

ON THE MULTIPLICITY OF EIGENVALUES OF
A VECTORIAL STURM-LIOUVILLE DIFFERENTIAL EQUATION
AND SOME RELATED SPECTRAL PROBLEMS

CHAO-LIANG SHEN AND CHUNG-TSUN SHIEH

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ABSTRACT. We prove that under certain conditions, a vectorial Sturm-Liouville differential equation of dimension $n \geq 2$ can only possess finitely many eigenvalues which have multiplicity n . For the case $n = 2$, we find a sufficient condition on the potential function $Q(x)$, and a bound m_Q depending on $Q(x)$, such that the eigenvalues of the equation with index exceeding m_Q are all simple. These results are applied to find some sufficient conditions which imply that the spectra of two potential equations, or two string equations, have finitely many elements in common, and an estimate of the number of elements in the intersection of two spectra is provided.

1. INTRODUCTION

It was known a long time ago that if $q(x)$ is a nice real-valued function on $0 \leq x \leq 1$, and $\rho(x)$ is a nice positive function on $0 \leq x \leq 1$, then the eigenvalues of the *potential equation* with *potential function* $q(x)$,

$$(1) \quad y''(x) + (\lambda - q(x))y(x) = 0, \quad y(0) = y(1) = 0,$$

and the eigenvalues of the *string equation* with *density function* $\rho(x)$,

$$(2) \quad y''(x) + \mu\rho(x)y(x) = 0, \quad y(0) = y(1) = 0,$$

are simple, i.e., of multiplicity 1, and the k^{th} eigenfunction has exactly $k - 1$ sign-changed zeros in the interval $0 < x < 1$. But for the following n -dimensional vectorial Sturm-Liouville differential equation

$$(3) \quad \vec{z}''(x) + (\lambda I - Q(x))\vec{z}(x) = \vec{0}, \quad \vec{z}(0) = \vec{z}(1) = \vec{0},$$

where $n \geq 2$, I is the n -by- n identity matrix, $Q(x)$ is a continuous n -by- n real symmetric matrix-valued function defined on $0 \leq x \leq 1$, and $\vec{z}(x)$ is an R^n -valued function, the multiplicity of the eigenvalues and properties of the eigenfunctions are not well-understood. One purpose of this paper is to investigate the multiplicity problem of the equation (3) for the case when $Q(x)$ is a Jacobi matrix-valued function, i.e., $Q(x) = [q_{ij}(x)]$, where $q_{ii}(x) = p_i(x)$, $i = 1, 2, \dots, n$, $q_{j,j+1}(x) = q_{j+1,j}(x) = -r_j(x)$, $j = 1, \dots, n - 1$, and $q_{lm}(x) = 0$ if $|l - m| \geq 2$. We shall denote

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this $Q(x)$ by $J[p_1(x), \dots, p_n(x); r_1(x), \dots, r_{n-1}(x)]$, where $p_i(x)$ and $r_j(x)$, $1 \leq i \leq n$, $1 \leq j \leq n-1$, are assumed to be continuous functions. In section two, using the theory of matrix differential equations (see [2]), we prove the following theorem among others:

Theorem 2.3. *Suppose $Q(x) = J[p_1(x), \dots, p_n(x); r_1(x), \dots, r_{n-1}(x)]$, $n \geq 2$. If*

$$\int_0^1 r_1(x) dx \neq 0,$$

then, except finitely many eigenvalues, the multiplicity of the eigenvalues of (3) are at most $n-1$.

The case $n=2$ in Theorem 2.3 (also see Theorem 2.2) is more interesting for application. Let $\sigma(q)$ denote the spectrum of the potential equation (1) with potential function $q(x)$, and let $\sigma_s(\rho)$ denote the spectrum of the string equation (2) with density function $\rho(x)$. In section three, as applications of Theorem 2.3, we prove the following results.

