PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 10, Pages 2995–3002
S 0002-9939(99)05048-0
Article electronically published on May 3, 1999

L^p ESTIMATES FOR OSCILLATORY INTEGRAL OPERATORS

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(Communicated by Christopher D. Sogge)

ABSTRACT. An endpoint boundedness result is established for a class of oscillatory integral operators.

1.

This paper deals with a problem left open in [6]. Let $\Delta = \{(x,x) : \quad x \in \mathbf{R}\}$, $K \in C^\infty(\mathbf{R}^2 \backslash \Delta)$ and satisfy

$$|\frac{\partial^{j+l} K}{\partial x^j \partial y^l}(x,y)| \leq A_{jl}|x-y|^{-j-l} \tag{1}$$

for $(x,y) \in \mathbf{R}^2 \backslash \Delta$ and $j, l \geq 0$.

For $a, b \geq 1$, define the non-convolutional oscillatory integral operator $T_{a,b}$:

$$(T_{a,b}f)(x) = \int_R e^{i|x|^a|y|^b} K(x,y) f(y) dy, \tag{2}$$

initially for $f \in \mathcal{S}(\mathbf{R})$.

Such operators often arise in harmonic analysis (see e.g. [7], [8], [9]). It should be noted that when $a = b = 1$ and $K \equiv 1$ the operator in (2) is essentially the Fourier transform.

The main problem under investigation concerns the L^p boundedness of the operators $\{T_{a,b}\}$. A quick examination of the operators $T_{a,b}$ with $K \equiv 1$ reveals that the L^p inequality

$$\|T_{a,b}f\|_p \leq C_{a,b,p}\|f\|_p \tag{3}$$

cannot hold (for all $f \in \mathcal{S}(\mathbf{R})$) unless $p = \frac{a+b}{a}$. Thus one is led to the problem of determining whether

$$\|T_{a,b}f\|_{\frac{a+b}{a}} \leq C_{a,b}\|f\|_{\frac{a+b}{a}} \tag{4}$$

holds for $f \in \mathcal{S}(\mathbf{R})$ and $a, b \geq 1$. The following is known:

Theorem A. *Suppose that K satisfies (1). Let $T_{a,b}$ be given as in (2). Then:*

(i) If $a > 1$ and $b > 1$, then $T_{a,b}$ is bounded from $L^{\frac{a+b}{a}}$ to itself.

(ii) If $a = b = 1$ (hence $\frac{a+b}{a} = 2$), then $T_{a,b} = T_{1,1}$ is bounded from L^2 to itself.

Received by the editors January 8, 1998.
1991 *Mathematics Subject Classification.* Primary 42A38; Secondary 42A50.
The second author was supported in part by NSF Grant DMS-9622979.

Part (i) of Theorem A was established in [6] (Theorem 3.1 on page 212), while part (ii) can be found in [5] (see also [7]). Apparently the remaining issue is to answer the following question:

Question. *Is $T_{a,b}$ bounded from $L^{\frac{a+b}{a}}$ to itself when either $a = 1$ or $b = 1$?*

It would seem quite reasonable to believe that the answer is "yes". In fact, this was shown to be true in the special case where $K(x, y) = |x - y|^{i\sigma}$ with $\sigma \in \mathbf{R}$ ([6]). However, for the operators with generic K's that satisfy (1) the question has remained unanswered. The main purpose of this paper is to completely resolve this issue. Namely we have:

Theorem B. *Let $b > 1$, K satisfy (1), and $T_b = T_{1,b}$ be given as in (2). Then there exists a constant $C_b > 0$ such that*

$$\|T_b f\|_{b+1} \leq C_b \|f\|_{b+1} \tag{5}$$

holds for $f \in \mathcal{S}(\mathbf{R})$.

For $a > 1$ and $b = 1$ one obtains (4) from (5) by duality. It follows from Theorems A and B that $T_{a,b}$ is bounded on $L^{\frac{a+b}{a}}$ for all $a, b \geq 1$.

