

## QUADRATIC $q$ -EXPONENTIALS AND CONNECTION COEFFICIENT PROBLEMS

MOURAD E. H. ISMAIL, MIZAN RAHMAN, AND DENNIS STANTON

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ABSTRACT. We establish expansion formulas of  $q$ -exponential functions in terms of continuous  $q$ -ultraspherical polynomials, continuous  $q$ -Hermite polynomials and Askey-Wilson polynomials. The proofs are based on solving connection coefficient problems.

### 1. INTRODUCTION

The  $q$ -exponential function on a  $q$ -quadratic grid is

$$(1.1) \quad \mathcal{E}_q(x; a, b) := \sum_{n=0}^{\infty} \frac{(aq^{(1-n)/2}e^{i\theta}, aq^{(1-n)/2}e^{-i\theta}; q)_n}{(q; q)_n} q^{n^2/4} b^n,$$

where  $x = \cos \theta$  [Is:Zh], [At:Su]. Ismail and Zhang [Is:Zh] gave a  $q$ -analogue of the expansion of the plane wave in spherical harmonics. Their formula is

$$(1.2) \quad \mathcal{E}_q(x; -i, b/2) = \frac{(q; q)_{\infty} (2/b)^{\nu}}{(-b^2/4; q^2)_{\infty} (q^{\nu}; q)_{\infty}} \sum_{n=0}^{\infty} i^n (1 - q^{n+\nu}) q^{n^2/4} J_{\nu+n}^{(2)}(b; q) C_n(x; q^{\nu}|q),$$

where  $J_{\nu+n}^{(2)}$  are  $q$ -Bessel functions [Is], [Ga:Ra] and  $C_n(x; \beta|q)$  are the continuous  $q$ -ultraspherical polynomials [As:Is], [Ga:Ra]. This formula has attracted some attention and two different proofs were given in [Fl:Vi] and [Is:Ra:Zh]. The proof by Floreanini and Vinet [Fl:Vi] is group theoretic and is of independent interest. Equation (1.2) was extended to continuous  $q$ -Jacobi polynomials in [Is:Ra:Zh], where the following expansion was established:

$$(1.3a) \quad \mathcal{E}_q(x; -i, r) = \sum_{m=0}^{\infty} a_m p_m(x; b, bq^{1/2}, -c, -cq^{1/2}),$$

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with the  $a_m$ 's given by

$$(1.3b) \quad a_m = \frac{(b^2c^2, b^2q^{1/2}; q)_m (irq^{1/2}; q)_\infty}{(q, bcq^{1/2}, bc; q)_m (-ir; q)_\infty} (ir/b)^m q^{m^2/4} \times {}_2\phi_1 \left( \begin{matrix} cq^{m/2+1/4}, -bq^{m/2+1/4} \\ bcq^{m+1/2} \end{matrix} \middle| q^{1/2}, ir \right).$$

The purpose of this paper is to extend (1.3) to an Askey-Wilson polynomial expansion (see (2.7)), and thereby also several special cases: (4.1), (4.4), (4.6). We also specialize (2.7) to find expansions of  $q$ -Bessel functions instead of the quadratic  $q$ -exponential function, see (3.2), (3.3). The fundamental technique is a connection coefficient result for Askey-Wilson polynomials (Theorem 1), which is established from the Nassrallah-Rahman integral [Ga:Ra, (6.3.9)]

It is known that expansions of the type treated in this paper are equivalent to inversions of certain lower triangular matrices [Fi:Is] and to Lagrange and  $q$ -Lagrange inversion [Ge:St]. In §6 we give the matrices and inverse relations lying behind the expansions established in this paper. We explain a positivity result for the connection coefficients for certain Askey-Wilson polynomials in §5.

We record here some the definitions and properties of the  $q$ -Bessel functions that will be required. The  $q$ -Bessel functions  $J_\nu^{(1)}(x; q)$  and  $J_\nu^{(2)}(x; q)$  are defined by

$$(1.4a) \quad J_\nu^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n},$$

$$(1.4b) \quad J_\nu^{(2)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n} q^{n(\nu+n)}.$$

They are related through

$$(1.5) \quad J_\nu^{(2)}(x; q) = (-x^2/4; q)_\infty J_\nu^{(1)}(x; q).$$

We will also need

$$(1.6) \quad {}_2\phi_1(-q^{\nu+1}, -q^{\nu+2}; q^{2\nu+2}; q^2, -b^2/4) = \frac{(q; q)_\infty}{(q^{\nu+1}; q)_\infty} \frac{(2/b)^\nu}{(-b^2/4; q^2)_\infty} J_\nu^{(2)}(b; q),$$

which follows from (1.23) of [Ra] (note a misprint) via [Ga:Ra, (III.4)].

