

## INFINITE HOMOGENEOUS ALGEBRAS ARE ANTICOMMUTATIVE

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(Communicated by Lance W. Small)

ABSTRACT. A (non-associative) algebra  $A$ , over a field  $k$ , is called homogeneous if its automorphism group permutes transitively the one dimensional subspaces of  $A$ . Suppose  $A$  is a nontrivial finite dimensional homogeneous algebra over an infinite field. Then we prove that  $x^2 = 0$  for all  $x$  in  $A$ , and so  $xy = -yx$  for all  $x, y \in A$ .

### INTRODUCTION

The algebras to be discussed are assumed to be finite dimensional over a field  $k$  and not necessarily associative. Thus the multiplication  $A \times A \rightarrow A$  can be an arbitrary bilinear map. Such an algebra  $A$  is *nontrivial* if  $\dim A > 1$  and  $A^2 \neq 0$ . Also,  $\text{Aut}(A)$  denotes the group of algebra automorphisms of  $A$ .

An algebra  $A$  is said to be *extremely homogeneous* if  $\text{Aut}(A)$  acts transitively on  $A \setminus \{0\}$ . The concept of an extremely homogeneous algebra arose from a particular problem in the structure of certain finite  $p$ -groups as studied by Boen, Rothaus, and Thompson [2]. Extremely homogeneous algebras were also investigated by Kostrikin [6].

An algebra  $A$  is *homogeneous* if  $\text{Aut}(A)$  permutes transitively the one dimensional subspaces of  $A$ . Homogeneous algebras over finite fields were studied by Shult [11] and Gross [4]. The classification of homogeneous algebras over finite fields was completed by Ivanov [5]. He showed that if  $A$  is a nontrivial homogeneous algebra over a finite field  $k$ , then  $k = \text{GF}(2)$  and  $A$  is of a type previously described by Kostrikin.

The condition of homogeneity is very strong indeed. For instance, if  $k$  is algebraically closed, then the second author [10] has shown that there are no nontrivial homogeneous algebras. When  $k$  is the real field, the first author [3] has classified all homogeneous algebras. In that case there are up to isomorphism exactly three nontrivial homogeneous algebras. Also, MacDougall and Sweet [7], [8], [9] have found all homogeneous algebras of dimension 2, 3, or 4 with no restrictions on the ground field.

If  $a \in A$ , we denote by  $\langle a \rangle$  the subalgebra of  $A$  generated by  $a$ . If  $A$  is homogeneous, then  $\langle a \rangle$  is also homogeneous and it is easy to show that  $\langle a \rangle = \langle x \rangle$  for any

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Received by the editors January 7, 1998 and, in revised form, February 6, 1998.

1991 *Mathematics Subject Classification*. Primary 17D99; Secondary 17A36, 15A69.

*Key words and phrases*. Non-associative algebras, automorphism group, hypersurface.

This work was supported in part by the NSERC Grant A-5285.

$x \in \langle a \rangle \setminus \{0\}$ . Indeed if  $x \in \langle a \rangle \setminus \{0\}$ , there is some  $\varphi \in \text{Aut}(A)$  taking  $k \cdot a$  to  $k \cdot x$ , hence carrying  $\langle a \rangle$  to  $\langle x \rangle$ , so  $\varphi$  yields an automorphism of  $A$ . It follows that if  $A$  is homogeneous and  $a \in A$ , then  $\langle a \rangle$  is an algebra with no proper subalgebras. Algebras with no proper subalgebras were investigated by Artamonov [1]. In particular he showed that if  $A$  is a nontrivial algebra with no proper subalgebras and  $\dim A$  is a prime, then  $\text{Aut}(A)$  is a finite group.

The purpose of this paper is to show that if  $A$  is a homogeneous algebra over an infinite field, then  $x^2 = 0$  for all  $x \in A$ . This of course implies that  $A$  is anticommutative. It also implies that if  $A$  is a nontrivial homogeneous algebra with no proper subalgebras, then the underlying field  $k$  must be finite (and so these algebras are known, they must be of Kostrikin type).

In the remainder of the paper we use the following notation:  $A$  is a homogeneous algebra over an infinite field  $k$ ,  $K$  is an algebraic closure of  $k$ ,  $A_K = K \otimes_k A$  is the  $K$ -algebra obtained from  $A$  by extension of scalars, and  $G_k$  (resp.  $G_K$ ) is the automorphism group of the algebra  $A$  (resp.  $A_K$ ). For  $x \in A_K$ ,  $L_x : A_K \rightarrow A_K$  denotes left multiplication by  $x$  (and similarly  $R_x$  denotes right multiplication by  $x$ ). If  $A$  is an algebra, then  $A^+$  is the commutative algebra obtained from  $A$  by introducing the new multiplication  $x \cdot y = xy + yx$  (note that  $\text{Aut}(A)$  is a subgroup of  $\text{Aut}(A^+)$  and so  $A$  homogeneous implies  $A^+$  homogeneous). By  $A^{\text{opp}}$  we denote the opposite algebra of  $A$  in which  $x \cdot y = yx$ . As usual we set  $k^* = k \setminus \{0\}$  and  $K^* = K \setminus \{0\}$ .

