Lp ESTIMATES FOR OSCILLATORY INTEGRAL OPERATORS

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Abstract. An endpoint boundedness result is established for a class of oscillatory integral operators.

1.

This paper deals with a problem left open in [6]. Let \( \Delta = \{(x, x) : x \in \mathbb{R}\} \), \( K \in C^\infty(\mathbb{R}^2 \setminus \Delta) \) and satisfy

\[
|\partial^{j+l}K(x, y)| \leq A_{jl}|x - y|^{-j-l}
\]

for \((x, y) \in \mathbb{R}^2 \setminus \Delta\) and \(j, l \geq 0\).

For \(a, b \geq 1\), define the non-convolutional oscillatory integral operator \(T_{a,b}\):

\[
(T_{a,b}f)(x) = \int e^{i|x|^a|y|^b}K(x, y)f(y)dy
\]

initially for \(f \in S(\mathbb{R})\).

Such operators often arise in harmonic analysis (see e.g. [7], [8], [9]). It should be noted that when \(a = b = 1\) and \(K = 1\) the operator in (2) is essentially the Fourier transform.

The main problem under investigation concerns the \(L^p\) boundedness of the operators \(\{T_{a,b}\}\). A quick examination of the operators \(T_{a,b}\) with \(K \equiv 1\) reveals that the \(L^p\) inequality

\[
\|T_{a,b}f\|_p \leq C_{a,b,p}\|f\|_p
\]

cannot hold (for all \(f \in S(\mathbb{R})\)) unless \(p = \frac{a+b}{a}\). Thus one is led to the problem of determining whether

\[
\|T_{a,b}f\|_{\frac{a+b}{a}} \leq C_{a,b}\|f\|_{\frac{a+b}{a}}
\]

holds for \(f \in S(\mathbb{R})\) and \(a, b \geq 1\). The following is known:

Theorem A. Suppose that \(K\) satisfies (1). Let \(T_{a,b}\) be given as in (2). Then:

(i) If \(a > 1\) and \(b > 1\), then \(T_{a,b}\) is bounded from \(L^{\frac{a+b}{a}}\) to itself.

(ii) If \(a = b = 1\) (hence \(\frac{a+b}{a} = 2\)), then \(T_{a,b} = T_{1,1}\) is bounded from \(L^2\) to itself.
Part (i) of Theorem A was established in [6] (Theorem 3.1 on page 212), while part (ii) can be found in [5] (see also [7]). Apparently the remaining issue is to answer the following question:

**Question.** Is \( T_{a,b} \) bounded from \( L^a+b \) to itself when either \( a = 1 \) or \( b = 1 \)?

It would seem quite reasonable to believe that the answer is “yes”. In fact, this was shown to be true in the special case where \( K(x,y) = |x - y|^{\sigma} \) with \( \sigma \in \mathbb{R} \) ([6]). However, for the operators with generic \( K \’s \) that satisfy (1) the question has remained unanswered. The main purpose of this paper is to completely resolve this issue. Namely we have:

**Theorem B.** Let \( b > 1 \), \( K \) satisfy (1), and \( T_b = T_{1,b} \) be given as in (2). Then there exists a constant \( C_b > 0 \) such that

\[
\|T_b f\|_{b+1} \leq C_b \|f\|_{b+1}
\]

(5)

holds for \( f \in S(\mathbb{R}) \).

For \( a > 1 \) and \( b = 1 \) one obtains (4) from (5) by duality. It follows from Theorems A and B that \( T_{a,b} \) is bounded on \( L^{a+b} \) for all \( a, b \geq 1 \).

Part of our argument presented here is similar to the method used in [6], which goes back to [7]. The key new ingredients are reduction to pseudodifferential operators and the establishment of related \( L^2 \) estimates. This combination allows us to tackle the above mentioned problem and may have applications elsewhere.

For a nonnegative locally integrable function \( w \) we shall write \( \|f\|_{p,w} \) for \( \left( \int \mathbb{R} |f(x)|^p w(x) \, dx \right)^{1/p} \). When \( w \equiv 1 \), we shall simply write \( \|f\|_p \) for \( \|f\|_{p,w} \).

2.

Let \( b > 1 \), \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(x) \equiv 1 \) when \( |x| \leq 1/2 \) and \( \eta(x) \equiv 0 \) when \( |x| \geq 1 \). For \( K(x,y) \) that satisfies (1) and \( m \in \mathbb{N} \) we define \( K_m(\cdot,\cdot) \) and \( \omega_m(\cdot,\cdot) \) by

\[
K_m(x,y) = (1 - \eta(x - y))K(x,y)\eta\left(\frac{x - y}{m}\right),
\]

(6)

\[
\omega_m(\tau,\xi) = \int \mathbb{R} e^{ix(\tau - \xi)} K_m(x + |\tau|^{1/b},|\tau|^{1/b}) \, dx.
\]

(7)

We shall begin with two propositions.

