PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 11, Pages 3339–3344
S 0002-9939(99)04912-6
Article electronically published on May 6, 1999

ON RANDOM ALGEBRAIC POLYNOMIALS

K. FARAHMAND

(Communicated by Stanley Sawyer)

ABSTRACT. This paper provides asymptotic estimates for the expected number of real zeros and K-level crossings of a random algebraic polynomial of the form $a_0\binom{n-1}{0}^{1/2} + a_1\binom{n-1}{1}^{1/2}x + a_2\binom{n-1}{2}^{1/2}x^2 + \cdots + a_{n-1}\binom{n-1}{n-1}^{1/2}x^{n-1}$, where $a_j (j = 0, 1, \ldots, n-1)$ are independent standard normal random variables and K is a constant independent of x. It is shown that these asymptotic estimates are much greater than those for algebraic polynomials of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}$.

1. Introduction

Let $(\Omega, \mathcal{A}, \Pr)$ be a fixed probability space and let $\{a_j(\omega)\}_{j=0}^{n-1}$ be a sequence of independent random variables defined on Ω. The random algebraic polynomial was introduced in the pioneer work of Littlewood and Offord [9] and [10] as

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega)x^j,$$

and since then has been greatly studied. Denote by $N_K(\alpha, \beta)$ the number of real roots of the equation $P(x) = K$ in the interval (α, β) and by $EN_K(\alpha, \beta)$ its expected value. In particular it is shown (for example see Kac [8] or Wilkins [12]) that if the coefficients are assumed to have a standard normal distribution and n is sufficiently large, $EN_0(-\infty, \infty) \sim (2/\pi)\log n$. Recently (see Farahmand [4]), it was shown that this asymptotic value remains valid for $EN_K(-\infty, \infty)$ as long as K is bounded. For K large such that $K^2/n \longrightarrow 0$ as $n \longrightarrow \infty$, $EN_K(-\infty, \infty)$ is asymptotically reduced to $(1/\pi)\log(n/K^2)$ in $(-1, 1)$ while it remains the same as for $K = 0$ in $(-\infty, -1) \cup (1, \infty)$ ([4]). In contrast, a random trigonometric polynomial

$$T(x) \equiv T_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega)\cos j\theta$$

has more roots in $(0, 2\pi)$. In fact $EN_K(0, 2\pi) \sim 2n/\sqrt{3}$ for $K = o(\sqrt{n})$; see [5] and [6].

Motivated by the interesting results obtained in Littlewood and Offord [10] we considered the case when the coefficients $a_j(\omega)$ have variance $1/j!$. It is presumably

Received by the editors July 9, 1997 and, in revised form, December 17, 1997 and February 5, 1998.

1991 *Mathematics Subject Classification.* Primary 60H99; Secondary 42Bxx.

Key words and phrases. Number of real roots, real zeros, random algebraic polynomials, random trigonometric polynomials, Kac-Rice formula.

the case, possibly under some mild conditions for K and for n sufficiently large, that $EN_K(-\infty,\infty)$ is $o(\sqrt{n})$. This author, however, was unable to make any substantial progress towards this conjecture. Instead in this paper we study the polynomials

$$P(x) \equiv P_n(x,\omega) = \sum_{j=0}^{n-1} a_j(\omega) \binom{n-1}{j}^{1/2} x^j.$$

This is, indeed, the same as saying that the jth coefficient of $Q(x)$ has variance $\binom{n-1}{j}$. Besides the mathematical interest, as reported in Edelman and Kostlan [3, page 11], these polynomials have some relationship with physics [1]. We prove the following theorems. Theorem 1 was known to Edelman and Kostlan, however to be complete we give a proof here. Theorem 3, and its comparison with Theorem 1, shows that for $P(x)$ there are as many extrema as the number of zero crossings. Therefore, unlike $Q(x)$, all the oscillations of $P(x)$ are between two zero crossings, asymptotically in expectation.