## 2. A BASIC EXPANSION FORMULA

We prove Theorem 1, which is the fundamental theorem in this paper. It allows one to expand any function of  $(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n$  in terms of Askey-Wilson polynomials which are defined by (see [As:Wi] or [Ga:Ra])

$$p_n(\cos \theta; a, b, c, d) := (ab, ac, ad; q)_n a^{-n} r_n(\cos \theta; a, b, c, d),$$

where

$$r_n(\cos \theta; a, b, c, d) := {}_4\phi_3 \left( \begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q, q \right).$$

The main difference between  $p_n$  and  $r_n$  is that  $p_n$  is symmetric in all four parameters while  $r_n$  is symmetric in  $b, c, d$  but not in  $a$ . The basic expansion formula that we

shall prove in this section is

$$(2.1) \quad (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n = \sum_{j=0}^n A_{n,j} p_j(x; a, b, c, d).$$

**Theorem 1.** *The coefficients  $A_{n,j}$  in (2.1) are given by*

$$(2.2) \quad A_{n,j} = \frac{(q, \alpha d; q)_n (\alpha/d; q)_{n-j} (abcd/q; q)_j}{(q, \alpha d; q)_j (q; q)_{n-j} (abcd/q; q)_{2j}} \times (-\alpha)^j q^{j(j-1)/2} {}_4\phi_3 \left( \begin{matrix} q^{j-n}, adq^j, bdq^j, cdq^j \\ \alpha dq^j, dq^{1-n+j}/\alpha, abcdq^{2j} \end{matrix} \middle| q, q \right).$$

*Proof.* From the orthogonality relation of the Askey-Wilson polynomials, (7.5.15)-(7.5.17) in [Ga:Ra], we find that

$$(2.3) \quad \begin{aligned} \kappa A_{n,j} &= h_j \int_{-1}^1 \frac{h(x; 1, -1, q^{1/2}, -q^{1/2}, \alpha)}{h(x; a, b, c, d, \alpha q^n)} p_j(x; a, b, c, d) \frac{dx}{\sqrt{1-x^2}} \\ &= (ab, ac, ad; q)_j a^{-j} h_j \sum_{k=0}^j \frac{(q^{-j}, abcdq^{j-1}; q)_k q^k}{(q, ab, ac, ad; q)_k} \\ &\quad \times \int_{-1}^1 \frac{h(x; 1, -1, q^{1/2}, -q^{1/2}, \alpha)}{h(x; aq^k, b, c, d, \alpha q^n)} \frac{dx}{\sqrt{1-x^2}}, \end{aligned}$$

where

$$h(\cos \theta; \alpha) := (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty, \quad h(x; a_1, \dots, a_n) := \prod_{j=1}^n h(x; a_j),$$

the  $h_j$ 's are

$$h_j = \frac{1 - abcdq^{2j-1}}{1 - abcd/q} \frac{(abcd/q; q)_j}{(q, ab, ac, ad, bc, bd, cd; q)_j}.$$

and  $\kappa$  is

$$\kappa := \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

This is valid if  $\max(|a|, |b|, |c|, |d|) < 1$ . The integral in (2.3) can be evaluated by the Nassrallah-Rahman integral [Ga:Ra, (6.3.8)], and is equal to

$$\begin{aligned} &\frac{2\pi(\alpha b, \alpha c, \alpha d, abcdq^k, \alpha bcdq^n; q)_\infty}{(q, bc, bd, cd, abq^k, acq^k, adq^k, \alpha bq^n, \alpha cq^n, \alpha dq^n, abcd; q)_\infty} \\ &\quad \times {}_8W_7(\alpha bcd/q; bc, bd, cd, \alpha q^{-k}/a, q^{-n}; q, a\alpha q^{n+k}) \\ &= \kappa (\alpha d, \alpha/d; q)_n (ab, ac; q)_k \\ &\quad \times \sum_{m=0}^n \frac{(q^{-n}, bd, cd; q)_m q^m}{(q, \alpha d, dq^{1-n}/\alpha; q)_m} \frac{(ad; q)_{k+m}}{(abcd; q)_{k+m}}, \end{aligned}$$

where (III.17) of [Ga:Ra] was used in the last step. Thus the  $A_{n,j}$ 's have the representation

$$(2.4) \quad A_{n,j} = (ab, ac, ad; q)_j a^{-j} h_j(\alpha d, \alpha/d; q)_n \sum_{m=0}^n \frac{(q^{-n}, ad, bd, cd; q)_m}{(q, d\alpha, dq^{1-n}/\alpha, abcd; q)_m} q^m \\ \times {}_3\phi_2 \left( \begin{matrix} q^{-j}, abcdq^{j-1}, adq^m \\ abcdq^m, ad \end{matrix} \middle| q, q \right).$$

The  ${}_3\phi_2$  in (2.4) is now summed by (II.12) in [Ga:Ra]. After some simplification we establish (2.2), completing the proof of Theorem 1.  $\square$