Theorem 3.1. *Suppose $q_1(x)$ and $q_2(x)$ are two real-valued continuous functions defined on $0 \leq x \leq 1$. If*

$$\int_0^1 q_1(x) dx \neq \int_0^1 q_2(x) dx,$$

then $\sigma(q_1)$ and $\sigma(q_2)$ only have finitely many elements in common.

Theorem 3.2. *Suppose $\rho_1(x)$ and $\rho_2(x)$ are two positive C^2 -functions defined on $0 \leq x \leq 1$. If*

$$\int_0^1 \rho_1^{1/2}(x) dx = \int_0^1 \rho_2^{1/2}(x) dx,$$

and

$$\int_0^1 (\rho_1^{-1/4})(x) \frac{d^2}{dx^2} (\rho_1^{-1/4})(x) dx \neq \int_0^1 (\rho_2^{-1/4})(x) \frac{d^2}{dx^2} (\rho_2^{-1/4})(x) dx,$$

then $\sigma_s(\rho_1)$ and $\sigma_s(\rho_2)$ only have finitely many elements in common.

Finally, in section four, for a two-by-two potential function $Q(x)$ satisfying the condition of Theorem 2.2, we study the problem of finding a number m_Q such that for $n \geq m_Q$ the n^{th} eigenvalue of (3) is simple. Suppose $Q(x) = J[p_1(x), p_2(x); r_1(x)]$, $\int_0^1 r_1(x) dx \neq 0$. We find two constants Q_* and Λ_* , which are expressed in terms of $p_1(x)$, $p_2(x)$, and $r_1(x)$; these two numbers play important roles for studying the above mentioned problem. To be more precise, we have

Theorem 4.1. *If λ_n is an eigenvalue of (3) which satisfies the conditions*

$$\sqrt{\lambda_n} \geq 2Q_*, \quad \lambda_n > \Lambda_*,$$

then λ_n is a simple eigenvalue.

Using Theorem 4.1, an estimate of m_Q is found (see Theorem 4.2), and the result is applied to find an upper bound for the number of elements in the intersection of the spectra of two potential equations (see Theorem 4.3).

2. THE MULTIPLICITY OF EIGENVALUES
OF A VECTORIAL STURM-LIOUVILLE EQUATION

To study the eigenvalue problem (3) we consider the following matrix differential equation:

$$(4) \quad Y''(x) + (\lambda I - Q(x))Y(x) = 0, \quad Y(0) = 0, \quad Y'(0) = I,$$

where $Y(x)$ is an n -by- n matrix-valued function. We use $Y(x; \lambda)$ to denote the solution of the initial value problem (4). Then λ_* is an eigenvalue of (3) if and only if $Y(1; \lambda_*)$ is a singular matrix. The *multiplicity* of λ_* as an eigenvalue of (3) is equal to the dimension of the null space of $Y(1; \lambda_*)$. It is known (see [7, pp 304-305]) that (3) has a countably infinite number of real eigenvalues which are bounded from below. Counting the multiplicity of eigenvalues, we arrange all eigenvalues of (3) as an ascending sequence $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots$. The multiplicity of an eigenvalue λ_* of (3) is at most n , and the multiplicity of λ_* is n if and only if $Y(1; \lambda_*)$ is the zero matrix. The following asymptotic formula of $Y(x; \lambda)$ (see [1, (1.2.2)]) shall be used later: for $\lambda > 0$,

$$(5) \quad Y(x; \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}}I + O\left(\frac{1}{\lambda}\right), \quad \lambda \rightarrow \infty.$$

To study the multiplicity of eigenvalues of (3) for

$$Q(x) = J[p_1(x), \dots, p_n(x); r_1(x), \dots, r_{n-1}(x)],$$

first we treat the case $n = 2$.