Theorem 1. *If the coefficients a_j of $P(x)$ are independent standard normal random variables, then*

$$EN_0(-\infty,\infty) = \sqrt{n-1}.$$

Theorem 2. *Denote $\Upsilon(n) = (2n-1)!!/(2n)!!$ and assume $K^2\Upsilon(n) \longrightarrow 0$. With the same assumptions as in Theorem 1 for the coefficients of $P(x)$, we have*

$$EN_K(-\infty,\infty) \sim \sqrt{n-1}.$$

Theorem 3. *Denote $M(-\infty,\infty)$ the number of extrema of $P(x)$ in $(-\infty,\infty)$. Then the expected value of $M(-\infty,\infty)$ satisfies*

$$EM(-\infty,\infty) \sim \sqrt{n-1}.$$

2. A formula for the expected number of real roots

Let

$$A^2 = \operatorname{var}\{P(x) - K\}, \tag{2.1}$$

$$B^2 = \operatorname{var}\{P'(x)\}, \tag{2.2}$$

$$C = \operatorname{cov}[\{P(x) - K\}, P'(x)] \tag{2.3}$$

and

$$\eta = -CK/A\sqrt{A^2B^2 - C^2}.$$

Then by using the expected number of level crossings given by Cramer and Leadbetter [2, page 285] for our equation $P(x) - K = 0$, we can obtain

$$EN_K(\alpha,\beta) = \int_\alpha^\beta \frac{B\sqrt{1 - C^2/A^2B^2}}{A} \phi\left(-\frac{K}{A}\right) [2\phi(\eta) + \eta\{2\Phi(\eta) - 1\}]\ dx,$$

where as usual

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^t \exp(-y^2/2)\ dy$$

and

$$\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2).$$

Let $\Delta^2 = A^2B^2 - C^2$ and $\mathrm{erf}(x) = \int_0^x \exp(-t^2)\,dt$; then we can write the extension of a formula obtained by Rice [11] for the case of $K = 0$ as

$$EN_K(\alpha,\beta) = I_1(\alpha,\beta) + I_2(\alpha,\beta), \tag{2.4}$$

where

$$I_1(\alpha,\beta) = \int_\alpha^\beta \frac{\Delta}{\pi A^2}\exp\left(-\frac{B^2K^2}{2\Delta^2}\right)\,dx \tag{2.5}$$

and

$$I_2(\alpha,\beta) = \int_\alpha^\beta \frac{\sqrt{2}KC}{\pi A^3}\exp\left(-\frac{K^2}{2A^2}\right)\mathrm{erf}\left(\frac{KC}{\sqrt{2}A\Delta}\right)\,dx. \tag{2.6}$$

3. Proof of Theorem 1

We need the following, where (3.2) and (3.3) are obtained by differentiation of (3.1),

$$A^2 = \sum_{j=0}^{n-1}\binom{n-1}{j}x^{2j} = (x^2+1)^{n-1}, \tag{3.1}$$

$$B^2 = \sum_{j=0}^{n-1} j^2\binom{n-1}{j}x^{2j-2} = (n-1)(x^2+1)^{n-3}(nx^2-x^2+1), \tag{3.2}$$

and

$$C = \sum_{j=0}^{n-1} j\binom{n-1}{j}x^{2j-1} = (n-1)x(x^2+1)^{n-2}. \tag{3.3}$$

Therefore, since from (3.1)-(3.3)

$$\Delta^2 = A^2B^2 - C^2 = (n-1)(x^2+1)^{2n-4}, \tag{3.4}$$

from (2.4)-(2.6) we obtain

$$\begin{aligned} EN_0(0,\infty) &= \pi^{-1}\int_0^\infty \frac{\Delta}{A^2}\,dx \\ &= \frac{\sqrt{n-1}}{\pi}\int_0^\infty \frac{dx}{x^2+1} = \frac{\sqrt{n-1}}{2}. \end{aligned} \tag{3.5}$$

This gives the proof of Theorem 1. It is, indeed, interesting to note that from (3.5)

$$\frac{\sqrt{n-1}}{\pi(x^2+1)}$$

is the density function of the number of real zeros of $P(x)$; see also [3, page 12]. Obtaining a closed form of the above density function is uncommon. An asymptotic result, for most cases, is the best that can be achieved.