Application of the Sears transformation [Ga:Ra, (III.15)] gives an alternate representation of  $A_{n,j}$ , which we will find useful later in this work. It is

$$(2.5) \quad A_{n,j} = \frac{(q, c\alpha, d\alpha, ab; q)_n (abcd/q; q)_j}{(q, c\alpha, d\alpha, ab; q)_j (q; q)_{n-j} (abcd; q)_{n+j}} \frac{1 - abcdq^{2j-1}}{1 - abcd/q} (-\alpha)^j q^{j(j-1)/2} \\ \times {}_4\phi_3 \left( \begin{matrix} q^{j-n}, cdq^j, \alpha/a, \alpha/b \\ \alpha cq^j, \alpha dq^j, q^{1-n}/ab \end{matrix} \middle| q, q \right).$$

It is clear from (2.1) that  $A_{n,j}$  must be symmetric in  $a, b, c, d$  which, however, is not obvious in either of the two forms (2.2) or (2.5). The symmetry becomes explicit when we apply the Watson transformation [Ga:Ra, (III.17)] to obtain the following representation:

$$(2.6) \quad A_{n,j} = \frac{(q, a\alpha, b\alpha, c\alpha, d\alpha; q)_n (abcd/q, \alpha^2; q)_j}{(q, a\alpha, b\alpha, c\alpha, d\alpha; q)_j (\alpha^2; q)_n (q; q)_{n-j}} \frac{1 - abcdq^{2j-1}}{1 - abcd/q} \frac{(-\alpha)^j q^{j(j-1)/2}}{(abcd; q)_{n+j}} \\ \times {}_8W_7(\alpha^2 q^{j-1}; q^{j-n}, \alpha/a, \alpha/b, \alpha/c, \alpha/d; abcdq^{n+j}).$$

In the subsequent sections we shall use Theorem 1 in the following way. We have

$$(2.7) \quad \sum_{n=0}^{\infty} c_n (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n = \sum_{j=0}^{\infty} p_j(\cos \theta; a, b, c, d) \sum_{n=0}^{\infty} A_{n+j,j} c_{n+j},$$

which is obtained from (2.1) by multiplying by  $c_n$  and summing over  $n$ , where  $\{c_n\}$  is an arbitrary sequence, provided that the left-hand side of (2.7) converges and interchanging the sums is justifiable. If we choose  $\alpha = eq^{(1-n)/2}$  for some constant  $e$  and  $c_n$  in accordance with (1.1), then (2.7) is an expansion for a  $q$ -exponential function  $\mathcal{E}_q$ . We will specialize the parameters in such a way that the  ${}_4\phi_3$  series in (2.2) and (2.5), or the  ${}_8W_7$  series in (2.6), can be summed. We will also choose  $\alpha$  independent of  $n$ , obtaining basic hypergeometric series expansions.

### 3. EXPANSIONS IN CONTINUOUS $q$ -ULTRASPHERICAL POLYNOMIALS

In this section we first find two general Askey-Wilson polynomial expansions with  $q$ -Bessel functions, (3.2) and (3.3). First, if  $\alpha = a$ , then  $A_{n,j}$  is summable by (2.5). Next we specialize to  $\alpha = -iq^{(1-n)/2}$  and the continuous  $q$ -ultraspherical polynomials, and prove (1.2).

If  $\alpha = a$ , and

$$c_n = \frac{(E_1, E_2; q)_n}{(q, ab, ac, ad; q)_n} (-B^2/4)^n,$$

then (2.5) and (2.7) imply

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(ae^{i\theta}, ae^{-i\theta}, E_1, E_2; q)_n}{(q, ab, ac, ad; q)_n} (-B^2/4)^n \\
 (3.1) \quad &= \sum_{j=0}^{\infty} \frac{(abcd/q, E_1, E_2; q)_j}{(q, ab, ac, ad; q)_j} \frac{1 - abcdq^{2j-1}}{1 - abcd/q} (aB^2/4)^j q^{j(j-1)/2} p_j(x; a, b, c, d) \\
 & \times {}_2\phi_1 \left( \begin{matrix} E_1q^j, E_2q^j \\ abcdq^{2j} \end{matrix} \middle| q, -B^2/4 \right).
 \end{aligned}$$

Setting  $abcd = q^{2\nu+1}$ ,  $E_1 = -q^{\nu+1/2}$ , and  $E_2 = -q^{\nu+1}$ , in (3.1) and using (1.6) we find

$$\begin{aligned}
 (3.2) \quad & {}_4\phi_3 \left( \begin{matrix} ae^{i\theta}, ae^{-i\theta}, -q^{\nu+1/2}, -q^{\nu+1} \\ ab, ac, ad \end{matrix} \middle| q, -\frac{B^2}{4} \right) \\
 &= \frac{(q^{1/2}; q^{1/2})_{\infty}}{(q^{\nu+1/2}; q^{1/2})_{\infty} (-b^2/4; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(q^{2\nu}; q)_j (q^{\nu+1/2}; q^{1/2})_{2j}}{(q, ab, ac, ad; q)_j} \frac{1 - q^{2\nu+2j}}{1 - q^{2\nu}} q^{j(j-1)/2} a^j \\
 & \times (2/b)^{2\nu} J_{2\nu+2j}^{(2)}(B; q^{1/2}) p_j(x; a, b, c, d).
 \end{aligned}$$

