PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 10, Pages 2899–2904
S 0002-9939(99)05266-1
Article electronically published on April 23, 1999

# FURTHER EXTENSION OF THE HEINZ-KATO-FURUTA INEQUALITY

MITSURU UCHIYAMA

(Communicated by David R. Larson)

Abstract. Let $T$ be a bounded operator on a Hilbert space $\mathfrak{H}$, and $A, B$ positive definite operators. Kato has shown that if $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in \mathfrak{H}$, then $|(Tx, y)| \leq ||f(A)x|| \, ||g(B)y||$, where $f(t),\ g(t)$ are operator monotone functions defined on $[0,\infty)$ such that $f(t)g(t) = t$. Furuta has shown that $|(T|T|^{\alpha+\beta-1}x, y)| \leq ||A^\alpha x|| \, ||B^\beta y||$, where $0 \leq \alpha, \beta \leq 1,\ 1 \leq \alpha + \beta$. Let $f(t),\ g(t)$ be any continuous operator monotone functions, and set $h(t) = f(t)g(t)/t$ for $t > 0$. We will show that $Th(|T|)$ is well defined and $|(Th(|T|)x, y)| \leq ||f(A)x|| \, ||g(B)y||$. Moreover, we will extend this result for unbounded closed operators densely defined on $\mathfrak{H}$.

## 1. Introduction

Let $B(\mathfrak{H})$ be the $C^*$-algebra of all bounded operators on a Hilbert space $\mathfrak{H}$. Let $f$ be a real valued continuous function defined on $[0,\infty)$. If $f$ satisfies $f(A) \leq f(B)$ whenever $0 \leq A \leq B$, $A, B \in B(\mathfrak{H})$, then $f$ is called an *operator monotone function.* $f$ is an operator monotone function if and only if $f$ has an analytic extension that is holomorphic on the upper half plane and maps the upper half plane into it [4]. Therefore, if $0 \leq f(t)$ is an operator monotone function on $[0,\infty)$, then so is $f(\sqrt{t})^2$. This implies that for $A, B \geq 0$,

$$(1) \qquad \|Ax\| \leq \|Bx\| \text{ for every } x \in \mathfrak{H} \to \|f(A)x\| \leq \|f(B)x\| \text{ for every } x \in \mathfrak{H}.$$

$t^\alpha$ $(0 < \alpha \leq 1)$ and $\log(1 + t)$ are operator monotone functions [2], [4]. Kato has shown the following.

**Theorem A** ([3]). *Let $T$ be a bounded operator on a Hilbert space $\mathfrak{H}$, and $A, B$ positive definite operators. If $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x,\ y \in \mathfrak{H}$, then*

$$(2) \qquad |(Tx, y)| \leq \|A^\alpha x\|\|B^{1-\alpha}y\| \quad \textit{for any} \quad \alpha \in [0, 1].$$

Actually, he has shown that the above inequality holds for (unbounded) closed operators $T, A, B$, and furthermore that $A^\alpha$ and $B^{1-\alpha}$ in the above inequality can

Received by the editors June 6, 1997 and, in revised form, December 9, 1997.
1991 *Mathematics Subject Classification.* Primary 47A63, 47A30, 47B25.
*Key words and phrases.* Löwner theory, Heinz inequality, operator monotone function, closed operator, non-negative operator.
This research was partially supported by Grant-in-Aid for Scientific Research.

be replaced by $f(A)$ and $g(B)$, where $f(t),\ g(t)$ are operator monotone functions defined on $[0,\infty)$ such that $f(t)g(t)=t$, that is,

$$|(Tx,y)| \leq ||f(A)x||\ ||g(B)y||. \tag{3}$$

Furuta has extended (2) as follows.

