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VANISHING OF A CERTAIN KIND OF VASSILIEV INVARIANTS OF 2-KNOTS

SEIICHI KAMADA

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Abstract. In knot theory, Vassiliev's 1-knot invariants are defined in a combinatorial way as finite type invariants. By a natural generalization of the combinatorial definition, one has a certain family of 2-knot invariants, which should be called finite type 2-knot invariants. They form a subspace of the whole space of "Vassiliev 2-knot invariants". In this paper we prove that it is 1-dimensional.

By a Σ-*immersion*, we mean an immersion $k : \Sigma \to R^4$ of an oriented, closed and connected 2-manifold Σ into 4-space. If there exist orientation-preserving diffeomorphisms of Σ and R^4 to themselves which make the following diagram commutative, then we say that k and k' are *equivalent*:

$$\begin{tikzcd}
\Sigma \arrow[r, "k"] \arrow[d] & R^4 \arrow[d] \\
\Sigma \arrow[r, "k'"] & R^4.
\end{tikzcd}$$

A Σ-immersion is *generic* if the singularities are transverse double points. In this case, the normal Euler number $e(k)$ of k is equal to $2(d_+(k) - d_-(k))$, where $d_+(k)$ is the number of positive double points of k and $d_-(k)$ is that of negative ones.

By the Smale and Hirsch theorem [10], [11], [4], two Σ-immersions k_0 and k_1 are related by a regular homotopy if and only if $e(k_0) = e(k_1)$. Moreover, if k_0 and k_1 are generic, then by taking a regular homotopy generically, we have a generic one-parameter family of Σ-immersions k_t ($t \in [0,1]$) being generic for all but a finite number of values t such that at each exceptional t a finger move or a Whitney trick occurs. (There exists an explicit procedure to construct such a one-parameter family, cf. [6].) A finger move (cf. [3], [7], [8], [9]) is a regular homotopy between Σ-immersions yielding a pair of positive and negative transverse double points, whose local image is illustrated in Figure 1 in the motion picture method. The inverse operation is a Whitney trick in dimension four. It is a difficult problem to determine whether or not a given pair of double points is a Whitney pair (cf. [3], [7], [9]).

Let $\mathcal{K}(\Sigma)_{(n)}$ be the set of equivalence classes of Σ-immersions whose singularities are (transverse or non-transverse) double points and the number of non-transverse

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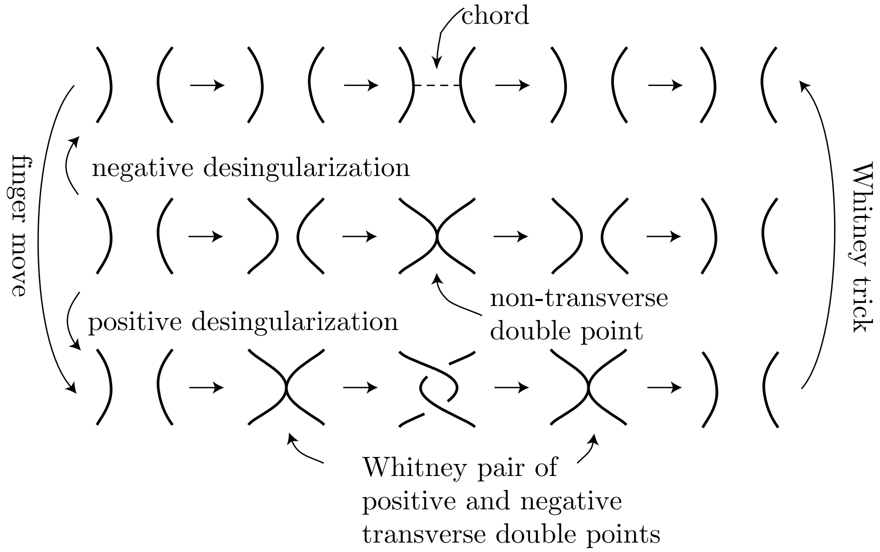

FIGURE 1

ones is n. In particular, $\mathcal{K}(\Sigma)_{(0)}$ (denoted simply by $\mathcal{K}(\Sigma)$) is the set of equivalence classes of generic Σ-immersions. If a Σ-immersion k has a non-transverse double point, then there exists a finger move (regular homotopy) k_t ($t \in [0,1]$) with $k_{0.5} = k$ as a bifurcation. We call the restricted homotopy k_t ($t \in [0.5,1]$) a *positive desingularization*, and the inverse of the restricted homotopy k_t ($t \in [0,0.5]$) a *negative desingularization*. The equivalence classes of k_0 and k_1 are uniquely determined from that of k.