We may find another  $q$ -Bessel expansion by choosing  $E_1 = E_2 = 0$  in (3.1), and using (1.4a) and (1.5). The result (again for  $abcd = q^{2\nu+1}$ ) is

$$\begin{aligned}
 (3.3) \quad & {}_4\phi_3 \left( \begin{matrix} ae^{i\theta}, ae^{-i\theta}, 0, 0 \\ ab, ac, ad \end{matrix} \middle| q, -\frac{B^2}{4} \right) \\
 &= \frac{(q; q)_{\infty} (2/B)^{2\nu}}{(q^{2\nu+1}, -B^2/4; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(q^{2\nu}; q)_j (q^{2\nu+1}; q)_{2j}}{(q, ab, ac, ad; q)_j} \frac{1 - q^{2\nu+2j}}{1 - q^{2\nu}} q^{j(j-1)/2} a^j \\
 & \times J_{2\nu+2j}^{(2)}(B; q) p_j(x; a, b, c, d).
 \end{aligned}$$

The continuous  $q$ -ultraspherical polynomials  $C_n(x; a^2|q)$  are obtained from the Askey-Wilson polynomials by

$$p_n(x; a, a\sqrt{q}, -a, -a\sqrt{q}) = \frac{(q; q)_n (a^4; q)_{2n}}{(a^2, a^4; q)_n} C_n(x; a^2|q).$$

Thus  $q$ -ultraspherical versions of (3.2) and (3.3) can be found by putting

$$\{a, b, c, d\} = \{q^{\nu/2}, q^{(\nu+1)/2}, -q^{\nu/2}, -q^{(\nu+1)/2}\}.$$

The left-hand sides then become other  $q$ -analogues of the Bessel function  $J_{\nu+1/2}$ .

Next we consider (2.7) for the continuous  $q$ -ultraspherical polynomials ( $b = a\sqrt{q}, c = -a, d = -a\sqrt{q}$ ) for an arbitrary  $\alpha$ . Using (3.4.7) in [Ga:Ra], we have

$$\begin{aligned}
 & {}_8W_7(\alpha^2q^{j-1}; q^{j-n}, \alpha q^{-1/2}/a, -\alpha q^{-1/2}/a, \alpha/a, -\alpha/a; q, a^4q^{n+j+1}) \\
 &= \frac{(\alpha^2q^j, a^4q^{2j+1}; q)_{\infty}}{(a^2\alpha^2q^{2j}, a^2q^{j+1}; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} \alpha^2/q a^2, q^{-n}/a^2 \\ \alpha^2q^n \end{matrix} \middle| q, a^4q^{n+j+1} \right).
 \end{aligned}$$

According to (III.2) in [Ga:Ra] the above  ${}_2\phi_1$  can be transformed to

$$\frac{(\alpha^2 q^j, a^2 \alpha^2 q^{2n}, a^4 q^{2j+1}; q)_\infty}{(\alpha^2 q^n, a^2 \alpha^2 q^{2j}, a^4 q^{n+j+1}; q)_\infty} {}_2\phi_1 \left( \begin{matrix} q^{j-n}, q^{-n}/a^2 \\ a^2 q^{j+1} \end{matrix} \middle| q, a^2 \alpha^2 q^{2n} \right).$$

From this we find that Theorem 1 reduces to

$$(3.4) \quad (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n = \sum_{j=0}^n \frac{(q; q)_n (-\alpha)^j q^{j(j-1)/2}}{(q; q)_{n-j} (a^2; q)_j} C_j(x; a^2|q) \times {}_2\phi_1 \left( \begin{matrix} q^{j-n}, q^{-n}/a^2 \\ a^2 q^{j+1} \end{matrix} \middle| q, a^2 \alpha^2 q^{2n} \right).$$

There are several choices for  $\alpha^2$  for which the  ${}_2\phi_1$  series in (3.4) is summable. The choice of interest here is  $\alpha = -iq^{(1-n)/2}$ , so that the Bailey-Daum formula [Ga:Ra, (II.9)] implies

$${}_2\phi_1 \left( \begin{matrix} q^{j-n}, q^{-n}/a^2 \\ a^2 q^{j+1} \end{matrix} \middle| q, -a^2 q^{n+1} \right) = \begin{cases} 0 & \text{if } n - j \text{ is odd,} \\ \frac{(q, q^{-2n}/a^4; q^2)_m}{(q^{-n}/a^2, q^{1-n}/a^2; q^2)_m} & \text{if } n - j = 2m. \end{cases}$$

The appropriate choice of  $c_n$  leads to

$$(3.5) \quad \mathcal{E}_q(x; -i, b/2) = \sum_{j=0}^\infty \frac{q^{j^2/4} (ib/2)^j}{(a^2; q)_j} C_j(x; a^2|q) \times {}_2\phi_1 \left( \begin{matrix} -a^2 q^{j+1}, -a^2 q^{j+2} \\ a^4 q^{2j+2} \end{matrix} \middle| q^2, -\frac{b^2}{4} \right).$$

Setting  $a = q^{\nu/2}$  in (3.5) and using (1.6), we see that (1.2) follows from (3.5).