Lemma 2.1. *If $Q(x) = J[p_1(x), p_2(x); r_1(x)]$, and λ_* is an eigenvalue of (3) which has multiplicity two, let $y_1(x) = \text{col}(y_{11}(x), y_{21}(x))$ and $y_2(x) = \text{col}(y_{12}(x), y_{22}(x))$ be two solutions of (3) corresponding to λ_* . Then we have the following identity:*

$$(6) \quad \int_0^1 r_1(x)\{y_{11}(x)y_{22}(x) - y_{12}(x)y_{21}(x)\}dx = 0.$$

Proof. Note that $y_{ij}(x)$, $i, j = 1, 2$, satisfy the following two relations:

$$(7) \quad y''_{11}(x) + \{\lambda_* - p_1(x)\}y_{11}(x) + r_1(x)y_{21}(x) = 0,$$

$$(8) \quad y''_{12}(x) + \{\lambda_* - p_1(x)\}y_{12}(x) + r_1(x)y_{22}(x) = 0.$$

By (7) and (8) we have

$$(9) \quad \begin{aligned} &\frac{d}{dx}\{y'_{11}(x)y_{12}(x) - y_{11}(x)y'_{12}(x)\} \\ &= r_1(x)\{y_{11}(x)y_{22}(x) - y_{12}(x)y_{21}(x)\}. \end{aligned}$$

Integrating (9) from $x = 0$ to $x = 1$ and using the boundary conditions $y_{ij}(0) = y_{ij}(1) = 0$, we obtain (6). □

Using Lemma 2.1 we prove the following result.

Theorem 2.2. *If $Q(x) = J[p_1(x), p_2(x); r_1(x)]$, and*

$$\int_0^1 r_1(x)dx \neq 0,$$

then, except finitely many eigenvalues, all eigenvalues of (3) are simple.

Proof. Suppose, on the contrary, there are infinitely many eigenvalues λ_{n_k} which have multiplicity two. Denote the solution of (4) for $\lambda = \lambda_{n_k}$ by

$$Y(x; \lambda_{n_k}) = [y_{ij}(x; \lambda_{n_k})]_{i,j=1}^2.$$

Then, by (5), we have

$$(10) \quad \det Y(x; \lambda_{n_k}) = \frac{1 - \cos(2\lambda_{n_k}^{1/2}x)}{2\lambda_{n_k}} + O(\lambda_{n_k}^{-3/2}).$$

(6) and (10) imply that

$$\int_0^1 r_1(x) \left\{ \frac{1 - \cos 2\lambda_{n_k}^{1/2}x}{2\lambda_{n_k}} + O(\lambda_{n_k}^{-3/2}) \right\} dx = 0,$$

which tells us that we have

$$(11) \quad \int_0^1 r_1(x) dx = \int_0^1 r_1(x) \cos(2\lambda_{n_k}^{1/2}x) dx + O(\lambda_{n_k}^{-1/2}).$$

By (11), the Riemann-Lebesgue Lemma tells us that if (3) has infinitely many eigenvalues of multiplicity two, then $\int_0^1 r_1(x) dx = 0$. Thus if $\int_0^1 r_1(x) dx \neq 0$, then all, except finitely many, eigenvalues of (3) are simple. \square

The result of Theorem 2.2 can be generalized as follows.

Theorem 2.3. *Suppose $Q(x) = J[p_1(x), \dots, p_n(x); r_1(x), \dots, r_{n-1}(x)]$, $n \geq 2$. If*

$$\int_0^1 r_1(x) dx \neq 0,$$

then, except finitely many eigenvalues, the multiplicity of the eigenvalues of (3) are at most $n - 1$.

Proof. By Theorem 2.2, the assertion of Theorem 2.3 holds for the case $n = 2$. Suppose there exists $Q(x) = J[p_1(x), \dots, p_n(x); r_1(x), \dots, r_{n-1}(x)]$, $n \geq 3$, such that the eigenvalue problem (3) has infinitely many eigenvalues λ_{m_k} of multiplicity n . Let $Y(x; \lambda_{m_k}) = [y_{ij}(x; \lambda_{m_k})]_{i,j=1}^n$ denote the corresponding matrix solution of (4). Then following the argument of Lemma 2.1, in particular using (7) and (8), we have (6) for this case. Then, by (5), we also have the identity (11), and hence, by the Riemann-Lebesgue Lemma, we have $\int_0^1 r_1(x) dx = 0$. Thus if $\int_0^1 r_1(x) dx \neq 0$, then, except finitely many eigenvalues, the multiplicity of the eigenvalues of (3) are at most $n - 1$. \square

