THE LENGTH AND THICKNESS OF WORDS
IN A FREE GROUP

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Abstract. In this paper we generalize the notion of a cut point of a graph. We assign to each graph a non-negative integer, called its thickness, so that a graph has thickness 0 if and only if it has a cut point. We then apply a method of J. H. C. Whitehead to show that if the coinitial graph of a given word has thickness $t$, then any word equivalent to it in a free group of rank $n$ has length at least $2nt$. We also define what it means for a word in a free group to be separable and we show that there is an algorithm to decide whether or not a given word is separable.

1. Introduction

Let $F$ be a free group of rank $n \geq 2$ with basis \{ $x_1, x_2, \ldots, x_n$ \}. We shall regard each element $x$ of $F$ as a reduced word in the letters $x_1, x_2, x_1^{-1}, x_2^{-1}, \ldots, x_n, x_n^{-1}$. Following the usual convention, the symbol $|x|$ denotes the length of $x$ as a word in our alphabet.

Definition 1. The symbol $[x]$ represents the set of all words in $F$ equivalent to $x$, i.e., \{ $\Phi(x) : \Phi \in Aut(F)$ \}. The length of the smallest element equivalent to $x$ is denoted as $\|x\|$.

It is clear that all cyclic conjugates of $x$ are in $[x]$ and that if $y \in [x]$, then $[x] = [y]$. By a result of J. H. C. Whitehead, [5], $\|x\|$ can be effectively computed. Now $\|x\| = 1$ if and only if $x$ is primitive, i.e. part of a basis for $F$. It is not difficult to prove that $\|x^m\| = |m|\|x\|$, however in general $\|$ behaves rather unpredictably with respect to multiplication in $F$ even when the multiplication does not involve cancellation. For example in the free group $F_2$ of rank 2 generated by $a$ and $b$, $\|ab^2a\| = 4$ and $\|b^3\| = 3$ yet $\|ab^2ab^3\| = 1$ since $\{ab^2a, ab^3\}$ is a basis for $F_2$.

We would like to introduce another integer, $t(x)$, which we call the thickness of $x \in F$. This integer will behave “better” under multiplication than $\|\|$.

Definition 2. Let $\Gamma$ be a finite graph without loops but with possibly multiple edges between vertices. By a path in $\Gamma$ between $v$ and $w$ we mean a sequence of vertices $v_1, v_2, \ldots, v_m$ where $v = v_1, w = v_m$ and there exists at least one edge between consecutive vertices. The thickness of a path is the largest integer $k$, such that there exist at least $k$ edges between consecutive vertices. We define $t(\Gamma)$, the
We see that $t(\Gamma)$ is in a certain way a generalization of a cut point of a graph. By that we mean $t(\Gamma) = 0$ if and only if $\Gamma$ is disconnected or has a cut point.

**Definition 3.** Let $x$ be a cyclically reduced word in $F$; the thickness of $x$, $t(x)$, is equal to $t(\Gamma_x)$ where $\Gamma_x$ is the coinitial(star) graph of the word $x$.

Let $y_1, y_2, \ldots, y_m$ be reduced words in our alphabet. The expression $z \equiv y_1y_2\ldots y_m$ means that word $y_1y_2\ldots y_m$ is reduced and that $z = y_1y_2\ldots y_m$. According to [4], if $t(x) > 0$, then $x$ is not primitive.

Several trivial facts about $t(x)$ are as follows:

1. $t(x^k) = |k|t(x)$.
2. If $y \equiv uxv$, then $t(y) \geq t(x) - 1$.

J. H. C. Whitehead [5] introduced geometric methods to study the relationship between $x$ and $\Phi(x)$. In this paper we will apply his methods to give a lower bound on $|x|$ in terms of $t(x)$.

We will now introduce rather briefly the method of J. H. C. Whitehead. The terminology employed will be that used in [1]; in particular the more descriptive term “endcap” introduced in the latter is used in place of Whitehead’s “two element”.