Let $v : \mathcal{K}(\Sigma) \to \mathrm{F}$ be a mapping of $\mathcal{K}(\Sigma)$ to a field F (which may be a commutative ring with unit or an abelian group). For a Σ-immersion $k_\times$ whose class $[k_\times]$ belongs to $\mathcal{K}(\Sigma)_{(1)}$, define $v([k_\times])$ by $v([k_+]) - v([k_-])$, where k_+ and k_- are Σ-immersions obtained from $k_\times$ by positive and negative desingularizations respectively. Inductively, the mapping $v : \mathcal{K}(\Sigma) \to \mathrm{F}$ is extended to $v : \amalg_{n=0}^{\infty} \mathcal{K}(\Sigma)_{(n)} \to \mathrm{F}$. (It seems reasonable that such a mapping is called Vassiliev's invariant of Σ-immersions, cf. [1]. The family of such mappings forms a linear space over F.) If there exists an integer n_0 such that v vanishes over all $\mathcal{K}(\Sigma)_{(n)}$ with $n \geq n_0$, we say that v is of *finite type*.

Theorem 1. *Let* $v : \mathcal{K}(\Sigma) \to \mathrm{F}$ *be a mapping of finite type. Then* $v([k]) = v([k'])$ *for any generic* Σ*-immersions* k *and* k' *with* $e(k) = e(k')$ *and* $d(k) = d(k')$.

Here $d(k)$ stands for the number of (transverse) double points of k. The assumption of the theorem is equal to $d_+(k) = d_+(k')$ and $d_-(k) = d_-(k')$.

Consider a special case that Σ is a 2-sphere S^2. We call a 2-knot invariant valued over F a *finite type* invariant if it is extended to a mapping $v : \mathcal{K}(S^2) \to \mathrm{F}$ of finite type. This is a natural analogue of finite type 1-knot invariants and Vassiliev's invariants in knot theory. Theorem 1 implies the following theorem.

Theorem 2. *The set of finite type 2-knot invariants is 1-dimensional, i.e., they are constant maps.*

1. The Vassiliev module of Σ-immersions

Let k be a Σ-immersion whose class $[k]$ belongs to $\mathcal{K}(\Sigma)_{(n)}$ and $v_1, \ldots, v_n$ the non-transverse double points of k. For each n-tuple of signs $(\epsilon_1, \ldots, \epsilon_n)$, let $k_{\epsilon_1,\ldots,\epsilon_n}$ denote a generic Σ-immersion obtained by desingularizations on $v_1, \ldots, v_n$ in directions according to $(\epsilon_1, \ldots, \epsilon_n)$. Then an element

$$\sum_{(\epsilon_1,\ldots,\epsilon_n)} \epsilon_1 \ldots \epsilon_n [k_{\epsilon_1,\ldots,\epsilon_n}] \in \mathbb{Z}\mathcal{K}(\Sigma)$$

is determined from $[k]$ up to sign. Denote by L_n the subspace of $\mathbb{Z}\mathcal{K}(\Sigma)$ spanned by such elements for all $[k] \in \mathcal{K}(\Sigma)_{(n)}$. Evidently, we have $L_1 \supset L_2 \ldots$. The Smale and Hirsch theorem implies that $[k] - [k'] \in L_1$ if and only if $e(k) = e(k')$. The n-th *Vassiliev module* is the ($\mathbb{Z}$-)module $\mathbb{Z}\mathcal{K}(\Sigma)/L_n$. (By tensor product with F, one may consider it a vector space or a module over F.) A mapping $v : \mathcal{K}(\Sigma) \to \mathrm{F}$ is of finite type if and only if its extension $v : \mathbb{Z}\mathcal{K}(\Sigma) \to \mathrm{F}$ is factored through this module for some n.