By taking the limit  $\nu \rightarrow \infty$  in (1.3) we find that

$$(3.6) \quad \mathcal{E}_q(x; -i, b) = \frac{1}{(-b^2; q^2)_\infty} \sum_{n=0}^\infty (ib)^n q^{n^2/4} \frac{H_n(x|q)}{(q; q)_n},$$

where

$$H_n(x|q) = (q; q)_n C_n(x; 0|q),$$

are the continuous  $q$ -Hermite polynomials.

#### 4. $\mathcal{E}_q$ EXPANSIONS IN SPECIAL ASKEY-WILSON POLYNOMIALS

In this section we evaluate  $A_{n,j}$  using special balanced  ${}_4\phi_3$ 's. This naturally leads to expansions in special families of Askey-Wilson polynomials, see (4.1), (4.2), (4.3), (4.4). We also choose two specializations so that  $A_{n,j}$  is transformable, yielding  $\mathcal{E}_q$  expansions, including (1.3). At the end of the section we also prove the Al-Salam-Chihara polynomial result by generating functions.

Andrews' terminating  $q$ -analogue of the Watson's  ${}_3F_2$  sum is [Ga:Ra, (II.17)]

$${}_4\phi_3 \left( \begin{matrix} q^{-n}, Aq^n, C, -C \\ \sqrt{Aq}, -\sqrt{Aq}, C^2 \end{matrix} \middle| q, q \right) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{C^n (q, qA/C^2; q^2)_{n/2}}{(qA, qC^2; q^2)_{n/2}} & \text{if } n \text{ is even.} \end{cases}$$

This suggests that after replacing  $n$  by  $n + j$  in (2.5), we take

$$\alpha = -iq^{(1-n)/2}, \quad a = -b, \quad c = -d, \quad C = \pm iq^{(1-n-j)/2}/a, \quad A = -c^2q^{j-n}.$$

Therefore  $A_{2m+j+1,j} = 0$ , and

$$\frac{A_{2m+j,j}}{(q; q)_{2m+j}} = \frac{(a^2c^2/q; q)_j (q; q)_{j+2m} (-a^2q^{j+1}, -c^2q^{j+1}; q^2)_m}{(q; q)_j (a^2c^2/q; q)_{2j} (q^2, a^2c^2q^{2j+1}; q^2)_m} i^j (-1)^m q^{-m^2-mj}.$$

This leads to the expansion

$$\begin{aligned} \mathcal{E}_q(x; -i, b) &= \sum_{j=0}^{\infty} \frac{(a^2c^2/q; q)_j}{(q; q)_j (a^2c^2/q; q)_j} p_j(x; a, -a, c, -c) \\ (4.1) \quad &\times (ib)^j q^{j^2/4} {}_2\phi_1 \left( \begin{matrix} -a^2q^{j+1}, -c^2q^{j+1} \\ a^2c^2q^{2j+1} \end{matrix} \middle| q^2, -b^2 \right). \end{aligned}$$

In [Is:Ma:Su] it was pointed out that

$$\left(\frac{r}{2}\right)^\nu \frac{(q^{\nu+1}, -r^2/4; q)_\infty}{(q; q)_\infty} {}_2\phi_1 \left( \begin{matrix} q^{(\nu+1)/2}e^{i\theta}, q^{(\nu+1)/2}e^{-i\theta} \\ q^{\nu+1} \end{matrix} \middle| q, -\frac{r^2}{4} \right)$$

is a  $q$ -analogue of  $J_\nu(xr)$ ,  $x = \cos \theta$ . The  ${}_2\phi_1$ 's in (4.1) can be expressed in terms of the above  $q$ -Bessel function.

If we set  $c = 0$ , then the  ${}_2\phi_1$  in (4.1) becomes a  ${}_1\phi_0$ , which can be summed by the  $q$ -binomial theorem [Ga:Ra, (II.3)]. We find

$$(4.2) \quad \mathcal{E}_q(x; -i, b) = \sum_{n=0}^{\infty} \frac{(-a^2; q)_n (a^2b^2q^{n+1}; q^2)_\infty}{(q; q)_n (-b^2; q^2)_\infty} \left(\frac{ib}{a}\right)^n q^{n^2/4} r_n(x; a, -a, 0, 0).$$

Since (7.5.34) of [Ga:Ra] implies

$$r_n(x; q^{1/2}, -q^{1/2}, 0, 0) = \frac{q^{n/2}}{(-q; q)_n} H_n(x|q^2),$$

we find that

$$(4.3) \quad \mathcal{E}_q(x; -i, b) = \sum_{n=0}^{\infty} \frac{(b^2q^{n+2}; q^2)_\infty}{(-b^2; q^2)_\infty (q; q)_n} (ib)^n q^{n^2/4} H_n(x|q^2),$$

which may be compared with (3.6).