If $D$ is a closed 3-ball smoothly embedded in $S^3$, $int(D)$ will denote the interior of $D$ and $bd(D)$ will denote its boundary. Now suppose that $D_1, D_1', D_2, D_2', \ldots, D_n, D_n'$ are disjoint 3-balls smoothly embedded in $S^3$, define $\tilde{M}$ to be

$$S^3 \setminus \bigcup_{i=1}^{n}(int(D_i) \cup int(D_i'))$$

and let $M$ be the quotient space of $\tilde{M}$ obtained when $bd(D_i)$ is identified to $bd(D_i')$ via an orientation reversing homeomorphism. $M$ is easily seen to be a connected sum of $n$ copies of $S^1 \times S^2$ and thus $\Pi_1(M) \cong F$. From now on the 3-balls $D_1, D_1', D_2, D_2', \ldots, D_n, D_n'$ will be fixed and shall be referred to as the standard elements. Let $S_i$ denote the image in $M$ of $bd(D_i)$. In $S^3$, let each $bd(D_i)$ be given a normal direction by the vector which points away from $int(D_i)$. The image of this vector is a normal direction to each $S_i$. The collection of 2-spheres $\{S_1, S_2, \ldots, S_n\}$, together with their outward normals is called the standard sphere basis of $M$ and is denoted as $S$. Let $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ be another family of disjoint 2-spheres embedded in $M$, where each $\Sigma_i$ has some predefined normal direction. If we denote this family as $\Sigma$, then we say that $\Sigma$ is a sphere basis of $M$ if there exists a homeomorphism of $M$ onto itself which sends $S_i$ onto $\Sigma_i$ for all $i$ and which preserves the outward normal directions. It is an easily proved fact that if $\Sigma$ is a sphere basis, then any choice of outward normals to the $\Sigma_i$ also gives a sphere basis.

Let $T$ be a smoothly embedded 2-sphere in $M$ which is transverse to each $S_i$ and let $\tilde{T}$ be its preimage in $\tilde{M}$. Either $\tilde{T}$ is connected and is thus a 2-sphere or else it is disconnected and consists of endcaps (2-cells) and 2-cells with holes. Let us assume that the latter is true and let $A$ be some component of $\tilde{T}$. The boundary of $A$ consists of disjoint circles each one of which lies on one of the 2-spheres $bd(D_i)$ or $bd(D_i')$. If each component of the boundary of $A$ lies on a different 2-sphere, we shall say that $A$ is normal with respect to $S$. We say that $T$ is normal with respect to $S$ if either $\tilde{T}$ is connected or else each component of $\tilde{T}$ is normal with respect to $S$. 


Definition 4. Let \( E \) be an endcap of \( \tilde{T} \) and assume that the boundary \( C \) of \( E \) lies on \( \text{bd}(D_i) \). Now \( C \) separates \( \text{bd}(D_i) \) into two discs. The proper disc, \( E_p \), is the disc so that, in \( S^3 \), \( E \cup E_p \) separates \( \text{int}(D_i) \) from \( \text{int}(D'_i) \). Exactly one of the discs is proper. The other disc is called the improper disc and is denoted as \( E_q \).

2. Some preliminary results

**Proposition 1.** Let \( T \) be a 2-sphere smoothly embedded in \( M \) which is transverse to \( S \); then exactly one of the following holds:

1. \( T \) is trivial, i.e. \( T \) bounds a three ball in \( M \).
2. \( T \) is primitive, i.e. there exists a homeomorphism of \( M \) onto itself which sends \( T \) to some \( S_i \).
3. \( T \) is separant, i.e. \( M \setminus T \) consists of two components, each homeomorphic to either \( S^1 \times S^2 \) with a closed 3-ball removed or a non-trivial connected sum of \( S^1 \times S^2 \)'s with a closed 3-ball removed.

**Proof.** Our proof follows the method in [5]. Let \( k \) be the minimum number of circles of intersection of \( T \) with \( \Sigma \) as \( \Sigma \) ranges over all sphere bases for \( M \) which are transverse to \( T \). Whitehead’s method is by induction on the number \( k \). We first assume that \( k = 0 \). In this situation we can find a homeomorphism of \( M \) onto itself which sends \( \Sigma \) onto \( S \). The image of \( T \), which we also denote as \( T \), does not intersect \( S \). The lift of \( T \) in \( S^3 \), which we also call \( T \), is a 2-sphere in \( S^3 \) and therefore either

1. \( T \) bounds a 3-ball which contains none of the balls \( D_i \) or \( D'_i \).
2. \( T \) separates some \( D_i \) from \( D'_i \).
3. \( T \) is nontrivial and no \( D_i \) is separated from \( D'_i \) by \( T \), i.e. \( T \) is separant.