Part of our argument presented here is similar to the method used in [6], which goes back to [7]. The key new ingredients are reduction to pseudodifferential operators and the establishment of related L^2 estimates. This combination allows us to tackle the above mentioned problem and may have applications elsewhere.

For a nonnegative locally integrable function w we shall write $\|f\|_{p,w}$ for $(\int_R |f(x)|^p w(x) dx)^{1/p}$. When $w \equiv 1$, we shall simply write $\|f\|_p$ for $\|f\|_{p,w}$.

2.

Let $b > 1$, $\eta \in C^\infty(\mathbf{R})$ such that $\eta(x) \equiv 1$ when $|x| \leq 1/2$ and $\eta(x) \equiv 0$ when $|x| \geq 1$. For $K(x, y)$ that satisfies (1) and $m \in \mathbf{N}$ we define $K_m(\cdot, \cdot)$ and $\omega_m(\cdot, \cdot)$ by

$$K_m(x, y) = [1 - \eta(x - y)] K(x, y) \eta(\frac{x - y}{m}), \tag{6}$$

$$\omega_m(\tau, \xi) = \int_{\mathbf{R}} e^{ix(\tau - \xi)} K_m(x + |\tau|^{1/b}, |\tau|^{1/b}) dx. \tag{7}$$

We shall begin with two propositions.

Proposition 1. *There exist constants $\tilde{A}_{j,l}$ independent of m such that*

$$|\frac{\partial^{j+l} K_m}{\partial x^j \partial y^l}(x, y)| \leq \tilde{A}_{j,l} [1 + |x - y|]^{-j-l} \tag{8}$$

for $j, l \geq 0$ and $(x, y) \in \mathbf{R}^2 \backslash \Delta$.

Proposition 2. *There exists $A > 0$ which is independent of m such that*

$$|\omega_m(\tau, \xi)| \leq A \min\{|\tau - \xi|^{-1}, |\tau - \xi|^{-2}\} \tag{9}$$

for all $\tau, \xi \in \mathbf{R}$.

The proof of Proposition 1 is elementary and hence will be omitted.

Proof of Proposition 2. If $|\tau-\xi|\geq 1$, then by Integration by Parts and Proposition 1

$$|\omega_m(\tau,\xi)| = \frac{1}{|\tau-\xi|^2}\Big|\int_{\mathbf{R}} e^{ix(\tau-\xi)}\frac{d^2}{dx^2}K_m(x+|\tau|^{\frac{1}{b}},|\tau|^{\frac{1}{b}})dx\Big|$$
$$\leq A|\tau-\xi|^{-2}\int_{|x|\geq\frac{1}{2}}\frac{dx}{|x|^2}$$
$$= A\min\{|\tau-\xi|^{-1},|\tau-\xi|^{-2}\}.$$

Now suppose that $|\tau-\xi|<1$ and write $\omega_m(\tau,\xi)=I_1+I_2$ where

$$I_1=\int_{\mathbf{R}} e^{ix(\tau-\xi)}K_m(x+|\tau|^{\frac{1}{b}},|\tau|^{\frac{1}{b}})\eta(x(\tau-\xi))dx$$

and

$$I_2=\int_{\mathbf{R}} e^{ix(\tau-\xi)}K_m(x+|\tau|^{\frac{1}{b}},|\tau|^{\frac{1}{b}})[1-\eta(x(\tau-\xi))]dx.$$