**Theorem B** ([1])**.** *Let $T$ be a bounded operator on a Hilbert space $\mathfrak{H}$, and $A,B$ positive definite operators. If $||Tx|| \leq ||Ax||,\ ||T^*y|| \leq ||By||$ for all $x,y \in \mathfrak{H}$, then*

$$|(T|T|^{\alpha+\beta-1}x,y)| \leq ||A^\alpha x||\ ||B^\beta y||, \tag{4}$$

*where $0 \leq \alpha,\beta \leq 1,\ 1 \leq \alpha+\beta$.*

The aim of this paper is to extend (3) and (4). Actually, we will show that $T((fg)/t)(|T|)$ is well defined for every operator monotone function $f(t),\ g(t)$ and

$$|(T((fg)/t)(|T|)x,y)| \leq ||f(A)x||\ ||g(B)y||.$$

We will treat bounded operators in the next section and unbounded closed operators in the last section.

## 2. Bounded operators

**Lemma 1.** *Let $A=\int_{-0}^{\infty}\lambda dE_\lambda$ be the spectral decomposition of $A \geq 0$. Then $A(A+\delta)^{-1}$ strongly converges to $I-E_0$ as $\delta \to +0$.*

*Proof.* $A(A+\delta)^{-1}E_0x=(A+\delta)^{-1}AE_0x=0$ for $x \in \mathfrak{H}$. Thus we have

$$||A(A+\delta)^{-1}x-(I-E_0)x||^2 = ||\{A(A+\delta)^{-1}-I\}(I-E_0)x||^2$$
$$= \int_{-0}^{\infty}|\frac{\lambda}{\lambda+\delta}-1|^2 d||E_\lambda(I-E_0)x||^2 = \int_{+0}^{\infty}|\frac{\delta}{\lambda+\delta}|^2 d||E_\lambda(I-E_0)x||^2.$$

From Lebesgue's dominated convergence theorem, this converges to 0 as $\delta \to +0$. □

**Lemma 2.** *Let $T=V|T|$ be the polar decomposition of $T$. Then*

$$T(|T|+\delta)^{-1} \to V \ (\textit{strongly}) \textit{ as } \delta \to +0.$$

*Proof.* Let $\{E_\lambda\}$ be the spectral family for $|T|$. By Lemma 1, $|T|(|T|+\delta)^{-1} \to I-E_0$ as $\delta \to +0$. Thus we have

$$T(|T|+\delta)^{-1} \to V(I-E_0) \text{ (strongly) as } \delta \to +0.$$

Since $E_0$ is the projection onto the eigen-space $\{x : Tx=0\}$, we get $VE_0=0$. Consequently,

$$T(|T|+\delta)^{-1} \to V \text{ (strongly) as } \delta \to +0.$$

□

**Definition 1.** For a function $h(t)$ we set $h_\delta(t)=h(t+\delta)$.

**Lemma 3.** *Let $h(t)$ be a continuous function defined on $(0,\infty)$. If $th(t)$ has the continuous extension $f(t)$ on $[0,\infty)$, then*

$$Th_\delta(|T|) \to Vf(|T|) \ (\textit{strongly}) \textit{ as } \delta \to +0.$$

*Proof.* We note that $Th_\delta(|T|)=T(|T|+\delta)^{-1}f(|T|+\delta)$. Since $f(|T|+\delta)$ converges to $f(|T|)$ in the norm toplogy, from Lemma 2 it follows that $Th_\delta(|T|) \to Vf(|T|)$ (strongly) as $\delta \to +0$. □

**Definition 2.** Under the same assumption as in Lemma 3, we denote the limit of $Th_\delta(|T|)$ as $\delta \to +0$ by $Th(|T|)$, which is equal to $Vf(|T|)$.

Let us remark that we defined not $h(|T|)$, but $Th(|T|)$. If $h(t)$ is continuous on $[0,\infty)$, then $h(|T|)$ is well defined, and the product of $T$ and $h(|T|)$ is coincident with $Th(|T|)$ defined above. It is clear that if $h(t) = t^{-\alpha}$ $(0 < \alpha \leq 1)$, then $Th(|T|)$ is well defined and $Th(|T|) = V|T|^{1-\alpha}$.