Another balanced  ${}_4\phi_3$  evaluation is [An, (4.3)]

$${}_4\phi_3 \left( \begin{matrix} q^{-2n}, A^2q^{2n}, B, Bq \\ Aq, Aq^2, B^2 \end{matrix} \middle| q^2, q^2 \right) = B^n \frac{(Aq/B, -q; q)_n}{(A, -B; q)_n} \frac{1-A}{1-Aq^{2n}}.$$

So (2.2) is summable if  $b = aq^{1/2}$ ,  $d = cq^{1/2}$ , and  $\alpha^2 = q^{1/2-n}$ , or  $\alpha^2 = q^{3/2-n}$ .

After replacing  $c$  by  $-c$ , the  $\alpha = q^{(1-2n)/4}$  result with  $c_n = q^{n(n+1)/4}(it/2)^n/(q; q)_n$  is

$$\begin{aligned}
 \mathcal{E}_q(x; q^{-1/4}, itq^{1/4}/2) &= \sum_{n=0}^{\infty} \frac{(a^2c^2; q)_n}{(q; q)_n (a^2c^2; q)_{2n}} q^{n^2/4} \left(-\frac{it}{2}\right)^n \\
 (4.4) \quad &\times p_n(x; a, aq^{1/2}, -c, -cq^{1/2}) \\
 &\times {}_2\phi_1 \left( \begin{matrix} aq^{(2n+1)/4}, -cq^{(2n+1)/4} \\ acq^{n+1/2} \end{matrix} \middle| q^{1/2}, \frac{it}{2} \right).
 \end{aligned}$$

The choice of  $\alpha = q^{(3-2n)/4}$  (thus expanding  $\mathcal{E}_q(x; q^{1/4}, itq^{-1/4}/2)$ ) gives the same expansion as (4.4). These two  $\mathcal{E}_q$  functions are identical from [Su, (3.4)].

If, instead, we take  $a = -iq^{(1-n)/2}$ , then the  ${}_4\phi_3$  series in (2.5) can no longer be summed. However the  ${}_4\phi_3$  series in (2.5) can still be transformed to another  ${}_4\phi_3$  series in base  $q^{1/2}$  by [Ga:Ra, (III.21)]. It turns out that the series over  $n$  in (2.7), a double sum, can be simplified further by an interchange of the order of summation and finally reduced to a single sum. The result is equivalent to (1.3),

$$\begin{aligned}
 \mathcal{E}_q(x; -i, t/2) &= \frac{(itq^{1/2}/2; q)_{\infty}}{(-it/2; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a^2c^2; q)_n q^{n^2/4}}{(q; q)_n (a^2c^2; q)_{2n}} \left(-\frac{it}{2}\right)^n \\
 (4.5) \quad &\times p_n(x; a, aq^{1/2}, -c, -cq^{1/2}) \\
 &\times {}_2\phi_1 \left( \begin{matrix} aq^{(2n+1)/4}, -cq^{(2n+1)/4} \\ acq^{n+1/2} \end{matrix} \middle| q^{1/2}, \frac{it}{2} \right).
 \end{aligned}$$

The right sides of (4.4) and (4.5) are identical. This follows from Suslov’s addition theorem for the  $\mathcal{E}_q$  functions; see [Su, Theorem 3.1].

We have another example when a  ${}_4\phi_3$  transformation, not a summation, leads to an  $\mathcal{E}_q$  expansion. The Al-Salam-Chihara polynomials are defined by

$$p_n(x; a, b) := r_n(x; a, b, 0, 0).$$

We have (see also [Fl:Le:Vi2], (3.17))

$$\begin{aligned}
 \mathcal{E}_q(x; -i, b) &= \sum_{n=0}^{\infty} \frac{(-\gamma^2 b^2 q^n; q^2)_{\infty}}{(-b^2; q^2)_{\infty}} (ib)^n q^{n^2/4} p_n(x; \gamma e^{i\phi}, \gamma e^{-i\phi}) \\
 (4.6) \quad &\times \gamma^{-n} e^{-in\phi} \frac{(\gamma^2; q)_n}{(q; q)_n} \mathcal{E}_q(\cos \phi; -i, \gamma b q^{n/2}).
 \end{aligned}$$

To prove (4.6) we find the coefficient of  $b^n p_j(x; \gamma e^{-i\phi}, \gamma e^{i\phi})$  on both sides. The left-side coefficient (by (2.7) and (2.5)) is a  ${}_3\phi_2$  on base  $q$  with one denominator parameter equal to 0. The right-side coefficient (by the  $q$ -binomial theorem) gives a  ${}_3\phi_2$  on base  $q^2$ . These are equal using [Ga:Ra, (3.10.13), (3.2.2)].

The preceding sketch of the proof of (4.6) is not transparent, so we also prove (4.6) from earlier results in this paper.