Theorem 2.4. *Suppose $Q(x) = J[p_1(x), \dots, p_n(x); r_1(x), \dots, r_{n-1}(x)]$, $n \geq 2$. If at least one $r_j(x)$ which is not zero, and $r_1(x), \dots, r_{n-1}(x)$ never change sign in $0 \leq x \leq 1$, then, except finitely many eigenvalues, the multiplicity of the eigenvalues of (3) are at most $n - 1$.*

Proof. We shall prove this theorem by mathematical induction. By Theorem 2.2, the assertion of this theorem holds for the case $n = 2$. Assume it holds for the case $n = k$, where $k \geq 2$, but there exists a $Q(x) = J[p_1(x), \dots, p_{k+1}(x); r_1(x), \dots, r_k(x)]$ such that (3) has infinitely many eigenvalues λ_{m_t} which have multiplicity $k + 1$. Then, by the argument in the proof of Theorem 2.3 and the assumption that $r_1(x), \dots, r_k(x)$ never change sign on $0 \leq x \leq 1$, we conclude that $r_1(x) = 0$. But then for $\tilde{Q}(x) = J[p_2(x), \dots, p_{k+1}(x); r_2(x), \dots, r_k(x)]$, the eigenvalues of (3),

counting multiplicity, consist of the eigenvalues in $\sigma(p_1)$ and the eigenvalues of the eigenvalue problem

$$(12) \quad \vec{z}''(x) + (\mu I - \tilde{Q}(x))\vec{z}(x) = \vec{0}, \quad \vec{z}(0) = \vec{z}(1) = \vec{0},$$

which implies (12) has infinitely many eigenvalues of multiplicity k , a contradiction to the induction assumption. This completes the proof of Theorem 2.4. \square

3. THE INTERSECTION OF THE SPECTRA OF TWO POTENTIAL EQUATIONS, OR TWO STRING EQUATIONS.

In this section we shall apply Theorem 2.2 to study the intersection of the spectra of two potential equations of the form (1), or two string equations of the form (2). Recall that for (1) with potential function $q(x)$, $\sigma(q)$ is used to denote its spectrum, i.e., the set of all eigenvalues of (1); and $\sigma_s(\rho)$ is used to denote the spectrum of (2) with density function $\rho(x)$.

Theorem 3.1. *Suppose $q_1(x)$ and $q_2(x)$ are two real-valued continuous functions on $0 \leq x \leq 1$. If*

$$(13) \quad \int_0^1 q_1(x)dx \neq \int_0^1 q_2(x)dx,$$

then $\sigma(q_1)$ and $\sigma(q_2)$ only have finitely many elements in common.

Proof. Denote

$$\Lambda(x) = \begin{bmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{bmatrix},$$

$$U(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

and

$$Q_\theta(x) = U^*(\theta)\Lambda(x)U(\theta) = J[p_1(x), p_2(x); r_1(x)],$$

where

$$r_1(x) = (q_1(x) - q_2(x))\sin\theta\cos\theta,$$

θ is a constant. Then for $Q(x) = Q_\theta(x)$, the sequence of eigenvalues, counting multiplicity, of (3), consists of elements of $\sigma(q_1)$ and $\sigma(q_2)$, counting multiplicity. If we choose θ so that $\sin\theta\cos\theta \neq 0$, then, by (13), $\int_0^1 r_1(x)dx \neq 0$. And it follows from Theorem 2.2 that, for $Q(x) = Q_\theta(x)$, (3) has only finitely many eigenvalues which have multiplicity two, which implies that $\sigma(q_1) \cap \sigma(q_2)$ is a finite set. \square

Applying Theorem 3.1 and using the Liouville transformation (see [3]) we obtain the following result.

