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SINGULAR SETS AND MAXIMAL TOPOLOGIES

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Abstract. Spaces which are maximal with respect to a semi-regular property are characterised. Furthermore, a method to construct such topologies is given. Consequently, new characterisations of maximal pseudocompact spaces and of maximal Q.H.C. spaces are presented. Known characterisations of maximal connected spaces and of maximal feebly compact spaces are given alternative proofs.

1. Introduction

The family of all topologies definable for an infinite set X is a complete atomic lattice (ordered by inclusion), and is often denoted $LT(X)$. When $\mathcal{T}_1, \mathcal{T}_2$ are members of $LT(X)$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_2$ is an *expansion* of $\mathcal{T}_1$. A topological property $\mathfrak{R}$ is called *contractive* (*expansive*) when, for a given member $\mathcal{T}$ of $LT(X)$ with property $\mathfrak{R}$, any weaker (stronger) member of $LT(X)$ has property $\mathfrak{R}$; then, given a contractive property $\mathfrak{R}$, a member $\mathcal{T}$ of $LT(X)$ is said to be *maximal with respect to* $\mathfrak{R}$ if and only if $\mathcal{T}$ has property $\mathfrak{R}$ but no stronger member of $LT(X)$ has property $\mathfrak{R}$. When appropriate, we shorten this to, $\mathcal{T}$ is maximal $\mathfrak{R}$. Given $\mathcal{T} \in LT(X)$, recall that a $\mathcal{T}$-open set V is $\mathcal{T}$-regular open if $V = int_{\mathcal{T}} cl_{\mathcal{T}} V$. The topology generated by the family of $\mathcal{T}$-regular open sets is called the *semi-regularisation* of $\mathcal{T}$ and denoted by $\mathcal{T}_s$. Furthermore, $\mathcal{T}$ and $\mathcal{T}_s$ have precisely the same regular open sets and consequently, $(\mathcal{T}_s)_s = \mathcal{T}_s$ [17]. A topological property $\mathfrak{R}$ is called *semi-regular* when $\mathcal{T} \in LT(X)$ is $\mathfrak{R}$ if and only if $\mathcal{T}_s \in LT(X)$ is $\mathfrak{R}$ [3]. Hausdorff and connectedness are the classic examples of semi-regular properties.

Given $\mathcal{T} \in LT(X)$ and a subset V of X, the boundary of V, $(cl_{\mathcal{T}} V - int_{\mathcal{T}} V)$, is denoted by $\partial_{\mathcal{T}} V$. If $\mathcal{D}$ is a family of subsets of X, the topology generated by $\mathcal{T} \cup \mathcal{D}$ is denoted by $\langle \mathcal{T} \cup \mathcal{D} \rangle$. In the special case when $\mathcal{D} = \{A\}$ for some $A \subseteq X$ we write $\langle \mathcal{T} \cup \mathcal{D} \rangle$ as $\mathcal{T}(A)$.

The concept of a singular set was introduced by Guthrie and Stone in 1977 to construct a maximal connected expansion of the real line [8, 10]. In 1986, Neumann-Lara and Wilson generalised the notion of a singular set to characterise T_1 maximal connected spaces [14]. In this paper we refine and further generalise the notion of a singular set to establish, for *any* contractive topological property, a characterisation

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of a submaximal space which is maximal with respect to the contractive property. Of course, for contractive semi-regular properties (which include connectedness, pseudocompactness, feebly compactness and Q.H.C.) we will have established a complete characterisation of maximality. Indeed, motivated by the construction of Guthrie and Stone [10], we provide a general method of constructing topologies which are maximal with respect to a semi-regular property.

Throughout this paper we assume no separation axioms unless they are explicitly stated. We conclude this section by listing some basic results which we use in subsequent sections.

Lemma 1.1. *Given $\mathcal{T} \in LT(X)$, $\mathcal{T} = \langle \mathcal{T}_s \cup \mathcal{D} \rangle$ where $\mathcal{D}$ is the filter of $\mathcal{T}$-dense and open sets, and therefore is a filterbase of $\mathcal{T}_s$-dense sets. Furthermore, $\langle \mathcal{T}_s \cup \mathcal{F} \rangle_s = \mathcal{T}_s$, when $\mathcal{F}$ is a filterbase of $\mathcal{T}_s$-dense sets* [2].

Definition 1.2. Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is *submaximal* if every $\mathcal{T}$-dense set is $\mathcal{T}$-open.