The Al-Salam-Chihara polynomials have the generating function [As:Is2]

$$\sum_{n=0}^{\infty} \frac{(t_1 t_2; q)_n}{(q; q)_n} (t/t_1)^n p_n(\cos \theta, t_1, t_2) = \frac{(tt_1, tt_2; q)_{\infty}}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}}.$$

The continuous  $q$ -Hermite polynomials, see (3.10), have the generating function [As:Is]

$$\sum_{n=0}^{\infty} H_n(\cos \theta|q)t^n / (q; q)_n = 1 / (te^{i\theta}, te^{-i\theta}; q)_{\infty}.$$

Evidently the above two generating functions imply

$$(4.7) \quad \frac{H_n(x|q)}{(q; q)_n} = \sum_{j=0}^n \frac{H_j(\cos \phi|q)}{(q; q)_j} p_{n-j}(x; \gamma e^{i\phi}, \gamma e^{-i\phi}) \gamma^{2j-n} e^{-i(n-j)\phi} \frac{(\gamma^2; q)_{n-j}}{(q; q)_{n-j}}.$$

We then combine (4.7) and (3.6) to obtain

$$\begin{aligned} \mathcal{E}_q(x; -i, b) &= \frac{1}{(-b^2; q^2)_{\infty}} \sum_{n=0}^{\infty} (ib)^n q^{n^2/4} p_n(x; \gamma e^{i\phi}, \gamma e^{-i\phi}) \gamma^{-n} e^{-in\phi} \frac{(\gamma^2; q)_n}{(q; q)_n} \\ &\quad \times \sum_{j=0}^{\infty} \frac{H_j(\cos \phi|q)}{(q; q)_j} \gamma^j q^{j^2/4} (ibq^{n/2})^j. \end{aligned}$$

The  $j$  sum can be expressed in terms of  $\mathcal{E}_q$  by (3.6) to obtain (4.6).

It is not hard to see that choosing  $\phi = \pi/2$ ,  $\gamma = -i\sqrt{q}$  in (4.6) gives (4.3).

### 5. CONNECTION COEFFICIENTS FOR ASKEY-WILSON POLYNOMIALS

From (2.2) and (2.7) we have

$$r_m(x; \alpha, \beta, \gamma, \delta) = \sum_{j=0}^m B_{m,j} r_j(x; a, b, c, d),$$

where

$$\begin{aligned} (5.1) \quad B_{m,j} &= \frac{(ab, ac, ad, q^{-m}, \alpha\beta\gamma\delta q^{m-1}, abcd/q; q)_j}{(\alpha\beta, \alpha\gamma, q, abcd/q; q)_j} \\ &\quad \times \left(-\frac{\alpha}{a}\right)^{j(j+1)/2} \sum_{n=0}^{m-j} \frac{(q^{j-m}, \alpha\beta\gamma\delta q^{m+j-1}, \alpha dq^j, \alpha/d; q)_n}{(q, \alpha\beta q^j, \alpha\gamma q^j, \alpha\delta q^j; q)_n} q^n \\ &\quad \times {}_4\phi_3 \left( \begin{matrix} q^{-n}, adq^j, bdq^j, cdq^j \\ dq^{1-n}/\alpha, \alpha dq^j, abcdq^{2j} \end{matrix} \middle| q, q \right). \end{aligned}$$

We shall assume that

$$\max(|a|, |b|, |c|, |d|) < 1$$

and that  $\alpha$  and  $a$  are of the same sign. Askey and Wilson [As:Wi] have shown that the connection coefficients are positive when

$$0 < \alpha < a < 1, \quad b = \beta, \quad c = \gamma, \quad d = \delta.$$

This can be iterated using the symmetry of  $p_n(x; a, b, c, d)$  in its parameters. Thus we have proven that the  $B_{m,j}$  of (5.1) is nonnegative for

$$0 < \alpha < a < 1, \quad 0 < \beta < b < 1, \quad 0 < \gamma < c < 1, \quad 0 < \delta < d < 1.$$

## 6. MATRIX INVERSION

In this section we prove Theorem 1 from an explicit matrix inversion.

Let

$$\phi_n(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_n, \quad x = \cos \theta.$$

It follows from [Ga:Ra, (II.12)] that

$$\phi_n(x, \alpha) = \sum_{k=0}^n C_{nk} \phi_k(x, a),$$

where

$$C_{nk} = \frac{q^k (q^{-n}; q)_k (a\alpha, \alpha/a; q)_n}{(q, a\alpha, aq^{1-n}/\alpha; q)_k}.$$

To establish Theorem 1, it remains to expand  $\phi_k(x, a)$  in terms of the Askey-Wilson polynomials  $p_j(x; a, b, c, d)$  by inverting

$$p_n(x; a, b, c, d) = \sum_{j=0}^n D_{nj}(t) c_j \phi_j(x, a),$$

where

$$D_{nj}(t) = (q^{-n}, tq^n; q)_j, \quad t = abcd/q, \quad 1/c_j = q^{-j}(q, ab, ac, ad; q)_j.$$

It is known (see for example (3.6.19) and (3.6.20) in [Ga:Ra]) that

$$(6.1) \quad D_{jk}^{-1}(t) = \frac{t^{-k} q^{j-k^2}}{(q, q^{1-2k}/t; q)_k (q, tq^{1+2k}; q)_{j-k}}.$$

Thus Theorem 1 follows from the matrix inversion (6.1). Since there is bibasic version of (6.1), there is also a bibasic version of Theorem 1.