Theorem 3.2. *Suppose $\rho_1(x)$ and $\rho_2(x)$ are two positive C^2 -functions defined on $0 \leq x \leq 1$. If $\rho_1(x)$ and $\rho_2(x)$ satisfy the following two conditions,*

$$(14) \quad \int_0^1 \rho_1^{1/2}(x)dx \neq \int_0^1 \rho_2^{1/2}(x)dx,$$

$$(15) \quad \int_0^1 (\rho_1^{-1/4})(x) \frac{d^2}{dx^2}(\rho_1^{-1/4})(x) dx \neq \int_0^1 (\rho_2^{-1/4})(x) \frac{d^2}{dx^2}(\rho_2^{-1/4})(x) dx,$$

then $\sigma_s(\rho_1)$ and $\sigma_s(\rho_2)$ only have finitely many elements in common.

Proof. Letting

$$\begin{aligned}
 K &= \int_0^1 \rho^{1/2}(t)dt, & z &= K^{-1} \int_0^x \rho^{1/2}(t)dt, \\
 u(z) &= \rho^{1/4}(x)y(x), & \lambda &= K^2\mu, \\
 \theta(z) &= \rho^{1/4}(x), & q(z) &= \theta''(z)/\theta(z),
 \end{aligned}$$

the equation (2) is transformed to equation (1) where y and x are replaced by u and z , respectively. And, by the method of changing variables, we find that

$$(16) \quad \int_0^1 q(z)dz = -K^2 \int_0^1 (\rho^{-1/4})(x) \frac{d^2}{dx^2} (\rho^{-1/4})(x) dx.$$

Applying (14), (15) and (16), Theorem 3.1 implies our assertion. □

As an application of Theorem 3.1, we prove the following result about Bessel functions. For information about Bessel functions, we refer to [4] and [5]. Recall that $J_n(x)$ (resp., $Y_n(x)$) is the Bessel function of the first kind (resp., the second kind) of order n .

Corollary 3.3. *Suppose $n, k \geq 0, n \neq k, b > 1$. Then the functions*

$$J_n(s)Y_n(sb) - J_n(sb)Y_n(s)$$

and

$$J_k(s)Y_k(sb) - J_k(sb)Y_k(s)$$

have at most finitely many positive zeros in common.

Proof. It is easy to show that if we set

$$f(x) = \sqrt{x}\{J_n(s)Y_n(sx) - J_n(sx)Y_n(s)\},$$

then $f(x)$ satisfies the following equation:

$$f''(x) + (s^2 - \frac{n^2 - 1/4}{x^2})f(x) = 0,$$

with $f(1) = 0, f'(1) \neq 0$. Thus the square of the positive zeros of the function $J_n(s)Y_n(sb) - J_n(sb)Y_n(s)$ are the eigenvalues of the following eigenvalue problem:

$$(17) \quad y''(x) + (\lambda - \frac{n^2 - 1/4}{x^2})y(x) = 0, \quad y(1) = y(b) = 0.$$

Then our assertion is just a consequence of Theorem 3.1. □

4. THE MULTIPLICITY OF EIGENVALUES OF A TWO-DIMENSIONAL VECTORIAL STURM-LIOUVILLE EQUATION

Let $Q(x) = J[p_1(x), p_2(x); r_1(x)]$, where $p_1(x), p_2(x)$ and $r_1(x)$ are C^1 -functions on $0 \leq x \leq 1$, and let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ denote the sequence of eigenvalues of (3). If $\int_0^1 r_1(x)dx \neq 0$, then, by Theorem 2.2, there exists an index m_Q such that for $n \geq m_Q$ the eigenvalues λ_n of (3) are simple. In this section we shall try to find an upper bound estimate of the number m_Q . This estimate is obviously applicable to the problem of estimating the number of elements in the intersection of the spectra of two potential equations, or two string equations.