Then

$$|I_1|\leq A_1\int_{|x|\leq|\tau-\xi|^{-1}}dx = 2A_1|\tau-\xi|^{-1}$$

and

$$|I_2|\leq\frac{1}{|\tau-\xi|^2}\int_{\mathbf{R}}\Big|\frac{d^2}{dx^2}\{K_m(x+|\tau|^{\frac{1}{b}},|\tau|^{\frac{1}{b}})[1-\eta(x(\tau-\xi))]\}\Big|dx$$
$$\leq\frac{A_2}{|\tau-\xi|^2}\Big(\int_{|x|\geq(2|\tau-\xi|)^{-1}}\frac{dx}{|x|^2}dx+\int_{(2|\tau-\xi|)^{-1}\leq|x|\leq|\tau-\xi|^{-1}}\Big[\frac{|\tau-\xi|}{|x|}+|\tau-\xi|^2\Big]dx\Big)$$
$$\leq 7A_2|\tau-\xi|^{-1}=7A_2\min\{|\tau-\xi|^{-1},|\tau-\xi|^{-2}\}.$$

Thus (9) always holds. Using the arguments given above, one easily obtains:

Proposition 3. *There exists $A>0$ independent of m such that*

$$\Big|\frac{\partial\omega_m}{\partial\xi}(\tau,\xi)\Big|\leq A|\xi-\tau|^{-2} \tag{10}$$

and

$$\Big|\frac{\partial\omega_m}{\partial\tau}(\tau,\xi)\Big|\leq A(1+|\tau|^{\frac{1}{b}-1})|\xi-\tau|^{-2} \tag{11}$$

for all $\xi\in\mathbf{R}$ and $\tau\neq 0$.

Define the kernels Ω_m and operators H_m by

$$\Omega_m(\tau,\xi)=\omega_m(\tau,\xi)\eta(\xi-\tau)[1-\eta(4^b|\tau|)] \tag{12}$$

and

$$H_mf(\xi)=\int_{\mathbf{R}}\Omega_m(\tau,\xi)f(\tau)d\tau. \tag{13}$$

Below is a uniform weighted norm estimate for H_m.

Theorem 4. *Let $p\in(1,\infty)$ and $w\in A_p$. Then there exists a constant $A_{p,w}>0$ independent of m such that*

$$\int_{\mathbf{R}}|H_mf(x)|^pw(x)dx\leq A_{p,w}\int_{\mathbf{R}}|f(x)|^pw(x)dx \tag{14}$$

holds for all $f\in\mathcal{S}(\mathbf{R})$.

Here $A_p=A_p(\mathbf{R})$ represents the collection of Muckenhoupt's A_p-weights (see [4]).

Proof. Let $a(\tau,x) = K_m(x+|\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}})[1-\eta(4^b|\tau|)]$. Then by Proposition 1 for $j,k \geq 0$ there exists $A_{j,k} > 0$ independent of m such that

$$|\frac{\partial^{j+k} a(\tau,x)}{\partial\tau^j \partial x^k}| \leq A_{j,k}(1+|x|)^{-k}.$$

Thus $a(\cdot,\cdot)$ is a symbol in the class $S^0_{1,0}$. Let W_a be the pseudodifferential operator with symbol $a(\cdot,\cdot)$ i.e.

$$(W_a f)(\tau) = \int_{\mathbf{R}} e^{i\tau x} a(\tau,x)\hat{f}(x)dx. \tag{15}$$

Then there exists $C > 0$ independent of m such that

$$\|W_a f\|_2 \leq C\|f\|_2 \tag{16}$$

for all $f \in \mathcal{S}(\mathbf{R})$ (see, for example, [10] on page 234).

For $u \in \mathbf{R}$ we shall let $f_u(\tau) = e^{i\tau u} f(-\tau)$. Then by (7), (12), (13) and (15)

$$H_m f(\xi) = \int_{\mathbf{R}} e^{i\xi\cdot u}\hat{\eta}(u) g(\xi,u)du \tag{17}$$

where $g(\xi,u) = \mathcal{F}\circ[W_a\circ\mathcal{F}^{-1}]^*(f_u)(\xi)$ and $\mathcal{F}$ represents the Fourier transform. By (16) and Plancherel's Theorem, one obtains:

$$(\int_{\mathbf{R}} |g(\xi,u)|^2 d\xi)^{\frac{1}{2}} \leq C\|f_u\|_2 = C\|f\|_2 \quad \forall \quad u \in \mathbf{R}. \tag{18}$$

It then follows from (17), (18) and Minkowski's Inequality that

$$\|H_m f\|_2 \leq C\|\hat{\eta}\|_1 \|f\|_2 \leq A\|f\|_2 \tag{19}$$

for all $f \in \mathcal{S}(\mathbf{R})$.