We can also find the inverse of the lower triangular matrix  $A_{n,j}$  of Theorem 1 by expanding the polynomials  $p_j(x; a, b, c, d)$  in terms of  $\phi_l(x; \alpha)$ . Using (6.1) we find

$$\begin{aligned} A_{j,l}^{-1} &= q^l a^{l-j} \alpha^{-l} (abq^l, acq^l, adq^l; q)_{j-l} \frac{(abcdq^{j-1}, q^{-j}; q)_l}{(q; q)_l} \\ &\quad \times {}_4\phi_3 \left( \begin{matrix} q^{l-j}, abcdq^{j+l-1}, a/\alpha, a\alpha q^l \\ abq^l, acq^l, adq^l \end{matrix} \middle| q, q \right), \end{aligned}$$

which gives a discrete orthogonality relation for a  ${}_4\phi_3$ .

## REFERENCES

- [Al:Ch] W. A. Al-Salam and T. S. Chihara, *Convolutions of orthogonal polynomials*, SIAM J. Math. Anal. **7** (1976), 16-28. MR **53**:3381
- [An] G. E. Andrews, *Connection coefficient problems and partitions*, Proc. Symp. Pure Math **34** (1979), 1-24. MR **80c**:33004
- [As:Is] R. A. Askey and M. E. H. Ismail, *A generalization of ultraspherical polynomials*, Studies in Pure Mathematics (P. Erdős, ed.), Birkhauser, Basel, 1983, pp. 55-78. MR **87a**:33015
- [As:Is2] R. A. Askey and M. E. H. Ismail, *Recurrence relations, continued fractions and orthogonal polynomials*, Memoirs Amer. Math. Soc. **49** (1984). MR **85g**:33008
- [As:Wi] R. A. Askey and J. A. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs Amer. Math. Soc. **54** (1985). MR **87a**:05023
- [At:Su] N. M. Atakishiev and S. K. Suslov, *Difference hypergeometric functions*, Progress in Approximation Theory, An International Perspective (A. A. Gonchar and E. B. Saff, ed.), Springer-Verlag, New York, 1992, pp. 1-35. MR **94k**:33039

- [Fi:Is] J. Fields and M. E. H. Ismail, *Polynomial expansions*, Mathematics of Computation **29** (1975), 894-902. MR **51**:8680
- [Fl:Vi] R. Floreanini and L. Vinet, *A model for the continuous  $q$ -ultraspherical polynomials*, J. Math. Phys. **36** (1995), 3800-3813. MR **96d**:33012
- [Fl:Le:Vi] R. Floreanini, J. LeTourneux and L. Vinet, *More on the  $q$ -oscillator algebra and  $q$ -orthogonal polynomials*, Journal of Physics A **28** (1995), L287-L293. MR **96e**:33043
- [Fl:Le:Vi2] R. Floreanini, J. LeTourneux and L. Vinet, *Symmetry techniques for the Al-Salam-Chihara polynomials*, J. Phys. A **30** (1997), 3107-3114. MR **98k**:33036
- [Ga:Ra] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990. MR **91d**:33034
- [Ge:St] I. Gessel and D. Stanton, *Applications of  $q$ -Lagrange inversion to basic hypergeometric series*, Trans. Amer. Math. Soc. **277** (1983), 173-201. MR **84f**:33009
- [Is] M. E. H. Ismail, *The zeros of basic Bessel functions the functions  $J_{\nu+ax}(x)$  and the associated orthogonal polynomials*, J. Math. Anal. Appl. **86** (1982), 1-19. MR **83c**:33010
- [Is:Ma:Su] M. E. H. Ismail, D. Masson and S. Suslov, *The  $q$ -Bessel functions on  $q$ -quadratic grid* (to appear).
- [Is:Ra:Zh] M. E. H. Ismail, M. Rahman and R. Zhang, *Diagonalization of certain integral operators II*, J. Comp. Appl. Math. **68** (1996), 163-196. MR **98d**:33011
- [Is:Zh] M. E. H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Advances in Math. **109** (1994), 1-33. MR **96d**:39005
- [Lu] Y. Luke, *The Special Functions and Their Approximations, volume 2*, Academic Press, New York, 1969. MR **40**:2909
- [Ra] M. Rahman, *An integral representation and some transformation properties of  $q$ -Bessel functions*, J. Math. Anal. Appl. **125** (1987), 58-71. MR **88h**:33020
- [Su] S. K. Suslov, *Addition theorems for some  $q$ -exponential and trigonometric functions*, Methods and Applications of Analysis **4** (1997), 11-32. CMP 97:14

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620-5700  
*E-mail address:* ismail@math.usf.edu

DEPARTMENT OF MATHEMATICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA K1S 5B6  
*E-mail address:* mrahman@math.carleton.ca

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455  
*E-mail address:* stanton@math.umn.edu