Let $A = [a_{ij}]_{i,j=1}^2$ be a two-by-two matrix. Define the *maximum norm* of A as follows:

$$\|A\| = \sup \{|a_{ij}| : 1 \leq i, j \leq 2\}.$$

If A and B are two two-by-two matrices, then we have

$$(18) \quad \|AB\| \leq 2\|A\|\|B\|.$$

Let $Y(x; \lambda)$ denote the matrix-valued solution of the equation (4). Then $Y(x; \lambda)$ satisfies the following integral equation (see [1]):

$$(19) \quad Y(x; \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}I + \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}Q(t)Y(x; \lambda)dt.$$

Denote

$$(20) \quad Y(x; \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}I + G(x; \lambda),$$

where

$$(21) \quad \begin{aligned} G(x, \lambda) &= [g_{ij}(x; \lambda)]_{i,j=1}^2 \\ &= \int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}Q(t)Y(t; \lambda)dt. \end{aligned}$$

Note that, by (18) and (19), we have

$$(22) \quad \|Y(x; \lambda)\| \leq \frac{1}{\sqrt{\lambda}} + 2 \int_0^x \frac{1}{\sqrt{\lambda}}\|Q(t)\|\|Y(t; \lambda)\|dt,$$

and, by Gronwall's inequality, (22) tells us that

$$(23) \quad \|Y(x; \lambda)\| \leq \frac{1}{\sqrt{\lambda}}\exp\left(\frac{1}{\sqrt{\lambda}} \int_0^x 2\|Q(t)\|dt\right).$$

Then, by (21) and (23), we have, for $1 \leq i, j \leq 2$,

$$\begin{aligned} |g_{ij}(x; \lambda)| &\leq \frac{1}{\sqrt{\lambda}}\left\{\exp\left(\frac{1}{\sqrt{\lambda}} \int_0^x 2\|Q(t)\|dt\right) - 1\right\} \\ &\leq \frac{1}{\sqrt{\lambda}}\left\{\exp\left(\frac{1}{\sqrt{\lambda}} \int_0^1 2\|Q(t)\|dt\right) - 1\right\}. \end{aligned}$$

Thus if

$$(24) \quad \sqrt{\lambda} \geq \int_0^1 2\|Q(t)\|dt,$$

then we have, for $1 \leq i, j \leq 2, 0 \leq x \leq 1$,

$$(25) \quad |g_{ij}(x; \lambda)| \leq \frac{2e}{\lambda} \int_0^1 \|Q(t)\|dt.$$

Using (21) and (20),

$$(26) \quad \begin{aligned} \det Y(x; \lambda) &= \frac{\sin^2\sqrt{\lambda}x}{\lambda} + \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\text{Trace}G(x; \lambda) + \det G(x; \lambda) \\ &= \frac{1 - \cos 2\sqrt{\lambda}x}{2\lambda} + h(x; \lambda), \end{aligned}$$

where

$$h(x; \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\text{Trace}G(x; \lambda) + \det G(x; \lambda),$$

and, by (25), if (24) is satisfied, then we have, for $0 \leq x \leq 1$,

$$(27) \quad \lambda|h(x; \lambda)| \leq \frac{8e^2}{\sqrt{\lambda}}Q_*,$$

where

$$Q_* = \int_0^1 \|Q(t)\| dt.$$

And by (27), we have

$$(28) \quad \left| \int_0^1 \lambda r_1(x)h(x; \lambda) dx \right| \leq \frac{8e^2Q_*}{\sqrt{\lambda}} \int_0^1 |r_1(x)| dx.$$

We shall also need the following inequality, which follows from integration by parts:

$$(29) \quad \left| \int_0^1 r_1(x) \cos 2\sqrt{\lambda}x dx \right| \leq \frac{|r_1(1)|}{2\sqrt{\lambda}} + \frac{1}{2\sqrt{\lambda}} \int_0^1 |r_1'(x)| dx.$$

Note that, by (28) and (29), we have

$$(30) \quad \left| \int_0^1 r_1(x) \cos 2\sqrt{\lambda}x dx - 2\lambda \int_0^1 r_1(x)h(x; \lambda) dx \right| \\ \leq \frac{1}{\sqrt{\lambda}} \left\{ \frac{|r_1(1)|}{2} + \frac{1}{2} \int_0^1 |r_1'(x)| dx + 16e^2Q_* \int_0^1 |r_1(x)| dx \right\},$$

if λ satisfies inequality (24). Denote

$$R_* = \frac{|r_1(1)|}{2} + \frac{1}{2} \int_0^1 |r_1'(x)| dx + 16e^2Q_* \int_0^1 |r_1(x)| dx.$$

Now we are ready to prove the following result.