We thank the referee for his suggestions.

## RESULTS AND PROOFS

The following three lemmas are needed for the proof of the main theorem.

**Lemma 1.** *Let  $A$  be a nontrivial homogeneous algebra over an infinite field  $k$ . If  $a^2 \in k \cdot a$ ,  $a \in A$ , then  $a^2 = 0$ .*

*Proof.* This result follows directly from Theorem 7 of [8]. □

**Lemma 2.** *Let  $A = \langle a_0 \rangle$  be a nontrivial homogeneous algebra over an infinite field  $k$ . If  $a \in A \setminus \{0\}$ , then the stabilizer  $H \subset G_K$  of the 1-dimensional subspace  $K \cdot a$  is trivial.*

*Proof.* Assume that  $H$  contains an element  $h \neq 1$ . Then  $h(a) = \lambda a$  for some  $\lambda \in K^*$ ,  $\lambda \neq 1$ . We define  $a^{2^r}$ ,  $r \geq 0$ , inductively by  $a^1 = a$ ,  $a^{2^{r+1}} = (a^{2^r})^2$ . Since  $A = \langle a_0 \rangle$  has dimension  $> 1$ , we must have  $a_0^2 \neq 0$ , hence by homogeneity all nonzero elements of  $A$  have nonzero square, hence  $a^{2^r} \neq 0$  for all  $r \geq 0$  and  $h(a^{2^r}) = \lambda^{2^r} a^{2^r}$ . As  $h$  has only finitely many eigenvalues,  $\lambda$  is a root of unity, say,  $\lambda^d = 1$ ,  $d > 1$ .

As  $A = \langle a \rangle$ ,  $A$  has a basis  $\{b_1, b_2, \dots, b_n\}$  where each  $b_i$  is a product of  $n_i$   $a$ 's under some bracketing. Let  $m$  be a positive integer such that  $t = dm > n_i$  for all  $i$ . Then  $h(a^t) = \lambda^t a^t = a^t$ , where  $a^t$  is any product of  $t$   $a$ 's. If  $a^t \neq 0$  for some bracketing, then  $A = \langle a^t \rangle$  which implies that  $h = 1$ , a contradiction. Hence we have  $a^t = 0$  for all bracketing. But then  $L_a^{t-n_i}(b_i) = 0$ , and so  $L_a^t(b_i) = 0$  for all  $i$ . Thus  $L_a^t = 0$ , and similarly  $R_a^t = 0$ . Hence  $A$  is a special nil algebra as defined in [10]. Since  $k$  is infinite, it follows from Theorem 2 of the above paper that  $A^2 = 0$ , which is a contradiction. □

**Lemma 3.** *Let  $A = \langle a_0 \rangle$  be a nontrivial commutative homogeneous algebra over an infinite field  $k$  where  $\text{char } k \neq 2$ . Then  $A$  has no zero divisors.*

*Proof.* Suppose  $ab = 0$  for some  $a, b \in A \setminus \{0\}$ . Since  $A = \langle a \rangle$  and  $\dim A > 1$ , we have  $a^2 \neq 0$  and so  $a$  and  $b$  are linearly independent. Hence there exist  $g_1, g_2 \in G_k$  such that

$$(1) \quad g_1(a) = \lambda(a + b), \quad g_2(a) = \mu(a - b), \quad \lambda, \mu \in k^*.$$

But then, using  $ab = ba = 0$ ,

$$g_1(a^2) = \lambda^2(a^2 + b^2), \quad g_2(a^2) = \mu^2(a^2 + b^2)$$

and so  $g_2^{-1}g_1$  is in the stabilizer of  $k \cdot a^2$ . Lemma 2 implies that  $g_1 = g_2$  and (1) gives a contradiction.  $\square$

We can now prove our main result.

**Theorem.** *Let  $A$  be a nontrivial homogeneous algebra over an infinite field  $k$ . Then  $x^2 = 0$  for all  $x \in A$ .*

*Proof.* Assume that  $x^2 \neq 0$  for some  $x \in A$ . Since  $A$  is homogeneous, this implies that  $x^2 \neq 0$  for all nonzero  $x \in A$ . Our goal is to obtain a contradiction. If  $\text{char } k \neq 2$ , by replacing  $A$  with  $A^+$ , we may assume that  $A$  is commutative.