**Theorem 4.** *Let $f$ and $g$ be any continuous operator monotone functions defined on $[0,\infty)$. Then $T((fg)/t)(|T|)$ is well defined for every $T$ in $B(\mathfrak{H})$ in the sense of Definition 2. Moreover, if*

$$||Tx|| \leq ||Ax||,\ ||T^*y|| \leq ||By|| \quad \textit{for all} \quad x, y \in \mathfrak{H},$$

*where $A, B \geq 0$, then*

$$|(T((fg)/t)(|T|)x, y)| \leq ||f(A)x||\ ||g(B)y|| \quad \textit{for all} \quad x, y \in \mathfrak{H}. \tag{5}$$

*Proof.* Set $h(t) = f(t)g(t)/t$ for $t > 0$. It is clear that $th(t)$ is continuous on $[0,\infty)$ and hence $Th(|T|) = T((fg)/t)(|T|)$ is well defined. Let $T = V|T|$ be the polar decomposition of $T$. Then by Lemma 3 we obtain

$$\begin{aligned} Th(|T|) &= V(fg)(|T|) = Vf(|T|)g(|T|), \\ |(Th(|T|)x, y)| &= |(Vf(|T|)g(|T|)x, y)| = |(g(|T^*|)Vf(|T|)x, y)| \\ &\leq ||f(|T|)x||\ ||g(|T^*|)y||, \end{aligned} \tag{6}$$

and by (1)

$$\leq ||f(A)x||\ ||g(B)y||.$$

□

We remark that (5) is an extension of both (3) and (4): in fact, we removed the conditions $f(t)g(t) = t$ and $1 \leq \alpha + \beta$ from (3) and (4), respectively.

## 3. Unbounded closed operators

From now on, we consider the case where $T, A, B$ are unbounded closed operators densely defined on $\mathfrak{H}$. Let us remember that for a self-adjoint operator $A$ and for continuous functions $f, g$, $f(A)g(A) \subseteq (fg)(A)$ and that if $x \in \mathfrak{D}(g(A)) \cap \mathfrak{D}((fg)(A))$, then $x \in \mathfrak{D}(f(A)g(A))$ and $f(A)g(A)x = (fg)(A)x$.

It is known [4], [5] that an operator monotone function $f$ on $[0,\infty)$ has an integral representation

$$f(t) = \alpha + \beta t - \int_0^\infty \left(\frac{1}{t+s} - \frac{s}{s^2+1}\right) dv(s) \ \text{ for } \ t > 0,$$

where $\alpha$ and $\beta$ are real, $\beta \geq 0$, and $v$ is a non-negative Borel measure such that

$$\int_0^\infty \frac{1}{1+s^2} dv(s) < \infty.$$

Since

$$\lim_{t\to+0} f(t) = f(0) \text{ and } \lim_{t\to+0} \int_{\frac{1}{t}}^\infty \left(\frac{s}{s^2+1} - \frac{1}{t+s}\right) dv(s) = 0,$$

it follows that

$$f(0) = \alpha - \lim_{t\to+0}\int_0^{\frac{1}{t}}(\frac{1}{t+s} - \frac{s}{s^2+1})dv(s)$$
$$= \alpha - \lim_{t\to+0}\int_0^{\infty}(\frac{1}{t+s} - \frac{s}{s^2+1})\chi_{[0,\frac{1}{t}]}dv(s).$$

From Fatou's theorem it follows that

$$\int_0^\infty (\frac{1}{s} - \frac{s}{s^2+1})dv(s) < \infty, \text{ and hence } f(0) = \alpha - \int_0^\infty(\frac{1}{s} - \frac{s}{s^2+1})dv(s).$$

Thus we have

$$f(t) = f(0) + \beta t + \int_0^\infty (\frac{1}{s} - \frac{1}{t+s})dv(s). \tag{7}$$

