^+(0, x)) - \partial J^+(0, x)$, a neighborhood U of p in $J^+(0, x)$ and a point $q \in I^-(p) \cap U$. Then $q \in J^+(0, x), p \in I^+(q)$ and, by (2.1), $p \in I^+(0, x)$, a contradiction.

Consider the line

$$R_{x'} = \{(t, x') : t \in \mathbb{R}\}$$

and the set

$$J = \{t : (t, x') \in R_{x'} \cap J^+(0, x)\}.$$

Note that J is closed and if $t \in J$, then $[t, \infty) \subset J$; thus, J must be an interval $J = [T_0, \infty)$, with $T_0 > 0$. Moreover, $T_0 < \infty$. To check it, consider any g_M-speed 1 curve $\gamma_M : [0, 1] \to M$ joining x and x' which does not vanish at any point. Consider the equation in $t'(s)$ for each $s \in [0, 1]$:

$$-\beta t^{'2} + 2t' g_M(\delta, g_M') + 1 = 0$$

and take its positive solution (explicitly in (2.3) below). Then, a future-directed lightlike curve (t, γ_M) joining $(0, x)$ and $R_{x'}$ is constructed, and $t(1)$ is an upper bound for the value of T_0.

So, necessarily, $(T_0, x') \in J^+(0, x) - I^+(0, x)$, and if $T < T_0$, then $(T, x') \notin J^+(0, x) (\supset I^+(0, x))$. As the curve $s \to (T_0 + s, x')$ is timelike, if $T > T_0$, then, by (2.1), $(T, x') \in I^+(0, x)$, and we are done.

(B) First, let us consider the continuity in the second variable, $T_0(x, \cdot)$. Note that, by the proof of *(A)*, $\partial J^+(0, x) = \{(T_0(x, x'), x') : x' \in M'\}$. On the other hand, $J^+(0, x)$ can be written as the union of the sets $\{(t, x') : t \geq T_0(x, x')\}$ obtained with any $x' \in M'$. So, if there exists a sequence $\{x'_n\} \to x'_0$ such that $\{T_0(x, x'_n)\} \to T' \neq T_0(x, x'_0)$, we would have set $\{(t, x'_0) : t \in (T', T_0(x, x'_0))\}$ or set $\{(t, x'_0) : t \in (T_0(x, x'_0), T')\}$ included in $\partial J^+(0, x)$, which is absurd.

For the continuity of $T_0(\cdot, x')$, assume first that there exists a sequence $\{x_n\} \to x_0$ such that:

$$\{T_0(x_n, x')\} \to T < T_0(x_0, x'). \tag{2.2}$$

Then $(T, x') \notin J^+(0, x_0)$ and, as $J^+(0, x_0)$ is closed, there exists a neighborhood K of (T, x') such that $K \cap J^+(0, x_0) = \emptyset$. Now, note that our spacetime is outer causally continuous [2, p. 24, 32]. Thus, if K is chosen compact, there exists a neighborhood V of x such that $J^+(0, x) \cap K = \emptyset, \forall x \in V$, in contradiction with (2.2). The case $T > T_0(x_0, x')$ is proven analogously, using $(T, x') \in I^+(0, x_0)$ and the inner causal continuity of the spacetime.

(C) Choose $\epsilon > 0$ as the minimum of $T_0(x, x')$ on the compact subset $K \times \partial K^*$. The result follows easily taking into account that $t' > 0$ for any future-directed causal curve (t, γ_M), and, so, T_0 cannot be less than ϵ in $K \times (M - K^*)$. □

Remark. It is straightforward to check that all the conclusions in Proposition 2.2 hold for τ-periodic metrics, even though the following detail must be taken into account for the finiteness of $T_0(x, x')$. If we reason as in *(A)* above to prove it, then the constructed lightlike curve (t, γ_M) satisfies:

(2.3)
$$t'(s) = \frac{1}{\beta}\left(g_M(\delta, \gamma_M') + \left(g_M(\delta, \gamma_M')^2 + \beta\right)^{1/2}\right), \quad s \in [0,1], \quad t(0) = 0.$$