Since $\Omega_m(\tau,\xi) = 0$ when $|\tau| \leq 2^{-(2b+1)}$, (9), (10), (11), and (12) give us that

$$|\Omega_m(\tau,\xi)| \leq \frac{A}{|\tau-\xi|} \tag{20}$$

and

$$|\frac{\partial}{\partial\tau}\Omega_m(\tau,\xi)| + |\frac{\partial}{\partial\xi}\Omega_m(\tau,\xi)| \leq \frac{A}{|\tau-\xi|^2} \tag{21}$$

hold for $(\xi,\tau) \in \mathbf{R}^2\backslash\Delta$ and some positive constant A independent of m. It follows from (19)-(21) that $\Omega_m(\tau,\xi)$ is a generalized Calderón-Zygmund kernel. By a minor modification of the arguments in [1] one obtains the weighted L^p inequality (14) with a constant $A_{p,w}$ independent of m (see also [10], page 221, and [3]).

Corollary 5. *If we let*

$$(J_m f)(\xi) = \int_{\mathbf{R}} \Omega_m(|\tau|^b,\xi) f(\tau)d\tau, \tag{22}$$

then there exists $A > 0$ independent of m such that

$$\int_{\mathbf{R}} |(J_m f)(\xi)|^{b+1}|\xi|^{b-1}d\xi \leq A\int_{\mathbf{R}} |f(y)|^{b+1}dy \tag{23}$$

for all $f \in \mathcal{S}(\mathbf{R})$.

Proof. Let

$$f_1(t) = \frac{1}{b} f(t^{\frac{1}{b}}) t^{\frac{1}{b}-1} \chi_{[0,\infty)}(t)$$

and

$$f_2(t) = \frac{1}{b} f(-t^{\frac{1}{b}}) t^{\frac{1}{b}-1} \chi_{[0,\infty)}(t).$$

Then $J_m f = H_m f_1 + H_m f_2$. Let $w(\xi) = |\xi|^{b-1} \in A_{b+1}$. By Theorem 4 we get

$$\|J_m f\|_{b+1,w} \leq C(\|f_1\|_{b+1,w} + \|f_2\|_{b+1,w}) \leq A\|f\|_{b+1}.$$

This proves (23).

Lemma 6. *Define L_m by:*

$$(L_m f)(\xi) = \int_{\mathbf{R}} e^{iy(|y|^b - \xi)} \Omega_m(|y|^b, \xi) f(y) dy \tag{24}$$

for $\xi \in \mathbf{R}$. Then there exists $A > 0$ independent of m such that

$$\int_{\mathbf{R}} |(L_m f)(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq A \int_{\mathbf{R}} |f(y)|^{b+1} dy \tag{25}$$

for $f \in \mathcal{S}(\mathbf{R})$.

This result follows from Corollary 5 by using the method employed in [6] on pages 217 and 218, which can be traced all the way back to [7] and [9]. Below we shall give a sketch of the proof.

Without loss of generality we may assume that *supp*$(f) \subset [0, \infty)$. For $j \geq 0$ let $I_j = [j^{1/b}, (j+1)^{1/b})$, $f_j(y) = f(y)\chi_{I_j}(y)$, and $g_j(y) = e^{ij^{1/b} y^b} f_j(y)$. Then

$$(L_m f)(\xi) = \sum_{j=0}^{\infty} (L_m f_j)(\xi). \tag{26}$$

By (12) and (9)