Since $f(1) < \infty$, we get

$$\int_0^\infty \frac{1}{s(1+s)}dv(s) < \infty. \tag{8}$$

**Lemma 5.** *Let $0 \le f(t),\ g(t)$ be operator monotone functions, and let $A \ge 0$ be an unbounded self-adjoint operator. Then $\mathfrak{D}(A) \subseteq \mathfrak{D}(f(A))$, where $\mathfrak{D}(A)$ stands for the domain of $A$. Moreover, $\mathfrak{D}(A^2) \subseteq \mathfrak{D}(f(A)g(A)) \subseteq \mathfrak{D}((fg)(A))$.*

*Proof.* By (7) and (8), it is clear that $f(t)/t$ and $g(t)/t$ are bounded on $[1,\infty)$. Hence we can see

$$\mathfrak{D}(A) \subseteq \mathfrak{D}(f(A)) \quad \text{and} \quad \mathfrak{D}(A) \subseteq \mathfrak{D}(g(A)).$$

Similarly, from the boundedness of $f(t)g(t)/t^2$ on $[1,\infty)$ it follows that $\mathfrak{D}(A^2) \subseteq \mathfrak{D}((fg)(A))$. Thus for an arbitrary $x \in \mathfrak{D}(A^2)$, we have

$$x \in \mathfrak{D}(g(A)) \quad \text{and} \quad x \in \mathfrak{D}((fg)(A)),$$

and hence

$$x \in \mathfrak{D}(f(A)g(A)) \quad \text{and} \quad f(A)g(A)x = (fg)(A)x.$$

□

**Proposition 6.** *Let $T$ be a densely defined closed operator, and let $f$ and $g$ be arbitrary operator monotone functions defined on $[0,\ \infty)$, such that $f \ge 0,\ g \ge 0$. Set $h(t) = ((fg)/t)$ for $t > 0$. Then*

$$\mathfrak{D}(Th_\delta(|T|)) = \mathfrak{D}((fg)(|T|)) \textit{ for every } \delta > 0,$$

$$Th_\delta(|T|)x \to V(fg)(|T|)x \quad (s) \quad \textit{as} \quad \delta \to +0 \textit{ for } x \in \mathfrak{D}((fg)(|T|)).$$

*Proof.* By (7),

$$0 \le f(t+\delta) - f(t) \le f(\delta) - f(0), \quad 0 \le g(t+\delta) - g(t) \le g(\delta) - g(0).$$

This implies

$$1 \le \frac{f(t+\delta)}{f(t)}\frac{g(t+\delta)}{g(t)} \le (1 + \frac{f(\delta)-f(0)}{f(t)})(1 + \frac{g(\delta)-g(0)}{g(t)})$$
$$\le (1 + \frac{f(\delta)-f(0)}{f(1)})(1 + \frac{g(\delta)-g(0)}{g(1)}) \quad (t > 1).$$

Thus we can see that

$$\mathfrak{D}((fg)(|T|+\delta)) = \mathfrak{D}((fg)(|T|)),$$

$$(fg)(|T|+\delta)x \to (fg)(|T|)x \quad (s) \quad (\delta \to +0) \text{ for } x \in \mathfrak{D}((fg)(|T|)). \tag{9}$$

By $(fg)(t+\delta) \geq \delta h(t+\delta)$, we obtain

$$\mathfrak{D}((fg)(|T|+\delta)) \subseteq \mathfrak{D}(h(|T|+\delta)).$$

Therefore, using $(|T|+\delta)(h(|T|+\delta)) \subseteq (fg)(|T|+\delta)$ we get

$$(|T|+\delta)(h(|T|+\delta)) = (fg)(|T|+\delta).$$

Thus we obtain

$$Th_\delta(|T|) = \{V|T|(|T|+\delta)^{-1}(|T|+\delta)\}h_\delta(|T|) = \{V|T|(|T|+\delta)^{-1}\}(fg)(|T|+\delta));$$

here we used $\mathfrak{D}(|T|h(|T|+\delta)) = \mathfrak{D}((|T|+\delta)(h(|T|+\delta)))$.