Theorem 4.1. *Suppose $Q(x) = J[p_1(x), p_2(x); r_1(x)]$, where $p_1(x)$, $p_2(x)$ and $r_1(x)$ are C^1 -functions on $0 \leq x \leq 1$, and $\int_0^1 r_1(x) dx \neq 0$. Let Λ_* be the positive number defined by*

$$\sqrt{\Lambda_*} = \frac{R_*}{\left| \int_0^1 r_1(x) dx \right|}.$$

If λ_n is an eigenvalue of (3) satisfying the following conditions,

$$(31) \quad \sqrt{\lambda_n} \geq 2Q_*,$$

$$(32) \quad \lambda_n > \Lambda_*,$$

then λ_n is a simple eigenvalue.

Proof. Suppose λ_n satisfies conditions (31) and (32), but the multiplicity of λ_n is two. Then, by Lemma 2.1,

$$\int_0^1 r_1(x) \det Y(x; \lambda) dx = 0,$$

and by (26), we have

$$(33) \quad \int_0^1 r_1(x) dx = \int_0^1 r_1(x) \cos 2\sqrt{\lambda_n}x dx - 2\lambda_n \int_0^1 r_1(x)h(x; \lambda_n) dx.$$

But then (30), (31), (32) and (33) imply that

$$|\int_0^1 r_1(x)dx| < |\int_0^1 r_1(x)dx|,$$

which is absurd. □

Note that if $s_1(x) \leq s_2(x)$ are the characteristic values of the matrix $Q(x)$, then for any scalar-valued functions $f_1(x)$ and $f_2(x)$ we have

$$\begin{aligned} p_1(x)f_1^2(x) - 2r_1(x)f_1(x)f_2(x) + p_2(x)f_2^2(x) \\ \geq s_1(x)[f_1^2(x) + f_2^2(x)]. \end{aligned}$$

By the maximum-minimum principle (see [6, Chapter VI]) we have

$$\lambda_n[Q] \geq \lambda_n[Q_1],$$

where $\lambda_n[Q]$ is the n^{th} eigenvalue of (3), and $Q_1(x) = s_1(x)I$, I is the two-by-two identity matrix. Since each eigenvalue of (3), with $Q_1(x)$ replacing $Q(x)$, is of multiplicity two, we have

$$(34) \quad \lambda_n[Q] \geq [\frac{n+1}{2}]^2 \pi^2 + s_{1,min},$$

where $s_{1,min}$ is the minimum value of $s_1(x)$ in $0 \leq x \leq 1$, and $[y]$ is the Gauss symbol of y . By (34), Theorem 4.1 implies the following result.

Theorem 4.2. *Suppose $Q(x) = J[p_1(x), p_2(x); r_1(x)]$, where $p_1(x)$, $p_2(x)$ and $r_1(x)$ are C^1 -functions on $0 \leq x \leq 1$, $\int_0^1 r_1(x)dx \neq 0$. Let $s_1(x)$ be the smaller characteristic value of the matrix $Q(x)$, and let m_Q be the smallest positive integer which satisfies the following conditions:*

$$(35) \quad [\frac{m_Q}{2}]^2 \pi^2 + s_{1,min} \geq 4Q_*^2,$$

$$(36) \quad [\frac{m_Q}{2}]^2 \pi^2 + s_{1,min} > \Lambda_*.$$

Then the n^{th} eigenvalues of (3) is simple if $n \geq m_Q$.