In the stationary case, the right member is independent of the time variable t, but it is not if the metric is τ-periodic. Nevertheless, there are no problems to ensure the existence of solutions to (2.3) in all $[0, 1]$ (in fact, the solutions to the inequality $\geq$ are sufficient), because the right member satisfies a suitable condition as a function of t and s –say, it is bounded by

$$\frac{1}{m_\beta}\left(M_\delta + \left(M_\delta^2 + M_\beta\right)^{1/2}\right),$$

where $M_\beta > 0$ (resp. $m_\beta > 0$) is the maximum (resp. minimum) of β, and M_δ is the maximum of the g_M-norm (depending also on t) of δ. If the dependence of β, δ, and g_M with t were non-periodic, then we cannot guarantee the finiteness of $T_0(x, x')$. Nevertheless, considering it as a function with values in $[0, \infty]$ the remainder of the conclusions of Proposition 2.2 still hold. Moreover, consider any globally hyperbolic spacetime. By using a celebrated theorem of Geroch [10], this spacetime is topologically a product $\mathbb{R} \times S$ where the slices $\{t\} \times S$ are Cauchy hypersurfaces . Then, fixing the homeomorphism with $\mathbb{R} \times S$, the function T_0 can be defined as before, and all its properties stated in Proposition 2.2, but its finiteness, hold again.

Now, fix $T > 0$ and consider the group of isometries G of $(\mathcal{M}, g)$ yielded by the timelike translation $(t, x) \to (t + T, x)$, and the correspondent quotient $\mathcal{M}/G$, with the induced metric. Clearly, each closed timelike geodesic of $\mathcal{M}/G$, $\alpha : \mathbb{R} \to \mathcal{M}/G$, normalized such that $\alpha'(0) = \alpha'(1)$, yields a timelike periodic trajectory of $\mathcal{M}$ of universal period T (and vice versa). On the other hand, Tipler's result in [24] asserts (see also [8, Section 3], [2, Theorem 3.15]):

> A compact spacetime which admits a spacetime with a compact Cauchy hypersurface as a regular covering contains a closed timelike geodesic.

Applied under the hypotheses of our Theorem 1.1, this result and Proposition 2.1 would yield timelike periodic trajectories of all universal periods $T > 0$; nevertheless, these trajectories may be trivial. To obtain non-trivial trajectories and the remainder of the assertions, we will modify in the next section Tipler's technique, solving certain new technicalities which appear in this case.

3. Proof of Theorem 1.1

Consider the universal coverings of $\mathcal{M}$ and M, $\bar{\Pi} : \mathcal{M}' \to \mathcal{M}, \Pi : M' \to M$, respectively, endowed with the induced metrics, which we also denote by g and g_M . Fix a point in M', and identify, as usual, the fundamental group $\pi(M)$ with the group of deck transformations of the covering. Choose $T > 0$ and, for each $\phi \in \pi(M)$, define the function $F_\phi : M' \to [0, \infty)$,

$$F_\phi(x) = d((0, x), (T, \phi(x))), \quad \forall x \in M' \tag{3.1}$$

(by using the timelike distance function d of $\mathcal{M}'$). Of course, $F_\phi(x) > 0$ if and only if $(T, \phi(x)) \in I^+(0, x)$. We will use in the proof the fact that the functions F_ϕ, when restricted to a fixed compact subset, are all identically null but a finite number. This result will be obtained in the last of the following three lemmas.

Lemma 3.1. *Consider a compact subset K of a globally hyperbolic spacetime. For any Cauchy hypersurface S, the set $J^+(K) \cap S$ is compact.*

Proof. If $J^+(K) \cap S$ is empty, there is nothing to prove; otherwise, choose any sequence $\{y_n\} \subset J^+(K) \cap S$. Construct another sequence $\{x_n\} \subset K$ such that $y_n \in J^+(x_n)$ for all n, and consider a sequence of inextendible causal curves $\{\gamma_n\}$, which join each x_n with y_n. Taking a subsequence if necessary, we can assume $\{x_n\} \to x_0 \in K$; so, by [2, Proposition 2.18], there exists a causal limit curve γ of $\{\gamma_n\}$ through x_0 . This limit curve crosses S at a point y_0.