4. Proof of Theorem 2

Because $|\mathrm{erf}\,u| < \sqrt{\pi}/2$, it follows that from (2.6), (3.1), and (3.3) that

$$\begin{aligned} 0 \le I_2(-\infty,\infty) &\le \frac{|K|}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{C}{A^3}\exp\left(-\frac{K^2}{2A^2}\right)\,dx \\ &= \frac{2}{\sqrt{\pi}}\int_0^{|K|/\sqrt{2}} e^{-\nu^2}\,d\nu, \end{aligned}$$

if $\nu = |K|/(A\sqrt{2})$. Therefore, because erf $u \le u$ when $u \ge 0$,

$$0 \le I_2(-\infty,\infty) \le \frac{2}{\sqrt{\pi}}\operatorname{erf}\left(\frac{K}{\sqrt{2}}\right) \le \min\left\{1, \frac{\sqrt{2}|K|}{\sqrt{\pi}}\right\}. \tag{4.1}$$

Moreover, it follows from (2.5), (3.1), (3.2) and (3.4) that

$$I_1(-\infty,\infty) = \frac{\sqrt{n-1}}{\pi}\int_{-\infty}^{\infty} \frac{e^{-s}}{x^2+1}\, dx,$$

in which $s = K^2(nx^2 - x^2 + 1)/\left\{2(x^2+1)^{n-1}\right\}$. If $x = \tan\theta$, we find that

$$I_1(-\infty,\infty) = \frac{2\sqrt{n-1}}{\pi}\int_0^{\pi/2} e^{-s}\, d\theta,$$

in which $s = (K^2/2)\left\{(n-1)\sin^2\theta + \cos^2\theta\right\}\cos^{2n-4}\theta$. Therefore,

$$I_1(-\infty,\infty) = \frac{2\sqrt{n-1}}{\pi}\int_0^{\pi/2} \left\{1 - (1-e^{-s})\right\}\, d\theta = \sqrt{n-1} - R, \tag{4.2}$$

in which from [7, page 369],

$$\begin{aligned} 0 \le R &= \frac{2\sqrt{n-1}}{\pi}\int_0^{\pi/2}(1-e^{-s})\, d\theta \le \frac{2\sqrt{n-1}}{\pi}\int_0^{\pi/2} s\, d\theta = K^2\beta_n, \\ \beta_n &= \frac{\sqrt{n-1}(3n-4)(2n-5)!!}{(2n-2)(2n-4)!!} = \sqrt{n-1}\Upsilon(n-2)\frac{3n-4}{2n-2}. \end{aligned}$$

A straight-forward algebraic calculation shows that $\beta_{n+1} < \beta_n$ when $n \ge 2$. We conclude that $0 \le R \le K^2\beta_2 = K^2/2$ and then that $R = o(\sqrt{n})$ because $K^2 = o\{\Upsilon^{-1}(n)\}$ and $\Upsilon(n) \sim \sqrt{n\pi}$. When this last result is combined with (4.1), (4.2) and (2.4), it is clear that Theorem 2 is true. In fact, we have actually proved the better result that

$$-\frac{K^2}{2} \le EN_k(-\infty,\infty) - \sqrt{n-1} \le \min\left\{1, \frac{\sqrt{2}|K|}{\sqrt{\pi}}\right\},$$

from which we can also infer not only Theorem 2 but also that

$$EN_K(-\infty,\infty) = \sqrt{n-1} + O(1)$$

when K is bounded.