By Theorem 4.2 and the proof of Theorem 3.1 we also have the following result. We shall use the notation $a \vee b$ to denote the maximum of two real numbers a and b . For $q_1(x)$ and $q_2(x)$, two continuous functions on $0 \leq x \leq 1$, $\int_0^1 (q_1(x) - q_2(x))dx \neq 0$, we shall also use the following abbreviation:

$$\begin{aligned} T_* &= |q_1(1) - q_2(1)| + \int_0^1 |q_1'(x) - q_2'(x)|dx \\ &\quad + 16e^2 \int_0^1 \{(|q_1(x) + q_2(x)|) \vee (|q_1(x) - q_2(x)|)\} dx \int_0^1 |q_1(x) - q_2(x)|dx, \\ \sqrt{\tilde{\Lambda}_*} &= \frac{T_*}{2|\int_0^1 (q_1(x) - q_2(x))dx|}. \end{aligned}$$

Theorem 4.3. *Suppose $q_1(x)$ and $q_2(x)$ are C^1 -functions on $0 \leq x \leq 1$, with $\int_0^1 (q_1(x) - q_2(x))dx \neq 0$. Let $s_* = \min\{q_{1,min}, q_{2,min}\}$, and let n_{q_1, q_2} be the*

smallest positive integer which satisfies the following conditions:

$$(37) \quad [n_{q_1, q_2}]^2 \pi^2 + s_* \geq \left\{ \int_0^1 (|q_1(x) + q_2(x)|) \vee (|q_1(x) - q_2(x)|) dx \right\}^2,$$

$$(38) \quad [n_{q_1, q_2}]^2 \pi^2 + s_* > \tilde{\Lambda}_*.$$

Then $\sigma(q_1)$ and $\sigma(q_2)$ have at most n_{q_1, q_2} elements in common.

Proof. In the proof of Theorem 3.1 we choose $\theta = \pi/4$. Then

$$Q_{\pi/4}(x) = \begin{bmatrix} \frac{1}{2}(q_1(x) + q_2(x)) & \frac{1}{2}(q_1(x) - q_2(x)) \\ \frac{1}{2}(q_1(x) - q_2(x)) & \frac{1}{2}(q_1(x) + q_2(x)) \end{bmatrix},$$

$$r_1(x) = -\frac{1}{2}(q_1(x) - q_2(x)),$$

$$\int_0^1 \|Q_{\pi/4}(s)\| ds = \frac{1}{2} \int_0^1 (|q_1(x) + q_2(x)|) \vee (|q_1(x) - q_2(x)|) dx,$$

and $\tilde{\Lambda}_*$ is the same as the number Λ_* defined in Theorem 4.1. Suppose, on the contrary, $\sigma(q_1)$ and $\sigma(q_2)$ have at least $n_{q_1, q_2} + 1$ elements in common. Since the sequence of eigenvalues of (3) with $Q(x) = Q_{\pi/4}(x)$ consists of elements of $\sigma(q_1)$ and $\sigma(q_2)$ counting multiplicity, there exists an index n_* , $n_* \geq 2n_{q_1, q_2} + 2$, such that $\lambda_{n_*}[Q_{\pi/4}]$ is not simple. But, by (34),

$$\lambda_{n_*}[Q_{\pi/4}] > (n_{q_1, q_2})^2 \pi^2 + s_*;$$

hence, by (37) and (38), $\lambda_{n_*}[Q_{\pi/4}]$ satisfies the conditions of Theorem 4.1, which implies that $\lambda_{n_*}[Q_{\pi/4}]$ is simple, a contradiction. Hence $\sigma(q_1)$ and $\sigma(q_2)$ have at most n_{q_1, q_2} elements in common. \square

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DEPARTMENT OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU, TAIWAN 30043, REPUBLIC OF CHINA

E-mail address: shen@math.nthu.edu.tw

Current address, Chung-Tsun Shieh: Department of Mathematics, Fu Jen Catholic University, Hsinchuang, Taipei, Taiwan 24205, Republic of China

E-mail address: ctshieh@math.fju.edu.tw