Let us fix a nonzero element  $b \in A$ . By Lemma 1,  $b^2 \notin k \cdot b$ . By replacing  $A$  with the subalgebra generated by  $b$ , we may assume that  $A = \langle b \rangle$ . Suppose  $L_b$  is nilpotent. If  $\text{char } k \neq 2$ , then  $R_b = L_b$  is also nilpotent and so  $A$  is a special nilalgebra. As noted in the proof of Lemma 2, this is impossible. If  $\text{char } k = 2$ , we obtain the same contradiction if  $L_b$  and  $R_b$  are both nilpotent. So, if necessary, by replacing  $A$  by  $A^{\text{pp}}$ , we can assume that  $L_b$  is not nilpotent.

Let us fix a nonzero eigenvalue  $\mu \in K$  of  $L_b$  and set  $a := \mu^{-1}b \in A_K$ . Let  $P : A_K \rightarrow K$  be the polynomial function defined by

$$P(x) := \det(1 - L_x)$$

and let  $P = P_1P_2 \cdots P_m$  be its prime factorization. Since 1 is an eigenvalue of  $L_a$ , we have  $P(a) = 0$ . We may assume that the factors are numbered so that  $P_1(a) = 0$  and normalized so that  $P_i(0) = 1$  for all  $i$ . Let  $H$  be the hypersurface in  $A_K$  defined by  $P(x) = 0$  and let  $H_1$  be its irreducible component defined by  $P_1(x) = 0$ .

For  $g \in G_K$  and  $x \in A_K$  we have  $g \circ L_x \circ g^{-1} = L_{g(x)}$  and consequently  $P$  is  $G_K$  invariant. Hence the orbit  $\mathcal{O} := G_K \cdot a$  is contained in  $H$ .

Let

$$\pi : A_K \setminus \{0\} \rightarrow \mathbf{P}(A_K)$$

be the canonical projection onto the projective space  $\mathbf{P}(A_K)$ . The image of  $A \setminus \{0\}$  under  $\pi$  will be identified with the projective space  $\mathbf{P}(A)$  whose points are the 1-dimensional  $k$ -subspaces of  $A$ . We have

$$\mathbf{P}(A) = \pi(G_k \cdot b) \subset \pi(\mathcal{O}) \subset \mathbf{P}(A_K).$$

Since  $k$  is infinite,  $\mathbf{P}(A)$  is Zariski dense in  $\mathbf{P}(A_K)$ . Hence  $\dim \mathcal{O} \geq n - 1$ , and since  $\mathcal{O} \subset H$ , we infer that  $\dim \mathcal{O} = n - 1$ .

We claim that the affine algebraic group  $G_K$  is connected (in the Zariski topology). Indeed let  $G_K^0$  be the identity component of  $G_K$  and let  $g \in G_K$  be arbitrary.

Both  $\pi(G_K^0 \cdot a)$  and  $\pi(gG_K^0 \cdot a)$  are nonempty open subsets of  $\mathbf{P}(A_K)$ . Hence they have nontrivial intersection, i.e., there exist  $g_1, g_2 \in G_K^0$  such that  $gg_1 \cdot Ka = g_2 \cdot Ka$ . By Lemma 2 we conclude that  $gg_1 = g_2$  and so  $g = g_2g_1^{-1} \in G_K^0$  and our claim is verified.

It follows that each factor  $P_i$  of the  $G_K$ -invariant polynomial  $P$  is  $G_K$ -invariant. Hence  $H_1$  is  $G_K$ -invariant and so  $\mathcal{O} \subset H_1$  and  $\mathcal{O}$  is open and dense in  $H_1$ . Let  $d$  be the degree of  $P_1$  and write  $P_1 = P_{10} + P_{11} + \dots + P_{1d}$  where  $P_{1i}$  is a homogeneous polynomial of degree  $i$ . We set  $f = -P_{1d}$ .

We claim that  $d > 1$ . Assume that  $d = 1$ . Then  $H_1$  is the hyperplane  $f(x) = 1$ ; hence  $f(a) = 1$  and  $f(b) = \mu$ . If  $g \in G_k$  and  $g \neq 1$ , then also  $f(g \cdot b) = \mu$ . Thus  $c = b - g \cdot b$  is a nonzero vector of  $A$  such that  $f(c) = 0$  contradicting the homogeneity. Hence our claim is proved.