$$|(L_m f_j)(\xi) - e^{-ij^{1/b}\xi}(J_m g_j)(\xi)|$$
$$\leq A \int_{|y^b - \xi| \leq 1} |e^{i(y - j^{1/b})(y^b - \xi)} - 1| |\xi - y^b|^{-1} |f_j(y)| dy \leq A(\tilde{M} f_j)(\xi),$$

where $(\tilde{M} f)(\xi)$ represents the Hardy-Littlewood maximal function of $g(t) = f(t^{\frac{1}{b}}) t^{\frac{1}{b}-1}$. By Corollary 5 and the $L^{b+1}(\mathbf{R}, |x|^{b-1}dx) \to L^{b+1}(\mathbf{R}, |x|^{b-1}dx)$ boundedness of the Hardy-Littlewood maximal function ([4]), we obtain:

$$\int_{\mathbf{R}} |L_m f_j(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq C(\int_0^\infty |g_j(y)|^{b+1} + \int_0^\infty [|f_j(y^{\frac{1}{b}})| y^{\frac{1}{b}-1}]^{b+1} y^{b-1} dy) \tag{27}$$
$$= A \int_{\mathbf{R}} |f_j(y)|^{b+1} dy.$$

By *supp*$(L_m f_j) \subseteq [j-1, j+2]$, (26), and (27), we get

$$\int_{\mathbf{R}} |L_m f(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq \tilde{A} \int_{\mathbf{R}} |f(y)|^{b+1} dy.$$

Lemma 7. *Let*

$$(Pf)(x) = \int_{\mathbf{R}} e^{ix|y|^b}[1-\eta(x-y)]K(x,y)[1-\eta(4^b|y|^b)]f(y)dy.$$

Then there exists $A > 0$ *such that*

$$\|Pf\|_{b+1} \le A\|f\|_{b+1} \tag{28}$$

holds for all $f \in \mathcal{S}(\mathbf{R})$*.*

Proof. Let $f \in \mathcal{S}(\mathbf{R})$. For $m \in N$ and $x \in \mathbf{R}$ let

(29)

$$P_m(x) = \int_0^\infty e^{ixy^b}[1-\eta(x-y)]K(x,y)\eta(\frac{x-y}{m})[1-\eta(4^b y^b)]f(y)dy.$$

Then by (29), (24) and (12) we obtain

$$\hat{P}_m(\xi) = \int_0^\infty e^{iy(y^b-\xi)}[1-\eta(4^b y^b)]\omega_m(y^b,\xi)f(y)dy$$

and

$$L_m(\chi_{(0,\infty)}f)(\xi) = \int_0^\infty e^{iy(y^b-\xi)}[1-\eta(4^b y^b)]\omega_m(y^b,\xi)\eta(\xi-y^b)f(y)dy.$$

By (9) we get that for $\xi \in \mathbf{R}$

$$|\hat{P}_m(\xi) - L_m(\chi_{(0,\infty)}f)(\xi)| \tag{30}$$

$$\le A\int_0^\infty \frac{|1-\eta(\xi-y^b)|}{|\xi-y^b|^2}|f(y)|dy \le A(\tilde{M}f)(\xi).$$

By Pitt's Inequality ([2]), (30), and Lemma 6,

$$\|P_m\|_{b+1} \le (\int_{\mathbf{R}} |\hat{F}_m(\xi)|^{b+1}|\xi|^{b-1}d\xi)^{\frac{1}{b+1}} \tag{31}$$

$$\le C[\|L_m(\chi_{(0,\infty)}f)\|_{b+1,|x|^{b-1}} + \|\tilde{M}f\|_{b+1,|x|^{b-1}}]$$

$$\le C\|f\|_{b+1} + C[\int_0^\infty |f(y^{\frac{1}{b}})y^{\frac{1}{b}-1}|^{b+1}y^{b-1}dy]^{\frac{1}{b+1}} = A\|f\|_{b+1}.$$

By letting $m \to \infty$ in (31), we obtain

$$\|\int_0^\infty e^{ix|y|^b}[1-\eta(x-y)]K(x,y)[1-\eta(4^b|y|^b)]f(y)dy\|_{b+1} \le A\|f\|_{b+1}.$$

Similarly,

$$\|\int_{-\infty}^0 e^{ix|y|^b}[1-\eta(x-y)]K(x,y)[1-\eta(4^b|y|^b)]f(y)dy\|_{b+1} \le A\|f\|_{b+1}.$$

This proves (28).