The following is our main theorem, which implies Theorem 1.

Theorem 3. *Let k and k' be generic Σ-immersions with $e(k) = e(k')$ and $d(k) = d(k')$. Then $[k] - [k'] \in L_n$ for every $n \in \mathbb{N}$.*

2. Unknotted Σ-immersions

A generic Σ-immersion is called an *unknotted* one if it is equivalent to a standard embedding of Σ with some (or no) "trivial kinks" (see Figure 2). We denote by u^e_d an unknotted Σ-immersion with $e(u^e_d) = e$ and $d(u^e_d) = d$. Notice that the equivalence class of u^e_d is unique.

Let $k : \Sigma \to R^4$ be a Σ-immersion. A *chord* γ attached to $k(\Sigma)$ means a smooth simple arc in R^4 whose endpoints are distinct points of $k(\Sigma)$ except the singularities and the interior of γ is disjoint from it. By a standard argument on general position, the ambient isotopy class of γ by isotopies of R^4 keeping $k(\Sigma)$ setwise fixed may be treated as a homotopy class in the sense of [2], [5]. If k' is a Σ-immersion obtained from k by a finger move along γ (see Figure 1), then $[k']$ is uniquely determined from $[k]$ and the class of γ. (The Σ-immersion appearing as a bifurcation in the finger move is obtained from k by shrinking γ.)

Lemma 1. *Let u be an unknotted Σ-immersion. Any Σ-immersion obtained from u by a finger move is also unknotted.*

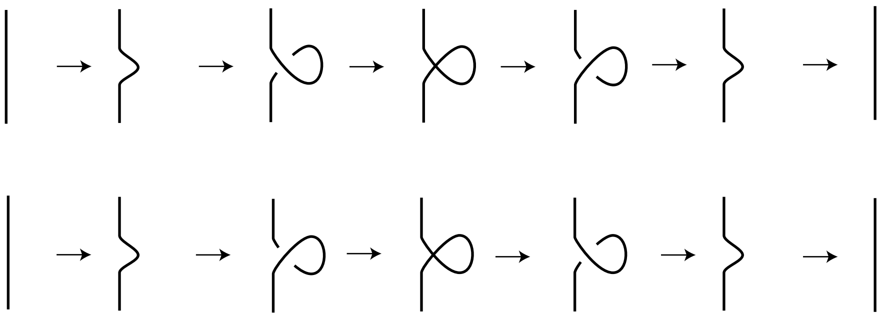

Figure 2

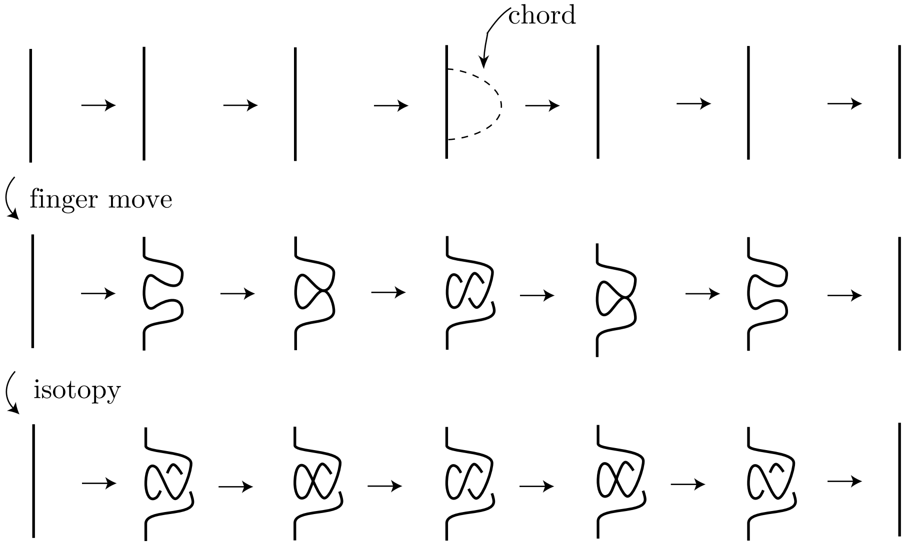

FIGURE 3

Proof. Let γ be a chord attached to $u(\Sigma)$ along which the finger move is applied. Since the fundamental group $\pi_1(R^4\backslash u(\Sigma),*)$ is an infinite cyclic group generated by the meridian of $u(\Sigma)$, by arguments of [5], [2], we may assume that γ is a chord attached to $u(\Sigma)$ trivially. The finger move brings the Σ-immersion u a pair of positive and negative trivial kinks, see Figure 3. ☐