**Proposition 1.** There exist constants \( \tilde{A}_{j,l} \) independent of \( m \) such that

\[
|\frac{\partial^{j+l} K_m}{\partial x^j \partial y^l}(x,y)| \leq \tilde{A}_{j,l}[1 + |x - y|]^{-j-l}
\]

(8)

for \( j, l \geq 0 \) and \( (x,y) \in \mathbb{R}^2 \setminus \Delta \).

**Proposition 2.** There exists \( A > 0 \) which is independent of \( m \) such that

\[
|\omega_m(\tau,\xi)| \leq A \min\{|\tau - \xi|^{-1},|\tau - \xi|^{-2}\}
\]

(9)

for all \( \tau, \xi \in \mathbb{R} \).

The proof of Proposition 1 is elementary and hence will be omitted.
Proof of Proposition 2. If $|\tau - \xi| \geq 1$, then by Integration by Parts and Proposition 1
\[
|\omega_m(\tau, \xi)| = \frac{1}{|\tau - \xi|^2} \int_{\mathbb{R}} e^{ix(\tau-\xi)} \frac{d^2}{dx^2} K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}}) dx
\]
\[
\leq A|\tau - \xi|^{-2} \int_{|x| \geq \frac{1}{2}} \frac{dx}{|x|^2}
\]
\[
= A \min\{|\tau - \xi|^{-1}, |\tau - \xi|^{-2}\}.
\]
Now suppose that $|\tau - \xi| < 1$ and write $\omega_m(\tau, \xi) = I_1 + I_2$ where
\[
I_1 = \int_{\mathbb{R}} e^{ix(\tau-\xi)} K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}}) \eta(\tau(x - \xi)) dx
\]
and
\[
I_2 = \int_{\mathbb{R}} e^{ix(\tau-\xi)} K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}})(1 - \eta(x(x - \xi))) dx.
\]
Then
\[
|I_1| \leq A_1 \int_{|x| \leq |\tau - \xi|^{-1}} dx = 2A_1|\tau - \xi|^{-1}
\]
and
\[
|I_2| \leq \frac{1}{|\tau - \xi|^2} \int_{\mathbb{R}} \frac{d^2}{dx^2} \{K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}})(1 - \eta(\tau(x - \xi))\})|dx
\]
\[
\leq \frac{A_2}{|\tau - \xi|^2} \left( \int_{|x| \geq (2|\tau - \xi|^{-1})} \frac{dx}{|x|^2} + \int_{(2|\tau - \xi|^{-1}) \leq |x| \leq |\tau - \xi|^{-1}} \frac{|\tau - \xi|}{|x|} + |\tau - \xi|^2 dx \right)
\]
\[
\leq 7A_2|\tau - \xi|^{-1} = 7A_2 \min\{|\tau - \xi|^{-1}, |\tau - \xi|^{-2}\}.
\]
Thus (9) always holds. Using the arguments given above, one easily obtains:

**Proposition 3.** There exists $A > 0$ independent of $m$ such that
\[
|\frac{\partial \omega_m}{\partial \xi}(\tau, \xi)| \leq A|\xi - \tau|^{-2}
\]
and
\[
|\frac{\partial \omega_m}{\partial \tau}(\tau, \xi)| \leq A(1 + |\tau|^{\frac{1}{b} - 1})|\xi - \tau|^{-2}
\]
for all $\xi \in \mathbb{R}$ and $\tau \neq 0$.

Define the kernels $\Omega_m$ and operators $H_m$ by
\[
\Omega_m(\tau, \xi) = \omega_m(\tau, \xi)\eta(\xi - \tau)[1 - \eta(4^b|\tau|)]
\]
and
\[
H_m f(\xi) = \int_{\mathbb{R}} \Omega_m(\tau, \xi)f(\tau)d\tau.
\]
Below is a uniform weighted norm estimate for $H_m$.

**Theorem 4.** Let $p \in (1, \infty)$ and $w \in A_p$. Then there exists a constant $A_{p,w} > 0$ independent of $m$ such that
\[
\int_{\mathbb{R}} |H_m f(x)|^p w(x) dx \leq A_{p,w} \int_{\mathbb{R}} |f(x)|^p w(x) dx
\]
holds for all $f \in \mathcal{S}(\mathbb{R})$.

