SPECTRUM PRESERVING LINEAR MAPPINGS FOR SCATTERED JORDAN-BANACH ALGEBRAS

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Abstract. Given two semisimple complex Jordan-Banach algebras with identity A and B, we say that T is a spectrum preserving linear mapping from A to B if T is surjective and we have Sp(Tx) = Sp(x), for all x ∈ A. We prove that if B is a scattered Jordan-Banach algebra, then T is a Jordan isomorphism.

The aim of this note is to provide a detailed proof of a question raised in [2], saying that surjective linear and spectrum preserving mappings on separable scattered Jordan-Banach algebras are Jordan isomorphisms. This is in fact an extension of Theorem 3.7 of [4] from the associative case to the more general situation of Jordan-Banach algebras. In the proof of our result we will use the extension of Harte’s theorem obtained by the author in [6, 7] and the structure theorem for scattered Jordan-Banach algebras of Aupetit-Baribeau [3]. In [2] Aupetit generalized some results previously obtained in [4], from the associative setting to the Jordan case.

The next theorem contains Theorem 2.1 and Corollary 2.4 of [2].

Theorem 1.1. Let T be a spectrum preserving linear mapping from A onto B. Then Tx^2 - (Tx)^2 ∈ Ann(Soc B), for every x ∈ A. Moreover T is a Jordan isomorphism from kh(Soc A) onto kh(Soc B).

We recall that a complex Jordan algebra A is non-associative and the product satisfies the identities ab = ba and (ab)a^2 = a(ba^2), for all a, b in A. A unital Jordan-Banach algebra is a Jordan algebra with a complete norm satisfying \|xy\| ≤ \|x\| \|y\|, for x, y ∈ A, and \|1\| = 1. An element a ∈ A is said to be invertible if there exists b ∈ A such that ab = 1 and a^2b = a. For an element x of a Jordan-Banach algebra A, the spectrum of x is by definition the set of λ ∈ \mathbb{C} for which λ - x is not invertible. It is a non-empty subset of \mathbb{C}. Moreover x → Sp(x) is upper semicontinuous on A. As in the associative case, the set Ω(A) of invertible elements is open, but unfortunately it is not anymore a multiplicative group (see [6, 7]). We also denote by Ω_1(A) the connected component of Ω(A), which contains the identity. At this stage it is appropriate to notice that if A is a Jordan-Banach algebra and we define exp(A) = \{e^x : x ∈ A\}, then exp(A) ⊂ Ω_1(A). It is clear that e^{-x} is the inverse of e^x for x ∈ A. It is also possible to give a concise notion of exponential spectrum of an element x, that we denote ε(x), as the compact set defined by λ ≠ ε(x) if and

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only if $\lambda - x \in \Omega_1(A)$. As in the associative case we have $\text{Sp}(x) \subseteq \varepsilon(x) \subseteq \sigma(x)$, where $\sigma(x)$ denotes the full spectrum of $x$. To see the first inclusion, suppose $\lambda \notin \varepsilon(x)$; then $\lambda - x \notin \Omega_1(A)$, which means $\lambda - x$ is invertible, so $\lambda \notin \text{Sp}(x)$. For the second one, if $\lambda \notin \sigma(x)$, then by the Holomorphic Functional Calculus $\lambda - x = e^{\eta}$ for $y$ in the closed associative subalgebra $C(1,x)$ generated by $1$ and $x$. Then $x(t) = e^{ty}$ defines a continuous path of invertible elements joining $\lambda - x$ and $1$. Hence $\lambda - x \notin \Omega_1(A)$ and $\lambda \notin \varepsilon(x)$.

In [6] we proved the following result for which another application is given in this note.

**Theorem 1.2** (Extension of Harte’s Theorem [6, 7]). Let $T$ be a continuous morphism from a Jordan-Banach algebra $A$ onto a Jordan-Banach algebra $B$. Then $T(\Omega_1(A)) = \Omega_1(B)$. Moreover,

$$\varepsilon(Tx) = \bigcap_{y \in \ker T} \varepsilon(x + y)$$

and

$$\text{Sp}(Tx) \subseteq \bigcap_{y \in \ker T} \text{Sp}(x + y) \subseteq \sigma(Tx).$$

We recall that for a semisimple Jordan-Banach algebra $A$, the socle $\text{Soc} A$ is the sum of all the quadratic minimal ideals of $A$. It is known that $\text{Soc} A$ is an ideal which is the sum of simple ideals generated by minimal projections. The theory of the socle has been extensively studied in recent years; for more details and references see [3, 7]. The notion of annihilator of a set of a Jordan algebra has been introduced by E. Zelmanov [8]; in particular the annihilator of an ideal is also an ideal. It is shown in [5] that $\text{Ann}(\text{Soc} A) = \{a \in A : a \cdot \text{Soc} A = \{0\}\}$.