Lemma 2. *If a Σ-immersion k' is obtained from another k by a Whitney trick followed by a finger move, then it is also obtained, up to equivalence, from k by a finger move followed by a Whitney trick.*

Proof. Let V be a 4-ball in R^4 such that a Whitney trick performed in V as in Figure 1 changes k into a Σ-immersion k'' and a finger move along a chord γ attached to $k''(\Sigma)$ changes k'' into k'. If γ is disjoint from V, the assertion is obvious. Since the inclusion-induced homomorphism $\pi_1(\partial V\backslash k''(\Sigma),*) \to \pi_1(V\backslash k''(\Sigma),*)$ is surjective (in fact, isomorphic), we may assume that γ is disjoint from V. ☐

Lemma 3 (Unknotting Lemma)**.** *For any generic Σ-immersion k, there exists a family of mutually disjoint chords attached to $k(\Sigma)$ such that finger moves along them change k into an unknotted one*

Proof. By the Smale and Hirsch theorem, there exists a generic regular homotopy between k and an unknotted Σ-immersion u. Using the previous lemma, we see that there exists a Σ-immersion k' such that (i) it is obtained from k by a sequence of finger moves and (ii) it is obtained also from u by a sequence of finger moves. By (ii) and Lemma 1, k' is unknotted. By a similar argument as in the proof of Lemma 2, it is easily seen that a sequence of finger moves may be replaced with the same number of simultaneous finger moves. Thus, from (i), we have the result. ☐

The *unknotting number* of a generic Σ-immersion k is defined by the minimum number of chords as in Lemma 3, which we denote by $\mathrm{u}(k)$. By the proof of

Lemma 3, we see that the unknotting number $\mathrm{u}(k)$ equals the minimum number of finger moves appearing in generic regular homotopies between k and unknotted Σ-immersions.

3. PROOF OF THEOREM 3

Let k be a generic Σ-immersion and $\gamma_1, \dots, \gamma_n$ mutually disjoint chords attached to $k(\Sigma)$. For each n-tuple of signs $(\epsilon_1, \dots, \epsilon_n)$, let $k_{\epsilon_1,\dots,\epsilon_n}$ denote a generic Σ-immersion obtained by finger moves along γ_i for i with $\epsilon_i = +1$ and by eliminating γ_i for i with $\epsilon_i = -1$. When one considers a Σ-immersion with n non-transverse double points obtained from k by shrinking the chords, $k_{\epsilon_1,\dots,\epsilon_n}$ is the same as before. Thus, in this situation, we have

$$\sum_{(\epsilon_1,\dots,\epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1,\dots,\epsilon_n}] \in L_n.$$

Proof of Theorem 3. It is sufficient to prove the following assertion for every $m \in \mathbb{N}$.

Assertion (m). Let k and k' be generic Σ-immersions with $e(k) = e(k')$ and $d(k) = d(k')$. If $\mathrm{u}(k) \leq m$ and $\mathrm{u}(k') \leq m$, then $[k] - [k'] \in L_n$ for any $n \geq m + 1$.