Take a basis $\{U_m\}$ of neighborhoods of y_0 such that $U_m \cap S$ is a Cauchy hypersurface of U_m (it can be easily obtained by using Cauchy developments, see [18, Theorem 14.38]). Each neighborhood U_m intersects infinite ranges of γ_n, $n \in \mathbb{N}$, and then U_m contains infinite y_n, by our assumption on $U_m \cap S$. So, a subsequence of $\{y_n\}$ converging to $y_0 \in J^+(x_0) \cap S \subset J^+(K) \cap S$ can be selected. □

The following elemental property is valid for all covering manifolds.

Lemma 3.2. *Let K, K^* be two compact subsets of M'. The set $\{\phi \in \pi(M) : \phi(K) \cap K^* \neq \emptyset\}$ is finite.*

Proof. Otherwise take a sequence of distinct elements $\{\phi_n\} \subset \pi(M)$, and a sequence $\{q_n\} \subset K$, such that $\{\phi_n(q_n)\} \subset K^*$. We can assume, up to subsequences:

$$\{q_n\} \to q \in K, \{\phi_n(q_n)\} \to q^* \in K^*. \tag{3.2}$$

Suppose first that $\Pi(q) = \Pi(q^*)$, and fix $\phi_0 \in \pi(M)$ such that $\phi_0(q) = q^*$. As the action of $\pi(M)$ on M' is properly discontinuous, we can take a neighborhood U of q such that $\phi_0(U) \cap \phi(U) = \emptyset, \forall \phi \in \pi(M) - \{\phi_0\}$, in contradiction with (3.2). If $\Pi(q) \neq \Pi(q^*)$, the contradiction is obtained analogously by taking two neighborhoods $U \ni q, U^* \ni q^*$ such that $\phi(U) \cap U^* = \emptyset, \forall \phi \in \pi(M)$. □

Then we can prove the previously mentioned result about F_ϕ. We will use the natural notation (t, C), for any $C \subseteq M'$, to denote the set of all the pairs which can be obtained with the fixed $t \in \mathbb{R}$ and any element of C.

Lemma 3.3. *For any $T > 0$ and any compact subset $K \subset M'$, the set $\{\phi \in \pi(M) : (T, \phi(K)) \cap J^+(0, K) \neq 0\}$ is finite.*

Thus, the restricted functions $F_\phi \mid_K$ obtained from (3.1) are all null but a finite number.

Proof. By using Proposition 2.1 and Lemma 3.1 the set $K^* \subseteq M'$ defined by $(T, K^*) = J^+(0, K) \cap (T, M')$ is compact. Thus, apply Lemma 3.2 to $\{T\} \times M'$ just taking into account that $(T, \phi(K)) \cap J^+(0, K) = (T, \phi(K)) \cap (T, K^*)$. □

As a consequence of this lemma, the following fundamental step can be taken. But first, fix a compact subset $K_0 \subseteq M'$ such that $\Pi(K_0) = M$ (which can be found because of the compactness of M).

Lemma 3.4. *The supremum of each function F_ϕ in (3.1) is reached at a point of M'.*

Moreover, the supremum of F_ϕ is equal to the supremum of $F_{\phi'}$, for all ϕ' in the conjugacy class $\mathcal{C} \subset \pi(M)$ of ϕ.

Proof. For the first assertion, if $F_\phi \equiv 0$, there is nothing to prove. Otherwise, consider the elements $\phi' \in \mathcal{C}$ such that, for the fixed compact subset K_0, $F_{\phi'} \mid_{K_0} \not\equiv 0$. By Lemma 3.3, there are a finite number of these elements, and we can take the point $x_0 \in K_0$ where the maximum of all $F_{\phi'} \mid_{K_0}$ is reached. This maximum is obtained for a function, say $F_{\phi_0} \mid_{K_0}, \phi_0 \in \mathcal{C}$.