Theorem 1.3. *Given $\mathcal{T} \in LT(X)$, the following statements are equivalent:*

(i) *$\mathcal{T}$ is submaximal.*
(ii) *The family of $\mathcal{T}$-dense open sets is an ultrafilter of $\mathcal{T}_s$-dense sets.*
(iii) *For any $\mathcal{H} \in LT(X)$ such that $\mathcal{T} \subsetneq \mathcal{H}, \mathcal{H}_s \neq \mathcal{T}_s$.*
(iv) *Every subset of X is the union of an open set and a closed set.*
(v) *Every subset of X is the intersection of an open set and a closed set.*
(vi) *For every subset A of X which is not open, there are non-empty proper closed sets F_1,F_2 such that $F_1 \subseteq A \subseteq F_2$.*
(vii) *Every subset A of X, for which $intA = \emptyset$ is closed.*
(viii) *Every subset A of X, for which $intA = \emptyset$ is discrete.*
(ix) *$cl(A) - A$ is closed, for every subset A of X.*
(x) *$cl(A) - A$ is discrete, for every subset A of X.*

Proof. [1], [2], [11] and [17]. □

So if $\mathfrak{R}$ is a contractive semi-regular property, then a maximal $\mathfrak{R}$ topology is submaximal. Also, every $\mathfrak{R}$ topology has a submaximal $\mathfrak{R}$ expansion (constructed by declaring open an ultrafilter of dense sets, thus ensuring condition (ii) of Theorem 1.3 is satisfied). Recall that a topology is T_{ES} if every singleton is either open or closed [12, 13]. A submaximal topology is T_{ES} [11].

Lemma 1.4. *If $\mathcal{T} \in LT(X)$ is submaximal and $B \subseteq X$, then $(int_{\mathcal{T}} cl_{\mathcal{T}} int_{\mathcal{T}} B) \cup \{x\}$ is $\mathcal{T}(B)$-open, for all $x \in B - int_{\mathcal{T}} B$.*

Proof. $(X - B) \cup (int_{\mathcal{T}} B) \cup \{x\}$ is $\mathcal{T}$-dense and so by hypothesis is $\mathcal{T}$-open. Now

$$(int_{\mathcal{T}} B) \cup \{x\} = B \cap [(X - B) \cup (int_{\mathcal{T}} B) \cup \{x\}]$$

and so is $\mathcal{T}(B)$-open. Thus, $(int_{\mathcal{T}} cl_{\mathcal{T}} int_{\mathcal{T}} B) \cup \{x\}$ is $\mathcal{T}(B)$-open. □

Note. Essentially this lemma is a refinement of [7, Lemma 5], and although simple, it is powerful when considering maximality. It essentially proclaims that an expansion of a submaximal topology is constructed by declaring open certain sets of the form $V \cup \{x\}$, where V is regular open and $x \in X$.

2. Singular set theory

Throughout this section we assume that $\mathfrak{R}$ is a contractive topological property.

Definition 2.1. Given $\mathcal{T} \in LT(X)$ has property $\mathfrak{R}$, V is $\mathcal{T}$-regular open and $x \in X$. Then, $V \cup \{x\}$ is said to be a $\mathcal{T}$-*singular* (*with respect to* $\mathfrak{R}$) *set* at x, if $\mathcal{T}(V \cup \{x\})$ has property $\mathfrak{R}$.

Note. Clearly, any regular open set is singular at each of its points. More generally, A is a *singular* (*w.r.t.* $\mathfrak{R}$) *set* when either A is regular open, or there exists $x \in A$ such that $A - \{x\}$ is regular open and $\mathcal{T}(A)$ enjoys the property $\mathfrak{R}$. It can be observed that, after Lemma 4.1 later, this definition refines and generalises that of Neumann-Lara and Wilson [14].

Example 2.2. Consider the real line with the usual topology. Let V be the following union of open intervals, $(-1,0) \cup \left\{\bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right)\right\}$. Then $V \cup \{0\}$ is a singular (w.r.t. connected) set at 0, but is not an open set.

Definition 2.3. Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is called *non-singular* (*w.r.t.* $\mathfrak{R}$) if $\mathcal{T}$ has property $\mathfrak{R}$ and every singular (w.r.t. $\mathfrak{R}$) set is $\mathcal{T}$-open.

Note. This generalises the definition of Guthrie and Stone [8].