Here $A_p = A_p(\mathbb{R})$ represents the collection of Muckenhoupt’s $A_p$-weights (see [4]).
Proof. Let \( a(\tau, x) = K_m(x + |\tau|^b, |\tau|^b)[1 - \eta(4^b|\tau|)] \). Then by Proposition 1 for \( j, k \geq 0 \) there exists \( A_{j,k} > 0 \) independent of \( m \) such that

\[
|\frac{\partial^{j+k}a(\tau,x)}{\partial \tau^j \partial x^k}| \leq A_{j,k}(1 + |x|)^{-k}.
\]

Thus \( a(\cdot, \cdot) \) is a symbol in the class \( S_{1,0}^0 \). Let \( W_a \) be the pseudodifferential operator with symbol \( a(\cdot, \cdot) \) i.e.

\[
(W_a f)(\tau) = \int f(x)e^{i\tau x}a(\tau, x)\hat{f}(x)dx.
\]

Then there exists \( C > 0 \) independent of \( m \) such that

\[
\|W_a f\|_2 \leq C\|f\|_2
\]

for all \( f \in \mathcal{S}(\mathbb{R}) \) (see, for example, [10] on page 234).

For \( u \in \mathbb{R} \) we shall let \( f_u(\tau) = e^{i\tau u}f(-\tau) \). Then by (7), (12), (13) and (15)

\[
H_m f(\xi) = \int e^{i\xi u}\hat{\eta}(u)g(\xi, u)du
\]

where \( g(\xi, u) = \mathcal{F}[W_a \mathcal{F}^{-1}(f_u)](\xi) \) and \( \mathcal{F} \) represents the Fourier transform. By (16) and Plancherel’s Theorem, one obtains:

\[
(\int_{\mathbb{R}} |g(\xi, u)|^2 d\xi)^{\frac{1}{2}} \leq C\|f_u\|_2 = C\|f\|_2 \quad \forall \ u \in \mathbb{R}.
\]

It then follows from (17), (18) and Minkowski’s Inequality that

\[
\|H_m f\|_2 \leq C\|\hat{\eta}\|_1\|f\|_2 \leq A\|f\|_2
\]

for all \( f \in \mathcal{S}(\mathbb{R}) \).

Since \( \Omega_m(\tau, \xi) = 0 \) when \( |\tau| \leq 2^{-(2^b+1)} \), (9), (10), (11), and (12) give us that

\[
|\Omega_m(\tau, \xi)| \leq \frac{A}{|\tau - \xi|}
\]

and

\[
|\frac{\partial}{\partial \tau} \Omega_m(\tau, \xi)| + |\frac{\partial}{\partial \xi} \Omega_m(\tau, \xi)| \leq \frac{A}{|\tau - \xi|^2}
\]

hold for \( (\xi, \tau) \in \mathbb{R}^2 \setminus \Delta \) and some positive constant \( A \) independent of \( m \). It follows from (19)-(21) that \( \Omega_m(\tau, \xi) \) is a generalized Calderón-Zygmund kernel. By a minor modification of the arguments in [1] one obtains the weighted \( L^p \) inequality (14) with a constant \( A_{p,w} \) independent of \( m \) (see also [10], page 221, and [3]).

**Corollary 5.** If we let

\[
(J_m f)(\xi) = \int_{\mathbb{R}} \Omega_m(|\tau|^b, \xi)f(\tau)d\tau,
\]

then there exists \( A > 0 \) independent of \( m \) such that

\[
\int_{\mathbb{R}} |(J_m f)(\xi)|^{b+1}|\xi|^{b-1}d\xi \leq A \int_{\mathbb{R}} |f(y)|^{b+1}dy
\]

for all \( f \in \mathcal{S}(\mathbb{R}) \).
Proof. Let
\[ f_1(t) = \frac{1}{b} f(t^{\frac{1}{b}}) t^{\frac{1}{b}-1} \chi_{[0,\infty)}(t) \]
and
\[ f_2(t) = \frac{1}{b} f(-t^{\frac{1}{b}}) t^{\frac{1}{b}-1} \chi_{[0,\infty)}(t). \]
Then \( J_m f = H_m f_1 + H_m f_2. \) Let \( w(\xi) = |\xi|^{b-1} \in A_{b+1}. \) By Theorem 4 we get
\[
\| J_m f \|_{b+1,w} \leq C(\| f_1 \|_{b+1,w} + \| f_2 \|_{b+1,w}) \leq A \| f \|_{b+1}.
\]
This proves (23).