In a fashion similar to Lemma 1, we can see

$$V|T|(|T|+\delta)^{-1} \to V(1-E_0) = V \text{ (strongly) as } \delta \to +0.$$

Since $V|T|(|T|+\delta)^{-1}$ is contractive, this and (9) imply

$$Th_\delta(|T|)x \to V(fg)(|T|)x \;\; (\delta \to +0) \text{ for } x \in \mathfrak{D}((fg)(|T|)).$$

□

**Definition 3.** Under the same condition as in Proposition 6, we denote the limit of $Th_\delta(|T|)$ by $Th(|T|)$. This is equal to $V(fg)(|T|)$, where $(fg)(t) = th(t)$.

For non-negative self-adjoint operators $A$ and $B$, $A \ll B$ means $\mathfrak{D}(B) \subseteq \mathfrak{D}(A)$ and $||Ax|| \leq ||Bx||$ for $x \in \mathfrak{D}(B)$.

If $0 \leq f$ is an operator monotone function defined on $[0, \infty)$, then $f(A) \ll f(B)$ whenever $A \ll B$ (see [3]).

**Theorem 7.** *Let $T$ be a densely defined closed operator, and let $A$ and $B$ be non-negative self-adjoint operators. Let $f$ and $g$ be arbitrary operator monotone functions defined on $[0, \infty)$, such that $f \geq 0$, $g \geq 0$. Then $T((fg)/t)(|T|)$ is well defined in the sense of Definition 3. Moreover, if $|T| \ll A$, $|T^*| \ll B$, then*

$$|(T((fg)/t)(|T|)x, y)| \leq ||f(A)x|| \; ||g(B)y||$$

*for every $x \in \mathfrak{D}(A) \cap \mathfrak{D}(fg(|T|))$ and for every $y \in \mathfrak{D}(B)$.*

*Proof.* By Proposition 6 and Definition 3, $T((fg)/t)(|T|)$ is well defined and equal to $V(fg)(|T|)$.

The assumption $|T| \ll A$, $|T^*| \ll B$ implies that $f(|T|) \ll f(A)$, $g(|T^*|) \ll g(B)$. Suppose

$$x \in \mathfrak{D}(A) \cap \mathfrak{D}(fg(|T|)) \quad \text{and} \quad y \in \mathfrak{D}(B).$$

By Lemma 5, we get

$$x \in \mathfrak{D}(f(|T|)) \quad \text{and} \quad y \in \mathfrak{D}(g(|T^*|)).$$

Therefore, $x \in \mathfrak{D}(f(|T|)) \cap \mathfrak{D}(fg(|T|))$ and hence $(fg)(|T|)x = g(|T|)f(|T|)x$. Consequently we get

$$\begin{aligned}|(Th(|T|)x, y)| &= |(V(gf)(|T|)x, y)| = |(Vg(|T|)f(|T|)x, y)| \\ &= |(g(|T^*|)Vf(|T|)x, y)| \leq ||f(|T|)x|| \; ||g(|T^*|)y|| \leq ||f(A)x|| \; ||g(B)y||. \quad \square\end{aligned}$$

## References

1. T. Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc. **120** (1994), 785–787. MR **94e:**47034
2. E. Heinz, *Beiträge zur Strüngstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415 – 438. MR **13:**471f
3. T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208 – 212. MR **14:**766e
4. K. Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177 – 216.
5. M. Rosenblum, J. Rovnyak, *Hardy classes and operator theory*, Oxford University Press, New York, 1985. MR **87e:**47001

Department of Mathematics, Fukuoka University of Education, Munakata, Fukuoka, 811-4192, Japan

*E-mail address*: uchiyama@fukuoka-edu.ac.jp