According to Singer [3], if we are in (2), then there exists a homeomorphism of \( M \) which sends \( S_i \) onto \( T \) and which maps each \( S_j \), \( i \neq j \), onto itself. Thus when \( k = 0 \) our lemma is proven.

We now assume that the lemma is proven for all \( k \leq m - 1 \) and we may assume, by the argument above, that \( T \cap S \) consists of \( m \) circles. Let \( E \) be an endcap of \( T \) and assume that the boundary of \( E \) lies on \( \text{bd}(D_i) \). Now \( E \cup E_p \) is a 2-sphere embedded in \( S^3 \) which separates \( \text{int}(D_i) \) from \( \text{int}(D'_i) \). Slightly push this 2-sphere into the component which contains \( \text{int}(D'_i) \) and call this new 2-sphere \( E^* \). Now \( T \cap E^* \) has fewer components than \( T \cap S_i \) and \( E^* \) separates \( D_i \) from \( D'_i \). Now replace \( S_i \) by \( E^* \) and call the new family \( \Sigma \). \( \Sigma \) is also a sphere basis for \( M \) and \( T \) intersects \( \Sigma \) in fewer than \( m \) circles. Applying our inductive hypothesis concludes the proof of the lemma. \( \square \)

We should point out that Proposition 1 is valid if \( S \) is replaced by an arbitrary sphere basis. The advantage, which we shall exploit in the next section, of replacing some sphere basis with \( S \) is that it allows arguments to take place in \( S^3 \).

3. Main results

**Definition 5.** Let \( \alpha \) be a directed simple closed curve, with base point, embedded in \( M \). If \( \Sigma \) is a sphere basis which is transverse to \( \alpha \), the word obtained by transversing \( \alpha \), see e.g. [1], is denoted as \( \alpha(\Sigma) \).

According to Whitehead [5], as \( \Sigma \) ranges over all sphere bases for \( M \) the set of elements obtained is just \([x]\) where \( x = \alpha(S) \). We shall thus denote this set of
elements as $[\alpha]$. If $T$ is a 2-sphere embedded in $M$ which is transverse to $\alpha$, we shall use the notation $\alpha \cdot T$ to denote the number of points of intersection of $\alpha$ and $T$.

**Theorem 1.** Let $T$ be a non-trivial 2-sphere smoothly embedded in $M$ and let $x$ be a cyclically reduced word in $F$ such that $t(x) \geq 1$. If $\alpha$ is a simple closed curve such that $x \in [\alpha]$ and $\alpha$ is transverse to $T$, then $\alpha \cdot T \geq 2t(x)$.

**Proof.** For each curve $\alpha$, which is transverse to $T$ and $x \in [\alpha]$ let $\Sigma_\alpha$ be a sphere basis with the following properties:

1. $\alpha(\Sigma_\alpha) = x$.
2. $\Sigma_\alpha$ is transverse to $T$.
3. The number of circles of intersection of $\Sigma_\alpha$ and $T$ is minimal with respect to (1) and (2).

Our proof is by induction on $k$, the number of circles of intersection of $T$ with $\Sigma_\alpha$, where $\alpha$ varies over all curves with the property that $x \in [\alpha]$.