3.

We shall finish the paper with a proof of our main result.

Proof of Theorem B. For $f \in \mathcal{S}(\mathbf{R})$ write

$$(T_b f)(x) = F_1(x) + F_2(x) + F_3(x) \tag{32}$$

where

$$F_1(x) = \int_{\mathbf{R}} e^{i|x||y|^b}[1-\eta(x-y)]K(x,y)[1-\eta(4^b|y|^b)]f(y)dy,$$

$$F_2(x) = \int_{\mathbf{R}} e^{i|x||y|^b}[1-\eta(x-y)]K(x,y)\eta(4^b|y|^b)f(y)dy,$$

and

$$F_3(x) = \int_{\mathbf{R}} e^{i|x||y|^b}\eta(x-y)K(x,y)f(y)dy.$$

By considering $x \geq 0$ and $x < 0$ separately and applying Lemma 7, it follows that

$$\|F_1\|_{b+1} \leq A\|f\|_{b+1}. \tag{33}$$

Since $\|F_3\|_{b+1} \leq A(|\eta| * |f|)(x)$,

$$\|F_3\|_{b+1} \leq A\|f\|_{b+1}. \tag{34}$$

For $(x,y) \in supp[(1-\eta(x-y)\eta(4^b|y|^b)]$ we have $|x-y| \geq \frac{1}{2}$ and $|y| \leq \frac{1}{4}$. Let

$$\mathrm{I}(x) = K(x,0)\int_{\mathbf{R}} e^{i|x||y|^b}[1-\eta(x-y)]\eta(4^b|y|^b)f(y)dy$$

and

$$\Pi(x) = \frac{\partial K}{\partial y}(x,0)\int_{\mathbf{R}} e^{i|x||y|^b}[1-\eta(x-y)]\eta(4^b|y|^b)yf(y)dy.$$

Then by (1)

$$|F_2(x) - [\mathrm{I}(x) + \Pi(x)]| \leq A\int_{\mathbf{R}} \frac{|f(y)|}{1+|x-y|^2}dy. \tag{35}$$

Clearly

$$|\mathrm{I}(x)| \leq A[|\check{g}(|x|)| + (|\eta| * |f|)(x)]$$

where

$$g(t) = \frac{1}{b}[f(t^{\frac{1}{b}}) + f(-t^{\frac{1}{b}})]t^{\frac{1}{b}-1}\eta(4^b t)\chi_{(0,\infty)}(t).$$

Hence by Pitt's inequality

$$\int_{\mathbf{R}} |\mathrm{I}(x)|^{b+1}dx \leq A[\int_0^\infty |\check{g}(x)|^{b+1}dx + \|\eta\|_1^{b+1}\|f\|_{b+1}^{b+1}]$$

$$\leq A[\int_{\mathbf{R}} |g(t)|^{b+1}|t|^{b-1}dt + \|\eta\|_1^{b+1}\|f\|_{b+1}^{b+1}] \leq A\|f\|_{b+1}^{b+1}.$$

Similarly,

$$\int_{\mathbf{R}} |\Pi(x)|^{b+1}dx \leq A\|f\|_{b+1}^{b+1}.$$

Using (35), we obtain:

$$\|F_2\|_{b+1} \leq A\|f\|_{b+1}. \tag{36}$$

It now follows from (32)-(34) and (36) that

$$\|T_b f\|_{b+1} \leq A\|f\|_{b+1}$$

for all $f \in \mathcal{S}(\mathbf{R})$. This completes the proof of Theorem B.

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