Theorem 2.4. *Given $\mathcal{T} \in LT(X)$ is submaximal and $\mathfrak{R}$. Then $\mathcal{T}$ is maximal $\mathfrak{R}$ if and only if $\mathcal{T}$ is non-singular (w.r.t. $\mathfrak{R}$).*

Proof. Suppose $\mathcal{T}$ is $\mathfrak{R}$ but not maximal $\mathfrak{R}$. Then there is a set $B \subset X$ such that $\mathcal{T}(B)$ is a proper $\mathfrak{R}$ expansion of $\mathcal{T}$. So there is a point $x \in B - int_{\mathcal{T}}B$. Now $V = int_{\mathcal{T}}cl_{\mathcal{T}}int_{\mathcal{T}}B$ is $\mathcal{T}$-regular open, however $V \cup \{x\}$ is not $\mathcal{T}$-open, since $\mathcal{T}$ is submaximal and $int_{\mathcal{T}}B \cup \{x\} = (V \cup \{x\}) \cap [int_{\mathcal{T}}B \cup (X - cl_{\mathcal{T}}int_{\mathcal{T}}B) \cup \{x\}]$. But, by Lemma 1.4 above, $V \cup \{x\}$ is $\mathcal{T}(B)$-open and so, as $\mathfrak{R}$ is contractive, $\mathcal{T}(V \cup \{x\})$ is $\mathfrak{R}$. That is, $V \cup \{x\}$ is a $\mathcal{T}$-singular (w.r.t. $\mathfrak{R}$) set which is not $\mathcal{T}$-open.

The converse is immediate. □

Corollary 2.5. *Suppose that $\mathcal{T} \in LT(X)$ and $\mathfrak{R}$ is a semi-regular property. Then $\mathcal{T}$ is maximal $\mathfrak{R}$ if and only if $\mathcal{T}$ is $\mathfrak{R}$, submaximal, and for every regular open set V and every non-isolated point x in the subspace $X - V$, $V \cup \{x\}$ is* not *singular (w.r.t. $\mathfrak{R}$) in the space X.*

Theorem 2.6. *Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is non-singular (w.r.t. $\mathfrak{R}$) if and only if $\mathcal{T}$ is $\mathfrak{R}$ and for every $x \in X$ the family of regular open neighbourhoods of x is an ultrafilter of singular (w.r.t. $\mathfrak{R}$) sets at x.*

Lemma 2.7. *Suppose $\mathcal{T} \in LT(X)$ is $\mathfrak{R}$, $A \subseteq X$ and $\mathcal{M}_x$ is a filterbase of $\mathcal{T}$-singular (w.r.t. $\mathfrak{R}$) sets at x, where $x \in X$. Let $\mathcal{T}^* = \langle \mathcal{T} \cup \mathcal{M}_x \rangle$. Then (using a convenient change of notation) the $\mathcal{T}^*$-closure of A is described by*

$$\overline{A}^* = \begin{cases} \overline{A}, & \text{if } x \in \overline{(M - \{x\}) \cap A} \text{ for every } M \in \mathcal{M}_x, \\ \overline{A} - \{x\}, & \text{if } x \notin \overline{(M - \{x\}) \cap A} \text{ for every } M \in \mathcal{M}_x. \end{cases}$$

Proof. Let $y \in \overline{A} - \{x\}$. Then a $(\mathcal{T}^* - \mathcal{T})$-neighbourhood of y contains a set of the form $G \cap M$ where $y \in G \in \mathcal{T}$ and $y \in M \in \mathcal{M}_x$. By definition of a singular set, either M or $M - \{x\}$ is $\mathcal{T}$-regular open so that $G \cap M$ is a $\mathcal{T}$-neighbourhood of y. But $y \in \overline{A}$, so $G \cap M \cap A \neq \emptyset$, that is, $y \in \overline{A}^*$. Hence, $\overline{A} - \{x\} \subseteq \overline{A}^* \subseteq \overline{A}$. Finally, it is clear that $x \in \overline{A}^*$ if and only if $x \in \overline{(M - \{x\}) \cap A}$ for every $M \in \mathcal{M}_x$. □

Corollary 2.8. *Suppose $\mathcal{T} \in LT(X)$ is $\mathfrak{R}$, and $\mathcal{M}_x$ is a filterbase of $\mathcal{T}$-singular (w.r.t. $\mathfrak{R}$) sets at x, where $x \in X$. Let $\mathcal{T}^* = \langle \mathcal{T} \cup \mathcal{M}_x \rangle$. If $G \in \mathcal{T}^*$ and $x \notin G$, then $G \in \mathcal{T}$.*