Since $\mathrm{u}(k) \leq m$, there exist m mutually disjoint chords attached to $k(\Sigma)$, say $\gamma_1, \dots, \gamma_m$, such that the finger moves along them change k into an unknotted one, which is u^e_{d+2m} where $e = e(k)$ and $d = d(k)$. Let n be an integer with $n \geq m+1$, and take parallel copies $\gamma_{m+1}, \dots, \gamma_n$ of γ_1 such that $\gamma_1, \dots, \gamma_n$ are mutually disjoint chords attached to $k(\Sigma)$. For each n-tuple of signs $(\epsilon_1, \dots, \epsilon_n)$, let $k_{\epsilon_1,\dots,\epsilon_n}$ denote a generic Σ-immersion obtained by finger moves along γ_i for i with $\epsilon_i = +1$ as before. Let $p = p(\epsilon_1, \dots, \epsilon_n)$ be the number of positive signs. If p is not zero, then by Lemma 1 we have $\mathrm{u}(k_{\epsilon_1,\dots,\epsilon_n}) \leq m-1$, $e(k_{\epsilon_1,\dots,\epsilon_n}) = e$ and $d(k_{\epsilon_1,\dots,\epsilon_n}) = d+2p$. We use the induction on m. If $m = 1$, then $k_{\epsilon_1,\dots,\epsilon_n} = u^e_{d+2p}$ for any $(\epsilon_1, \dots, \epsilon_n)$ with $p = p(\epsilon_1, \dots, \epsilon_n) \neq 0$. Thus,

$$\sum_{(\epsilon_1,\dots,\epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1,\dots,\epsilon_n}] = \sum_{p=1}^{n} (-1)^{n-p} \binom{n}{p} [u^e_{d+2p}] + (-1)^n [k].$$

Since the left-hand side belongs to L_n and a similar equation holds for k', we have $[k] - [k'] \in L_n$. If $m \geq 2$, then by the induction hypothesis we have $[k_{\epsilon_1,\dots,\epsilon_n}] - [u^e_{d+2p}] \in L_n$ for any $(\epsilon_1, \dots, \epsilon_n)$ with $p = p(\epsilon_1, \dots, \epsilon_n) \neq 0$. Thus,

$$\sum_{(\epsilon_1,\dots,\epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1,\dots,\epsilon_n}] \equiv \sum_{p=1}^{n} (-1)^{n-p} \binom{n}{p} [u^e_{d+2p}] + (-1)^n [k] \pmod{L_n}.$$

Therefore, we see that $[k] - [k'] \in L_n$. ☐

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REFERENCES

[1] J. S. Birman and X.-S. Lin, *Knot polynomials and Vassiliev's invariant*, Invent. Math. **11** (1993), 225–270. MR **94d:**57010

[2] J. Boyle, *Classifying 1-handles attached to knotted surfaces*, Trans. Amer. Math. Soc. **306** (1988), 475–487. MR **89f:**57032

[3] A. J. Casson, *Three lectures on new-infinite constructions in 4–dimensional manifolds*, A la Recherche de la Topologie Perdue, Progress in Math., 62, Birkhäuser, 1986, pp. 201–214. CMP 19:16

[4] M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276. MR **22:**9980

[5] F. Hosokawa and A. Kawauchi, *Proposals for unknotted surfaces in four-spaces*, Osaka J. Math. **16** (1979), 233–248. MR **81c:**57018

[6] S. Kamada, *Unknotting operations for 1- and 2-dimensional links*, The Proceedings of the fifth Korea-Japan School of Knots and Links, held in Taejon, Korea, 1997, to appear.

[7] R. Kirby, *The topology of* 4*-manifolds*, Lecture Notes in Mathematics, 1374, Springer-Verlag, 1989. MR **90j:**57012

[8] P. Kirk, *Link maps in the four sphere*, Differential Topology, Lect. Notes Math., 1350, Springer-Verlag, 1988, pp. 31–43. MR **90e:**57043

[9] G.-S. Li, *An invariant of link homotopy in dimension four*, Topology **36** (1997), 881–897. MR **98g:**57038

[10] S. Smale, *A classification of immersions of the two-sphere*, Trans. Amer. Math. Soc. **90** (1959), 281–290. MR **21:**2984

[11] ———, *The classification of immersions of spheres in Euclidean spaces*, Ann. Math. **69** (1959), 327–344.

[12] V. A. Vassiliev, *Cohomology of knot space*, Theory of Singularities and Its Applications, Advances in Soviet Math. vol. 1, Amer. Math. Soc., 1990. MR **92a:**57016

Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka 558, Japan
E-mail address: `kamada@sci.osaka-cu.ac.jp`