Lemma 6. Define \( L_m \) by:
\[
(L_m f)(\xi) = \int_{\mathbb{R}} e^{i y (|y|^{b} - \xi)} \Omega_m(|y|^b, \xi) f(y) dy
\]
for \( \xi \in \mathbb{R}. \) Then there exists \( A > 0 \) independent of \( m \) such that
\[
\int_{\mathbb{R}} \|(L_m f)(\xi)\|_{b+1} \|\xi|^{b-1} d\xi \leq A \int_{\mathbb{R}} \|f(y)\|_{b+1} dy
\]
for \( f \in \mathcal{S}(\mathbb{R}). \)

This result follows from Corollary 5 by using the method employed in [6] on pages 217 and 218, which can be traced all the way back to [7] and [9]. Below we shall give a sketch of the proof.

Without loss of generality we may assume that \( \text{supp}(f) \subset [0,\infty). \) For \( j \geq 0 \) let
\[
I_j = [j^{1/b}, (j + 1)^{1/b}), f_j(y) = f(y) \chi_{I_j}(y), \text{ and } g_j(y) = e^{iy^{1/b}y} f_j(y).
\]
Then
\[
(L_m f)(\xi) = \sum_{j=0}^{\infty} (L_m f_j)(\xi).
\]
By (12) and (9)
\[
\|(L_m f_j)(\xi) - e^{-iy^{1/b} \xi} (L_m g_j)(\xi)\|
\]
\[
\leq A \int_{|y|^{b} - \xi| \leq 1} |e^{i y^{1/b} (y^{b} - \xi)} - 1| |\xi - y^{b}|^{b-1} |f_j(y)| dy
\]
\[
\leq A (M f_j)(\xi),
\]
where \( (M f)(\xi) \) represents the Hardy-Littlewood maximal function of \( g(t) = f(t^{\frac{1}{b}}) t^{\frac{1}{b}-1}. \) By Corollary 5 and the \( L^{b+1}(\mathbb{R}, |x|^{b-1} dx) \to L^{b+1}(\mathbb{R}, |x|^{b-1} dx) \) boundedness of the Hardy-Littlewood maximal function ([4]), we obtain:
\[
\int_{\mathbb{R}} \|(L_m f_j)(\xi)\|_{b+1} \|\xi|^{b-1} d\xi \leq C \int_{0}^{\infty} |g_j(y)|_{b+1} + \int_{0}^{\infty} ||f_j(y^{\frac{1}{b}})| y^{\frac{1}{b}-1}|_{b+1} y y^{-1} dy
\]
\[
= A \int_{\mathbb{R}} |f_j(y)|_{b+1} dy,
\]
By \( \text{supp}(L_m f_j) \subset [j-1, j+2], \) (26), and (27), we get
\[
\int_{\mathbb{R}} \|(L_m f)(\xi)\|_{b+1} \|\xi|^{b-1} d\xi \leq \tilde{A} \int_{\mathbb{R}} |f(y)|_{b+1} dy.
\]
Lemma 7. Let

\[(Pf)(x) = \int_{\mathbb{R}} e^{ix|y|^b} [1 - \eta(x - y)] K(x, y)[1 - \eta(4^b|y|^b)] f(y) dy.\]

Then there exists \(A > 0\) such that

\[\|Pf\|_{b+1} \leq A\|f\|_{b+1}\]

holds for all \(f \in \mathcal{S}(\mathbb{R})\).

Proof. Let \(f \in \mathcal{S}(\mathbb{R})\). For \(m \in \mathbb{N}\) and \(x \in \mathbb{R}\) let

\[(P_m f)(x) = \int_{\mathbb{R}} e^{ix|y|^b} [1 - \eta(x - y)] K(x, y)\eta\left(\frac{x - y}{m}\right)[1 - \eta(4^b|y|^b)] f(y) dy.

Then by (29), (24) and (12) we obtain

\[\hat{P}_m(\xi) = \int_{\mathbb{R}} e^{iy(\xi - \xi)} [1 - \eta(4^b|y|^b)] \hat{\omega}_m(y, \xi) f(y) dy

and

\[L_m(\chi_{(0,\infty)} f)(\xi) = \int_{\mathbb{R}} e^{iy(\xi - \xi)} [1 - \eta(4^b|y|^b)] \hat{\omega}_m(y, \xi) \eta(\xi - y) f(y) dy.

By (9) we get that for \(\xi \in \mathbb{R}\)

\[|\hat{P}_m(\xi) - \hat{L}_m(\chi_{(0,\infty)} f)(\xi)| \leq A \int_{\mathbb{R}} \frac{|1 - \eta(\xi - y)|}{|\xi - y|^2} |f(y)| dy \leq A(\hat{M}f)(\xi).