Lemma 2.9. *Suppose $\mathcal{T} \in LT(X)$ is $\mathfrak{R}$, and that every singleton $\mathcal{T}$-singular set is $\mathcal{T}$-open, while $\mathcal{M}_x$ is an ultrafilter of $\mathcal{T}$-singular (w.r.t. $\mathfrak{R}$) sets at x, where $x \in X$. Let $\mathcal{T}' = \langle \mathcal{T} \cup \mathcal{M}_x \rangle$. If $\mathcal{T}'$ is $\mathfrak{R}$, then every $\mathcal{T}'$-singular set at x is $\mathcal{T}'$-open.*

Proof. Suppose $V \cup \{x\}$ is $\mathcal{T}'$-singular at x but is not $\mathcal{T}'$-open, so we may assume that V is $\mathcal{T}'$-regular open and that $x \in \partial_{\mathcal{T}'} V$. By Corollary 2.8 V is $\mathcal{T}$-open and by Lemma 2.7 $cl_{\mathcal{T}'} V = cl_{\mathcal{T}} V$. Then since $\mathcal{T} \subseteq \mathcal{T}'$, $V \subseteq int_{\mathcal{T}} cl_{\mathcal{T}'} V \subseteq int_{\mathcal{T}'} cl_{\mathcal{T}'} V = V$ and therefore V is $\mathcal{T}$-regular open. Now for each $M \in \mathcal{M}_x$, $M \cap (V \cup \{x\}) \neq \emptyset$ because $x \in \partial_{\mathcal{T}'} V$ and also $\mathcal{T}(M \cap (V \cup \{x\})) \subseteq \mathcal{T}'(V \cup \{x\})$. But $\mathfrak{R}$ is contractive and the intersection of any two regular open sets is regular open, thus $V \cup \{x\}$ meets each member of $\mathcal{M}_x$ at a $\mathcal{T}$-singular set at x, yet $V \cup \{x\} \notin \mathcal{M}_x$ (since $V \cup \{x\} \notin \mathcal{T}'$). That is, $\mathcal{M}_x$ is not an ultrafilter of $\mathcal{T}$-singular sets at x. □

Theorem 2.10. *Suppose $\mathfrak{R}$ is a semi-regular property and that $\mathcal{T} \in LT(X)$ is $\mathfrak{R}$, and every singleton $\mathcal{T}$-singular set is $\mathcal{T}$-open. Let $\mathcal{D}$ be an ultrafilter of $\mathcal{T}$-dense sets. Given $x \in X$, let $\mathcal{M}_x$ be an ultrafilter of $\mathcal{T}$-singular (w.r.t. $\mathfrak{R}$) sets at x. Let $\mathcal{T}' = \langle \mathcal{T} \cup \mathcal{D} \cup (\cup_{x \in X} \mathcal{M}_x) \rangle$. If $\mathcal{T}'$ has the property $\mathfrak{R}$, then $\mathcal{T}'$ is a maximal $\mathfrak{R}$ expansion of $\mathcal{T}$.*

Proof. $\mathcal{T}^* = \langle \mathcal{T} \cup \mathcal{D} \rangle$ is submaximal, so $\mathcal{T}'$ is submaximal. Suppose $V \cup \{x\}$ is $\mathcal{T}'$-singular at x but is not $\mathcal{T}'$-open. As every singleton $\mathcal{T}$-singular set is $\mathcal{T}$-open, if $M \cup \{x\} \in \mathcal{M}_x$, then $x \in cl_{\mathcal{T}} M$ and so necessarily $x \in cl_{\mathcal{T}^*} M$. Thus $int_{\mathcal{T}^*} V$ is $\mathcal{T}^*$-regular open and so must be $\mathcal{T}$-regular open. Now as $\mathcal{T}^*$ is submaximal and $(int_{\mathcal{T}^*} V) \cup \{x\} = (V \cup \{x\}) \cap [(int_{\mathcal{T}^*} V) \cup (X - V) \cup \{x\}]$ we have

$$\langle \mathcal{T}^* \cup \mathcal{M}_x \cup \{(int_{\mathcal{T}^*} V) \cup \{x\}\} \rangle \subseteq \mathcal{T}'(V \cup \{x\}).$$