As, by our choice of K_0, $\Pi(K_0) = M$, given any $x \in M'$ there exists $\psi \in \pi(M)$ such that $\psi(x) \in K_0$. By using the fact that the deck transformations of M' are isometries for d:

$$F_\phi(x) = d((0,x),(T,\phi(x))) = d((0,\psi(x)),(T,\psi(\phi(x)))) = F_{\phi'}(\psi(x)), \tag{3.3}$$

where $\phi' = \psi \circ \phi \circ \psi^{-1}$. From (3.3) it follows that $F_\phi(x) \leq F_{\phi_0}(x_0)$ for all $x \in M'$. Moreover, take $\psi_0 \in \pi(M)$ such that $\phi_0 = \psi_0 \circ \phi \circ \psi_0^{-1}$; then $F_\phi(\psi_0^{-1}(x_0)) = F_{\phi_0}(x_0)$. Thus, $\psi_0^{-1}(x_0)$ is the required point for the first assertion. The second one is obvious from (3.3). □

Proof of Theorem 1.1. We will follow the following three steps:

(A) Given the non-zero conjugacy class $\mathcal{C}$, define $T_{\mathcal{C}}$ as the minimum of the $T_0(x, \phi(x)) =: T_0^\phi(x), \forall x \in K_0$ and $\forall \phi \in \mathcal{C}$, where $T_0(x, \phi(x))$ is yielded in Proposition 2.2(A).

Necessarily $T_{\mathcal{C}} > 0$. It is obvious from the continuity of T_0^ϕ (Proposition 2.2(B)) if $\mathcal{C}$ is finite. Otherwise, take a compact subset $K_0^* \supset K_0$ and $\epsilon > 0$ with $T_0(x, x') \geq \epsilon > 0, \forall x \in K_0, \forall x' \in M - K_0^*$, as yielded in Proposition 2.2(C). Then, by Lemma 3.2, such ϵ is a lower bound for all but a finite number of the $T_0^\phi, \phi \in \mathcal{C}$, and we are done.

(B) If $T \leq T_{\mathcal{C}}$, the functions $F_\phi, \phi \in \mathcal{C}$, constructed for such T is identically null. Let us check that this implies not only *(ii)* in Theorem 1.1, but also the stronger assertion *(ii')* in the Remark (3) below it. Otherwise, assume that $\gamma : [0,1] \to \mathcal{M}, \gamma(s) = (t(s), \gamma_M(s))$, is a piecewise smooth future-directed timelike curve with $t(1) - t(0) = T, \gamma_M \in \mathcal{C}$, and let $\gamma^* : [0,1] \to \mathcal{M}', \gamma^*(s) = (t(s), \gamma_M^*(s))$ be a lifting of γ. If $\phi_0 \in \mathcal{C}$ is such that $\phi_0(\gamma_M^*(0)) = \gamma_M^*(1)$, then $F_{\phi_0}(\gamma_M^*(0)) > 0$, which is a contradiction.

(C) If $T > T_{\mathcal{C}}$, then $F_\phi \not\equiv 0$ and the maximum is reached at a point, $y \in M'$ for each $\phi \in \mathcal{C}$, Lemma 3.4. In particular, $(T, \phi(y)) \in I^+(0, y)$, and, as $\mathcal{M}'$ is globally hyperbolic, there exists a maximizing future-directed timelike geodesic $\gamma^* : [0,1] \to \mathcal{M}'$ joining $(0, y)$ and $(T, \phi(y))$.

If we consider the natural covering map $\mathcal{M}' \to \mathcal{M}/G$ ($\mathcal{M}/G$ defined at the end of Section 2), clearly the projection γ of γ^* satisfies $\gamma(0) = \gamma(1)$. Moreover, $\gamma'(0)$ must be equal to $\gamma'(1)$ because, otherwise, we could deform γ slightly in its extremes to obtain a timelike closed curve α in $\mathcal{C}$ of greater length. Lifting α to $\mathcal{M}'$ we would obtain a curve joining two points $(0, y'), (T, \phi'(y')), \phi' \in \mathcal{C}$, of greater length than the maximum of $F_{\phi'}$, which is a contradiction. Then, the timelike periodic trajectory associated to γ is the one required. $\square$

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