By Pitt’s Inequality ([2]), (30), and Lemma 6,

\[\|P_m\|_{b+1} \leq \left(\int_{\mathbb{R}} |\hat{P}_m(\xi)|^{b+1} |\xi|^{b-1} d\xi\right)^{\frac{1}{b+1}}

\leq C\|L_m(\chi_{(0,\infty)} f)\|_{b+1, |x|^{b-1}} + \|\hat{M}f\|_{b+1, |x|^{b-1}}

\leq C\|f\|_{b+1} + C\int_{\mathbb{R}} |f(y)| y^{\frac{b-1}{2}-1} y^{b-1} dy = A\|f\|_{b+1}.

By letting \(m \to \infty\) in (31), we obtain

\[\|\int_{\mathbb{R}} e^{ix|y|^b} [1 - \eta(x - y)] K(x, y)[1 - \eta(4^b|y|^b)] f(y) dy\|_{b+1} \leq A\|f\|_{b+1}.

Similarly,

\[\|\int_{-\infty}^{\infty} e^{ix|y|^b} [1 - \eta(x - y)] K(x, y)[1 - \eta(4^b|y|^b)] f(y) dy\|_{b+1} \leq A\|f\|_{b+1}.

This proves (28).
3.

We shall finish the paper with a proof of our main result.

Proof of Theorem B. For \( f \in S(\mathbb{R}) \) write

\[
(T_b f)(x) = F_1(x) + F_2(x) + F_3(x)
\]

where

\[
F_1(x) = \int_{\mathbb{R}} e^{i|x||y|^b} [1 - \eta(x - y)] K(x, y) [1 - \eta(4^b|y|^b)] f(y) dy,
\]

\[
F_2(x) = \int_{\mathbb{R}} e^{i|x||y|^b} [1 - \eta(x - y)] K(x, y) \eta(4^b|y|^b) f(y) dy,
\]

and

\[
F_3(x) = \int_{\mathbb{R}} e^{i|x||y|^b} \eta(x - y) K(x, y) f(y) dy.
\]

By considering \( x \geq 0 \) and \( x < 0 \) separately and applying Lemma 7, it follows that

\[
\|F_1\|_{b+1} \leq A\|f\|_{b+1}.
\]

Since \( \|F_2\|_{b+1} \leq A(\|\eta\| \ast |f|)(x) \),

\[
\|F_3\|_{b+1} \leq A\|f\|_{b+1}.
\]

For \( (x, y) \in \text{supp}[(1 - \eta(x - y)\eta(4^b|y|^b)] \) we have \( |x - y| \geq \frac{1}{2} \) and \( |y| \leq \frac{1}{4} \). Let

\[
I(x) = K(x, 0) \int_{\mathbb{R}} e^{i|x||y|^b} [1 - \eta(x - y)] \eta(4^b|y|^b) f(y) dy
\]

and

\[
\Pi(x) = \frac{\partial K}{\partial y}(x, 0) \int_{\mathbb{R}} e^{i|x||y|^b} [1 - \eta(x - y)] \eta(4^b|y|^b) y f(y) dy.
\]

Then by (1)

\[
|F_2(x) - [I(x) + \Pi(x)]| \leq A \int_{\mathbb{R}} \frac{|f(y)|}{1 + |x - y|^2} dy.
\]

Clearly

\[
|I(x)| \leq A(\|\hat{g}(|x|)\| + (\|\eta\| \ast |f|)(x))
\]

where

\[
g(t) = \frac{1}{b} [f(t^+) + f(-t^+)] t^{b-1} \eta(4^b t) \chi_{(0, \infty)}(t).
\]

Hence by Pitt’s inequality

\[
\int_{\mathbb{R}} |I(x)|^{b+1} dx \leq A \int_{0}^{\infty} |\hat{g}(t)|^{b+1} dt + \|\eta\|_1^{b+1} \|f\|_{b+1}^{b+1}
\]

\[
\leq A \int_{0}^{\infty} |g(t)|^{b+1} |t|^{b-1} dt + \|\eta\|_1^{b+1} \|f\|_{b+1}^{b+1} \leq A\|f\|_{b+1}^{b+1}.
\]

Similarly,

\[
\int_{\mathbb{R}} |\Pi(x)|^{b+1} dx \leq A\|f\|_{b+1}^{b+1}.
\]

Using (35), we obtain:

\[
\|F_2\|_{b+1} \leq A\|f\|_{b+1}.
\]
It now follows from (32)-(34) and (36) that
\[ \|T_b f\|_{b+1} \leq A \|f\|_{b+1} \]
for all \( f \in S(\mathbb{R}) \). This completes the proof of Theorem B.

REFERENCES


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15260