But $\mathfrak{R}$ is contractive, and $V \cup \{x\}$ is $\mathcal{T}'$-singular, so $\langle \mathcal{T} \cup \mathcal{M}_x \cup \{(int_{\mathcal{T}^*} V) \cup \{x\}\} \rangle$ is $\mathfrak{R}$. Now $(int_{\mathcal{T}^*} V) \cup \{x\}$ cannot be $\langle \mathcal{T} \cup \mathcal{M}_x \rangle$-regular open (otherwise, $V \cup \{x\}$ is $\mathcal{T}^*$-open), so by Corollary 2.8 above, $int_{\mathcal{T}^*} V$ is $\langle \mathcal{T} \cup \mathcal{M}_x \rangle$-regular open, and therefore, $(int_{\mathcal{T}^*} V) \cup \{x\}$ is $\langle \mathcal{T} \cup \mathcal{M}_x \rangle$-singular set at x, which is not $\langle \mathcal{T} \cup \mathcal{M}_x \rangle$-open (since $V \cup \{x\}$ is not $\mathcal{T}'$-open). This contradicts the statement of Lemma 2.9 above. □

Note. A connected T_{ES} topology is an example of a topology in which every singleton singular (w.r.t. connected) set is open. Also, a Q.H.C. almost H-space (feebly compact almost E_1-space) is an example of a topology in which every singleton singular (w.r.t. Q.H.C. (feebly compact)) set is open (see Definition 4.3 later).

The key to this construction is in finding suitable ultrafilters of singular sets which ensure the expanded topology possesses the property $\mathfrak{R}$. Although this method appears difficult, Lemma 1.4 confirms that it is necessary for expansions of submaximal $\mathfrak{R}$ topologies. Guthrie, Stone and Simon showed that it is possible to find suitable ultrafilters of singular sets on the real line which preserve connectedness [10, 18].

Question. Is there a Tychonoff non-singular (w.r.t. connected) topology?

If not, then there is no Tychonoff topology which is maximal connected. Tkačenko *et al.* have shown that no metrizable space is maximal connected and that certain classes of Tychonoff spaces are not maximal connected [19].

3. The semi-regular nature of non-singularity

In general, if $\mathcal{T} \in LT(X)$, and $\mathcal{T}_s$ is non-singular, then $\mathcal{T}$ is non-singular.

Theorem 3.1. *Suppose $\mathfrak{R}$ is contractive, semi-regular, and that $\mathcal{T} \in LT(X)$ is non-singular (w.r.t $\mathfrak{R}$). Then every $\mathcal{T}_s$-singular set $V \cup \{x\}$ such that $x \in \partial_{\mathcal{T}^*} V$ is $\mathcal{T}_s$-open.*

Proof. Suppose $\mathcal{T}_s(V \cup \{x\})$ is $\mathfrak{R}$, where V is $\mathcal{T}_s$-regular open and $x \in \partial_{\mathcal{T}_s} V$. Of course, V is $\mathcal{T}$-regular open and $\partial_{\mathcal{T}_s} V = \partial_{\mathcal{T}} V$. Now $\mathcal{T} = \langle \mathcal{T}_s \cup \mathcal{D} \rangle$, where $\mathcal{D}$ is a filterbase of $\mathcal{T}_s$-dense sets. But $x \in \partial V$, so $\mathcal{D}$ is a filterbase of $\mathcal{T}_s(V \cup \{x\})$-dense sets, and because $\mathfrak{R}$ is semi-regular, $\langle \mathcal{T}_s \cup \mathcal{D} \cup \{V \cup \{x\}\} \rangle = \mathcal{T}(V \cup \{x\})$ is also $\mathfrak{R}$. Hence $V \cup \{x\}$ is $\mathcal{T}$-singular at x, and so by hypothesis is $\mathcal{T}$-open. But $x \in \partial V$ so $x \in V$, that is, $V \cup \{x\} = V \in \mathcal{T}_s$. □

Theorem 3.2. *Suppose $\mathcal{T} \in LT(X)$ and every singleton is either $\mathcal{T}$-open or $\mathcal{T}_s$-closed. Then $\mathcal{T}$ is non-singular if and only if $\mathcal{T}_s$ is non-singular.*

Corollary 3.3. *The property 'non-singular and every singleton is either regular open or regular closed' is a semi-regular property.*

4. Applications of singular set theory

It transpires that a singular set may readily be characterised in terms particular to the existing topology and without reference to any expansions (see Lemmas 4.1, 4.4 and 4.6 below). As a consequence of Corollary 2.5, a complete internal characterisation of a topology which is maximal w.r.t. a contractive semi-regular property is easily established. This considerably enlarges Cameron's 'class of maximal topologies' [4].