5. Extrema

The expected number of extrema of $P(x)$, denoted by $EM(-\infty,\infty)$, is simply the expected number of real zeros of $P'(x) = \sum_{j=1}^{n-1} a_j j \binom{n-1}{j}^{1/2} x^{j-1}$. Therefore we apply $EN_0(-\infty,\infty)$ for $P'(x)$. To this end, by successive differentiation of (2.2) we obtain

$$A^2 = \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2}$$

$$(5.1) \qquad = (n-1)(x^2+1)^{n-3}(nx^2 - x^2 + 1),$$

$$C = \sum_{j=1}^{n-1} j^2 (j-1) \binom{n-1}{j} x^{2j-3}$$

$$(5.2) \qquad = (n-1)(n-2)x(x^2+1)^{n-4}\left\{(n-1)x^2 + 2\right\}$$

and

$$B^2 = \sum_{j=1}^{n-1} j^2 (j-1)^2 \binom{n-1}{j} x^{2j-4}$$

$$(5.3) \qquad = (n-1)(n-2)(x^2+1)^{n-5}\left\{(n-1)(n-2)x^4 + 4(n-2)x^2 + 2\right\}.$$

Therefore, from (5.1)-(5.3) we obtain

$$(5.4) \qquad \frac{\Delta}{A^2} = \frac{\sqrt{(n-2)\left\{(n-1)(n-2)x^4 + 2(n-1)x^2 + 2\right\}}}{(x^2+1)\left\{(n-1)x^2+1\right\}}.$$

From (5.4) it then follows that, for all non-zero x

$$\lim_{n\to\infty} \frac{\Delta}{A^2\sqrt{n-2}} = \frac{1}{x^2+1}.$$

Then since $\Delta/A^2\sqrt{n-2} \leq \sqrt{2}/(x^2+1)$ for all real x, the dominated convergent theorem for Lebesgue integrals shows that

$$\int_{-\infty}^{\infty} \frac{\Delta}{A^2\sqrt{n-2}}\,dx = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi.$$

Theorem 3 is then an immediate consequence of this result.

Acknowledgment

The author wishes to thank the referee and the probability editor for their comments on the earlier version of this paper.

References

[1] E. Bogomolny, O. Bohigas, and P. Leboeuf. Distribution of roots of random polynomials. *Phys. Rev. Lett.*, 68:2726–2729, 1992. MR **92m:**81054

[2] H. Cramér and M.R. Leadbetter. *Stationary and Related Stochastic Processes.* Wiley, N.Y., 1967. MR **36:**949

[3] A. Edelman and E. Kostlan. How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc.*, 32:1–37, 1995. MR **95m:**60082

[4] K. Farahmand. On the average number of real roots of a random algebraic equation. *Ann. Probab.*, 14:702–709, 1986. MR **87k:**60140

[5] K. Farahmand. On the average number of level crossings of a random trigonometric polynomial. *Ann. Probab.*, 18:1403–1409, 1990. MR **91i:**60140

[6] K. Farahmand. Level crossings of a random trigonometric polynomial. *Proc. Amer. Math. Soc.*, 111:551–557, 1991. MR **91f:**60092

[7] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series and Products.* Academic Press, London, 1980, xv+1160 pp. MR **81g:**33001

[8] M. Kac. On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.*, 49:314–320, 1943. MR 4,196d

[9] J.E. Littlewood and A.C. Offord. On the number of real roots of a random algebraic equation. *J. London Math. Soc.*, 13:288–295, 1938.

[10] J.E. Littlewood and A.C. Offord. On the number of real roots of a random algebraic equation II. *Proc. Camb. Phil. Soc.*, 35:133–148, 1939.

[11] S.O. Rice. Mathematical theory of random noise. *Bell. System Tech. J.*, 25:46–156, 1945. Reprinted in: Selected Papers on Noise And Stochastic Processes (ed. N. Wax), Dover, New York, 1954, 133-294.

[12] J.E. Wilkins. An asymptotic expansion for the expected number of real zeros of a random polynomial. *Proc. Amer. Math. Soc.*, 103:1249–1258, 1988. MR **90f:**60105

Department of Mathematics, University of Ulster, Jordanstown, Co. Antrim BT37 0QB, United Kingdom

E-mail address: `k.farahmand@ulst.ac.uk`