Since  $k$  is infinite and  $f \neq 0$ ,  $f$  cannot vanish identically on  $\mathbf{P}(A)$ . As  $\pi(G_k \cdot b) = \mathbf{P}(A)$ , it follows that  $f(b) \neq 0$ , and so  $f(a) \neq 0$ . We conclude that the polynomial  $P_1(ta)$  in the variable  $t$  must also have degree  $d$ . But  $P_1(a) = 0$  and so  $t = 1$  is a root of the polynomial  $P_1(ta)$ . Let  $\lambda \in K$  be an arbitrary root of that polynomial. Then  $\lambda \neq 0$  and  $\lambda\mathcal{O}$  and  $\mathcal{O}$  are both open dense subsets of  $H_1$ . Hence  $\lambda\mathcal{O} \cap \mathcal{O} \neq \emptyset$ , and consequently  $\lambda\mathcal{O} = \mathcal{O}$ . Now Lemma 2 implies that  $\lambda = 1$ . We conclude that  $t = 1$  is the only root of the polynomial  $P_1(ta)$  and, by using  $P_1(0) = 1$ , we infer that

$$P_1(ta) = (1 - t)^d.$$

Since  $P_1$  is  $G_K$ -invariant and  $\mathcal{O}$  is dense in  $H_1$ , we have

$$P_1(tx) = (1 - t)^d, \quad \forall x \in H_1.$$

This polynomial identity (in the variable  $t$ ) implies that each of the polynomials

$$P_{1i}(x) - (-1)^i \binom{d}{i}, \quad 0 \leq i \leq d,$$

vanishes on  $H_1$ . Since  $P_1$  is irreducible,  $P_1$  must divide each of these polynomials. It follows that  $P_{1i} = 0$  for  $0 < i < d$ ,  $\text{char } k = p > 0$ , and

$$\binom{d}{i} \equiv 0 \pmod{p}, \quad 0 < i < d.$$

The last condition implies that  $d = p^s$  for some integer  $s \geq 1$ . Since  $P_{10} = P_1(0) = 1$ , we have  $P_1 = 1 - f$  where  $f = -P_{1d}$  is homogeneous of degree  $d$ .

For  $x \in A_K$  and  $1 \leq i \leq p$ , we define the powers  $x^i$  recursively by  $x^1 = x$ , and  $x^i = x \cdot x^{i-1}$  for  $1 < i \leq p$ . Next we define the powers  $x^{p^i}$  for  $i > 1$  by

$$x^{p^i} = (x^{p^{i-1}})^p.$$

It follows from these definitions that for all  $x \in A_K$  and  $g \in G_K$  we have

$$g(x^{p^i}) = g(x)^{p^i}.$$

We claim that  $x^{p^i} \neq 0$  for nonzero  $x \in A$  and  $i \geq 0$ . If  $p > 2$ , this follows from Lemma 3, and for  $p = 2$  from our assumption that  $A = \langle b \rangle$ .

Let  $N$  be the smallest positive integer such that

$$a, a^p, a^{p^2}, \dots, a^{p^N}$$

are  $K$ -linearly dependent. Then

$$(2) \quad \sum_{i=0}^N \alpha_i a^{p^i} = 0$$

with  $\alpha_i \in K$  and  $\alpha_N = 1$ .

Assume that  $\alpha_0 = 0$ . As  $b = \mu a \in A \setminus \{0\}$ , there exists  $g \in G_k$  such that  $g(a^p) = \lambda a$  for some  $\lambda \in K^*$ . Equation (2) implies that

$$\sum_{i=1}^N \alpha_i g(a^p)^{p^{i-1}} = 0.$$

But then

$$\sum_{i=0}^{N-1} \alpha_{i+1} \lambda^{p^i} a^{p^i} = 0$$

and this contradicts our choice of  $N$ . So we have shown that  $\alpha_0 \neq 0$ .

As  $\mathcal{O}$  is dense in  $H_1$ , it follows from (2) that

$$\sum_{i=0}^N \alpha_i x^{p^i} = 0, \quad \forall x \in H_1.$$

Hence each coordinate function of the polynomial map

$$x \rightarrow \sum_{i=0}^N \alpha_i x^{p^i}$$

must be divisible by  $P_1 = 1 - f$ , and so

$$\sum_{i=0}^N \alpha_i x^{p^i} = (1 - f(x)) \sum_{i=1}^{p^N - d} Q_i(x)$$

where  $Q_i$  is a homogeneous polynomial map of degree  $i$ . By extracting the homogeneous components of degrees congruent to  $1 \pmod{p}$ , we obtain that

$$\alpha_0 x = (1 - f(x)) \sum_{i \equiv 1 \pmod{p}} Q_i(x).$$

It follows that  $Q_1(x) = \alpha_0 x \neq 0$ . Hence the right hand side has degree  $\geq d + 1$ , and we have a contradiction.  $\square$

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