We first assume that $k = 0$. In this case we have a closed curve $\alpha$, a sphere basis $\Sigma_\alpha$ such that $\alpha(\Sigma_\alpha) = x$ and $T \cap \Sigma_\alpha = \emptyset$. Thus by taking a homeomorphism we may assume that $T \cap S = \emptyset$ and $\alpha(S) = x$. Let $\hat{\alpha}$ be the lift of $\alpha$ to $S^3$. $\hat{\alpha}$ consists of disjoint arcs with endpoints on the boundary of the $D_i$ and $D'_i$ which miss the interior of these balls. The graph obtained by collapsing each $D_i$ and $D'_i$ to individual vertices and naming these vertices $v_i$ and $x_i^{-1}$ is isomorphic to $\Gamma_x$. Since $k = 0$, $T$ can be regarded as a 2-sphere in $S^3$. $T$ separates $S^3$ into two 3-balls, $B$ and $B'$. Both $B$ and $B'$ must contain some $D_i$ or $D'_i$ since $T$ is assumed to be non-trivial. Let the corresponding vertices in $\Gamma_x$ be $v$ and $v'$ respectively. Now there exists a path $P$ in $\Gamma_x$ between $v$ and $v'$ whose thickness is at least $t(x)$. Thus there exist 2 consecutive vertices in $P$, such that the corresponding standard elements, $D$ and $D'$, have the property that $D$ lies in $B$ and $D'$ lies in $B'$. The edges connecting these vertices correspond injectively to a collection of edges in $\hat{\alpha}$ which meet $T$, each of which meets $T$ at least once. Since $n \geq 2$, there must be another standard element inside $B$ or $B'$. Without loss of generality, assume this standard element, denoted $D''$, lies in $B$. If $w$ denotes the vertex in $\Gamma_x$ which corresponds to $D''$, then there is a path $P''$ between $w$ and $v'$ which misses $v$ and whose thickness is greater than or equal to $t(x)$. This implies that there is a second pair of standard elements, one of which lies in $B$ and the other in $B'$, which is not identically equal to $D$ and $D'$ and that there are at least $t(x)$ edges between them. These edges are disjoint from the first set. Hence we have at least $2t(x)$ subarcs of $\alpha$ which meet $T$ at least once. Our theorem is therefore proven in the case that $T \cap \Sigma_\alpha$ is empty.

We now assume that the theorem is valid for all $\alpha$ such that $T$ intersects $\Sigma_\alpha$ in $m-1$ or fewer circles. Let us assume that the number of circles of intersection of $T$ and $\Sigma_\alpha$ is $m$. Again we may assume that $\Sigma_\alpha$ is the standard basis $S$. Our argument again takes place in $S^3$. Let $E$ be an endcap of $T$. Now $E$ meets a unique standard element in some circle; without loss of generality we shall assume it meets the standard element labelled $D_i$. We will denote the circle of intersection as $C$. Now let us consider $E \cup E_q$, which we regard as a 2-sphere in $S^3$. This 2-sphere separates $S^3$ into two 3-balls, $B$ and $B'$, where $B'$ refers to the component which contains both $D_i$ and $D'_i$.

Suppose that $B$ does not contain any standard element. We first isotope, if necessary, $\alpha$, to another closed curve called $\beta$, in the following manner. If a component of $\hat{\alpha}$ has neither endpoint on $E_q$, it is isotoped, keeping its endpoints fixed, to an
The length and thickness of words in a free group

Let $F$ be a free group.

Proof. Suppose that $x \in F$ is cyclically reduced and that $y \in [x]$. The number of times that $x_i$ or $x_i^{-1}$ appears in $y$ is greater than or equal to $2t(x)$.

We shall now list some easily proven facts:

1. $t(x_1^2 x_2^2 \cdots x_n^2) = 1$.
2. If $n$ is even, then $t([x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]) = 1$.
3. $t(x^m) = mt(x)$ when $x$ is cyclically reduced.
4. If $x \equiv uv$, then $t(x) \geq t(u) - 1$.
5. If $t(x) \geq 1$, then $t(x^my)$ goes to infinity as $m$ goes to infinity for fixed $y$. We do not require that the word $xy$ is reduced.

A proper factorization of $F$ is a pair of proper subgroups, $F_1$ and $F_2$, so that the inclusions induce an isomorphism from $F_1 * F_2$ onto $F$. We say that $x$ is separable if there exists a proper factorization of $F$ and elements $u_i \in F_i$ such that $x = u_1 u_2$.

**Corollary 3.** Let $x \in F$; if $t(x) \geq 2$, then $x$ is not separable.

**Proof.** Suppose $x$ were separable. Choose a basis $\{y_1, y_2, \ldots, y_m\}$ for $F_1$ and choose a basis for $F_2$, say $\{z_1, z_2, \ldots, z_{n-m}\}$. We can find a sphere basis

$$\Sigma = \Sigma_1, \Sigma_2, \ldots, \Sigma_m, \Sigma_1', \Sigma_2', \ldots, \Sigma_{n-m}$$

so that if $\alpha$ is a closed curve with the property that $\alpha(S) = x$, then $\alpha(\Sigma)$ is the representation of $x$ with respect to the basis

$$\{y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{n-m}\}.$$
Let $T$ be 2-sphere which separates the $\Sigma$-spheres from the $\Sigma'$-spheres. Since $x$ was assumed to be separable, $\alpha$ can be isotoped to meet $T$ at most twice: this is a contradiction since $t(x) \geq 2$ implies it meets $T$ at least 4-times.