Lemma 4.1. *Suppose $\mathcal{T} \in LT(X)$ is connected, and that V is non-empty $\mathcal{T}$-regular open, while x is non-isolated in the subspace $X - V$. Then $V \cup \{x\}$ is* not *$\mathcal{T}$-singular (w.r.t connected) at x if and only if there is a non-empty clopen subset, C, of $X - \{x\}$ such that $x \notin cl_{\mathcal{T}}(V \cap C)$.*

Proof. $\mathcal{T}^* = \mathcal{T}(V \cup \{x\})$ is disconnected if and only if there is a non-empty proper $\mathcal{T}^*$-clopen set C. We may assume $x \notin C$, so by Corollary 2.8, C is $\mathcal{T}$-open. Also, C is $\mathcal{T}^*$-closed, but $\mathcal{T}$ is connected, so by Lemma 2.7, $cl_{\mathcal{T}} C = C \cup \{x\}$. Hence, C is non-empty and $\mathcal{T}$-clopen in $X - \{x\}$ and there is a $\mathcal{T}$-neighbourhood N of x such that $N \cap (V \cup \{x\}) \cap C = \emptyset$. □

Theorem 4.2. *Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is maximal connected if and only if $\mathcal{T}$ is connected, submaximal, and for every regular open set V and for all $x \in \partial V$ there is an open set C such that $\partial C = \{x\}$ and there is a regular open neighbourhood N of x such that $N \cap V \cap C = \emptyset$ (that is, $\mathcal{T}$ is submaximal and nearly maximal connected* [6]*).*

Proof. This is immediate by Lemmas 1.1, 4.1 and Corollary 2.5. □

Note. Recall Theorem 3.1 above and observe that Clark and Schneider's term *nearly maximal connected* is a semi-regular property. In addition it is evident that, if Neumann-Lara and Wilson [14, Theorem 2.5] had relaxed their initial hypothesis from T_1 to T_{ES}, they could have obtained a complete characterisation of maximal connectedness.

Definition 4.3. Given $\mathcal{T} \in LT(X)$,

(i) $\mathcal{T}$ is *Quasi-H-Closed* (*feebly compact*) if every (countable) $\mathcal{T}$-open cover of X has a finite subfamily whose union is $\mathcal{T}$-dense in X (or, equivalently, every (countable) open filterbase has a cluster point) [5, 15].

(ii) $p \in X$ is an *almost H-point* (*almost E_1-point*) if there is a (countable) filterbase of non-empty $\mathcal{T}$-regular open sets $\mathcal{W}$ such that $\{p\} = \bigcap\{cl_{\mathcal{T}}W : W \in \mathcal{W}\}$ [15].

(iii) $(X, \mathcal{T})$ is an *almost H-space* (*almost E_1-space*) if every point is an almost H-point (almost E_1-point).

Lemma 4.4. *Suppose $\mathcal{T} \in LT(X)$ is Q.H.C. (feebly compact). If V is a $\mathcal{T}$-regular open and x is non-isolated in the subspace $X - V$, then $V \cup \{x\}$ is* not *singular if and only if x is an almost H-point (almost E_1-point) in the subspace $X - V$.*

Proof. $\mathcal{T}^* = \mathcal{T}(V \cup \{x\})$ is not Q.H.C. if and only if there is a $\mathcal{T}^*$-open filterbase $\mathcal{G} = \{G_i : i \in I\}$ such that $\bigcap\{cl_{\mathcal{T}^*}G_i : i \in I\} = \emptyset$. Now there is some $G \in \mathcal{G}$ such that $x \notin G$, and so for any $i, j \in I$, $G_i \cap G_j \neq \{x\}$ (otherwise, $G \cap G_i \cap G_j = \emptyset$). By Lemma 2.8, for each $i \in I$, $G_i - \{x\} \subseteq int_{\mathcal{T}}G_i \subseteq G_i$, so that $\mathcal{G}' = \{int_{\mathcal{T}}G_i : i \in I\}$ is a filterbase of $\mathcal{T}$-open sets. But $\mathcal{T}$ is Q.H.C. so $\mathcal{G}'$, and therefore $\mathcal{G}$, has a $\mathcal{T}$-cluster point. However, if p is a $\mathcal{T}$-cluster point of $\mathcal{G}$, then there is a set $G_0 \in \mathcal{G}$ such that $p \in (cl_{\mathcal{T}}G_0) - (cl_{\mathcal{T}^*}G_0)$, so by Corollary 2.8 $p = x$, and there is a $\mathcal{T}$-neighbourhood N of x such that $N \cap V \cap G_0 = \emptyset$. Now $G_0 \nsubseteq V$, since $x \in cl_{\mathcal{T}}G_0$, and because V is $\mathcal{T}$-regular open, $G_0 \cap (X - cl_{\mathcal{T}}V) \neq \emptyset$. It follows that, for all $i \in I$,