As an application we note that if $x = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ where $m_k \geq 2$, then $x$ is separable but $x^p$ is not separable if $p \geq 2$.

**Theorem 2.** Let $x \in F$. If $x$ is separable, then there exists $y \in [x]$ such that $|y| = ||x||$ and $y \equiv w$ where $u$ is a word in the first $m$ letters and $v$ is a word in the last $n - m$ letters for some $1 \leq m < n$.

**Proof.** Let us assume that $|x| = ||x||$. If $x$ were in a proper free factor, then the result follows from a theorem of A. Shenitzer [2]. Thus we may assume there exists an automorphism $\Phi$ such that $\Phi(x) = u_1 u_2$, where $u_1$ is a non-trivial word in the first $m$-letters and $u_2$ is a non-trivial word in the last $n - m$-letters. Using Whitehead’s method we can find a sphere basis $\Sigma$ and a directed simple closed curve with base point, $\alpha$, such that $\alpha(S) \equiv x$ and $\alpha(\Sigma) \equiv u_1 u_2$. Thus there exists a separant sphere $T$ such that $\alpha$ intersects $T$ twice. Our proof is by induction on the number of circles, $k$, in $S \cap T$. If $k = 0$, then by a change of basepoints (cyclic permutation of $x$), the theorem is proven.

Our inductive step is the following assumption. Suppose that $x$ is a cyclically reduced word and $\alpha$ is a simple closed curve in $M$ so that $\alpha(S) = x$. Suppose that $T$ is a separant sphere such that $\alpha \cdot T = 2$ and $T \cap S$ consists of $p - 1$ or fewer circles. Then there exists an automorphism $\Phi$ so that $|\Phi(x)| \leq |x|$ and $\Phi(x) \equiv u_1 u_2$ where $u_1$ is a word in the first $m$ letters and $u_2$ is a word in the last $n - m$ letters. We now assume that an $\alpha$ and $T$ are given as above and that $T \cap S$ consists of $p$ circles. Let $E$ be an endcap of $T$. If $\alpha \cap E$ is empty, then replacing $S_i$ by $E^*$ as in the previous arguments proves the theorem.

We thus may assume that $E$ intersects $\alpha$ exactly once, since there are at least 2 endcaps and $T$ intersects $\alpha$ exactly twice. Now $B$ either contains some standard element $D$ ($D'$) or the standard elements in $B$ come in pairs $D$, $D'$, etc. If the latter is true, then $\alpha$ must meet $E_q$, since the only places $\alpha$ can enter or leave $B$ is at $E$ or $E_q$. By our previous arguments if we replace $S_i$ by $E^*$, we obtain our result. If the former is true and $\alpha$ meets $E_q$, then we use the same argument as in the line above. If $\alpha$ does not meet $E_q$, then it meets the 2-sphere $\Sigma_1 = E \cup E_q$ transversely in one point. This 2-sphere cannot separate $M$ since if it did any simple closed curve would have to meet it transversely an even number of times. Thus according to Proposition 1, $\Sigma_1$ is primitive. Let $\Sigma'$ be a sphere basis in which $\Sigma_1$ is the first sphere. The letter $x_1$ appears exactly once in the word $\Phi'(x)$ where $\Phi'$ is the automorphism of $F$ determined by $\Sigma'$. This implies that $\Phi'(x)$ and hence $x$ is primitive. Our result clearly follows in this case and our theorem is proven.

Since it is a decidable problem to list all the words of minimal length in $[x]$, it is therefore decidable whether or not $x$ is separable. Unlike the Shenitzer result not all separable words of minimal length can be factored even after taking a cyclic permutation. For example no cyclic permutation of $x = aba^{-1}b$ can be factored. The coinitial graph of $x$ has thickness 1 and thus $x$ is of minimal length. However $x$ is separable since the automorphism which sends $a$ to $a$ and $b$ to $ab$ sends $x$ to $a^2 b^2$. 


References