$$(int_{\mathcal{T}}G_i) \cap (X - cl_{\mathcal{T}}V) \neq \emptyset$$

and that

$$\mathcal{H} = \{(int_{\mathcal{T}}G_i) \cap (X - cl_{\mathcal{T}}V) \neq \emptyset : i \in I\}$$

is a $\mathcal{T}$-open filterbase and that x is the only $\mathcal{T}$-cluster point of $\mathcal{H}$. Furthermore,

$$\{x\} = \bigcap\{cl_{\mathcal{T}}int_{\mathcal{T}}cl_{\mathcal{T}}[G_i \cap (X - cl_{\mathcal{T}}V)] : i \in I\},$$

so that x is an H-point in $X - V$. (Note, by letting $|I| = \aleph_0$ in the above argument we have established that $V \cup \{x\}$ is not singular (w.r.t. feebly compact) if and only if x is an almost E_1-point in $X - V$.) □

We can now prove directly the recent characterisation of a maximal feebly compact topology [15], and provide an alternative to Cameron's characterisation of a maximal Q.H.C. topology [5].

Theorem 4.5. *Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is maximal Q.H.C. (maximal feebly compact) if and only if $\mathcal{T}$ is Q.H.C.* (*feebly compact*)*, submaximal, and every regular closed subspace is an almost H-space* (*almost E_1-space*)*.*

Note. Again recall Theorem 3.1 above and notice that the property *every regular closed subspace is an almost H-space* (*almost E_1-space*) is a semi-regular property [15].

Lemma 4.6. *Suppose $\mathcal{T} \in LT(X)$ is pseudocompact, and that V is $\mathcal{T}$-regular open, while x is non-isolated in the subspace $X - V$. Then $V \cup \{x\}$ is* not *singular if and only if there is an unbounded function $f : X \to \mathbb{R}$ which is continuous on $X - \{x\}$, and for any open interval O_x containing $f(x)$ there is a $\mathcal{T}$-neighbourhood N of x such that $N \cap V \subseteq f^{-1}(O_x)$.*

Proof. $\mathcal{T}^* = \mathcal{T}(V \cup \{x\})$ is not pseudocompact if and only if there is an unbounded function $f : X \to \mathbb{R}$ which is $\mathcal{T}^*$-continuous. Now if G is an open subset of $f(X - \{x\})$, then $f^{-1}(G)$ is a $\mathcal{T}^*$-open subset of $X - \{x\}$. By Corollary 2.8, $f^{-1}(G)$ is $\mathcal{T}$-open so that f is $\mathcal{T}$-continuous on $X - \{x\}$. Since f is $\mathcal{T}^*$-continuous at x, for any interval O_x containing $f(x)$, there is a $\mathcal{T}$-neighbourhood N of x such that $N \cap (V \cup \{x\}) \subseteq f^{-1}(O_x)$. □

Theorem 4.7. *Given $\mathcal{T} \in LT(X)$, $\mathcal{T}$ is maximal pseudocompact if and only if $\mathcal{T}$ is pseudocompact, submaximal and for every $\mathcal{T}$-regular open set V and for every non-isolated point x in the subspace $X - V$ there is an unbounded function $f : X \to \mathbb{R}$ which is continuous on $X - \{x\}$, and for any open interval O_x containing $f(x)$, there is a $\mathcal{T}$-neighbourhood N of x such that $N \cap V \subseteq f^{-1}(O_x)$.*

Note. This is a complete characterisation of maximal pseudocompact topologies which makes no reference to expansions, and so answers Cameron's question [3]. However, see also [16, Theorem 2.4] and [9, Corollary 11A].

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