In the proof of the following important lemma, we use implicitly that a projection which is in $kh(\text{Soc} B)$ is actually in $\text{Soc} B$ if and only if $u$ is in $\text{Soc} B$, then its non-zero spectrum consists of isolated points. These two assertions follow from Corollaries 2.5, 2.6 and 2.7 of [6].

**Lemma 1.3.** The ideal $I = kh(\text{Soc} B) \cap \text{Ann}(\text{Soc} B) = \{0\}$. 

**Proof.** First we prove that $u \in I$ implies $\rho(u) = 0$ where $\rho$ denotes the spectral radius. Suppose $\rho(u) \neq 0$; then there exists $\alpha \neq 0, \alpha \in \text{Sp} u$. Now, the non-zero Riesz projection $p$ associated to $\alpha$ and $u$ is in the socle of $B$ and we have $p = \frac{u}{2\pi i} \int_{\Gamma} \frac{1}{\lambda - u}^{-1} d\lambda$, where $\Gamma$ is a small circle centred at $\alpha$. Since $u \in \text{Ann}(\text{Soc} B)$, which is an ideal, we deduce that $p \in \text{Ann}(\text{Soc} B)$, so $p \in \text{Soc}(B) \cap \text{Ann}(\text{Soc} B) = \{0\}$, and this is absurd. Consequently, the ideal $I$ contains only quasi-nilpotent elements, so $I \subseteq \text{Rad} B = \{0\}$.

We now prove the main result of this note for scattered Jordan-Banach algebras, that is, Jordan-Banach algebras for which the spectrum of every element is finite or countable. By Aupetit-Baribeau’s theorem their socle is non-empty and they have a very particular algebraic structure [3].

**Theorem 1.4** (Aupetit-Baribeau). Let $J$ be a complex Jordan-Banach algebra with identity such that the spectrum of every element is at most countable. Moreover, suppose that $J$ is separable. Then there exists an ordinal $\alpha_0$ of the first or second class and a sequence $(I_{\alpha})_{\alpha \leq \alpha_0}$ of closed ideals of $J$ such that $I_0 = \text{Rad} J$, $I_{\alpha_0} = J$ and $I_{\alpha+1}/I_{\alpha}$ is a modular annihilator for $\alpha \leq \alpha_0$. 


We are ready now to state and prove the main result of this note, which is an adaptation to the Jordan case of the proof of Theorem 3.7 of [4].

**Theorem 1.5.** Let $T$ be a spectrum preserving linear mapping from a Jordan-Banach algebra $A$ onto a separable scattered Jordan-Banach algebra $B$. Then $T$ is a Jordan isomorphism (i.e., $Tx^2 = (Tx)^2$ for all $x \in A$).

**Proof.** Because $\text{Sp}(Tx) = \text{Sp}(x)$ for every $x \in A$ and $B$ is a scattered Jordan-Banach algebra, it is clear that $A$ is also a scattered Jordan-Banach algebra. Now let $I_1 = kh(\text{Soc} A)$ and $J_1 = kh(\text{Soc} B)$. By Theorem 1.1 we have $J_1 = T(I_1)$. Define semisimple Jordan-Banach algebras $A_1$ and $B_1$ by $A_1 = A/I_1, B_1 = B/J_1$. Let $\pi_1 : A \to A_1$ and $\gamma_1 : B \to B_1$ be the corresponding canonical maps. Define a linear map $T_1 : A_1 \to B_1$ by $T_1(\pi_1) = T\pi_1$. By Theorem 1.2 and since the spectrum of every element of $A$ and $B$ is at most countable, we have $\text{Sp}(x) = \sigma(x)$ and $\text{Sp}(Tx) = \sigma(Tx)$. Then

$$\text{Sp}(\pi_1) = \bigcap_{x \in I_1} \text{Sp}(x + a) = \bigcap_{y \in J_1} \text{Sp}(Ta + y) = \text{Sp}(T_1 \pi_1),$$

and hence $T_1$ is spectrum preserving. Furthermore if $a \in I_1$, then $Ta^2 - (Ta)^2 \in J_1 = kh(\text{Soc} B)$ and also $Ta^2 - (Ta)^2 \in \text{Ann}(\text{Soc} B)$. Since $\text{Soc} B \subset kh(\text{Soc} B)$ implies $\text{Ann}(kh(\text{Soc} B)) \subset \text{Ann}(B)$ we conclude that $Ta^2 - (Ta)^2 \in \text{Ann}(\text{Soc} B) \cap \text{Ann}(\text{Soc} B)$ which is zero by Lemma 1.3. Then $Ta^2 = (Ta)^2$ for every $a \in I_1$. Continuing inductively, we define

$$A_n = A_{n-1}/kh(\text{Soc} A_{n-1}), \quad B_n = B_{n-1}/kh(\text{Soc} B_{n-1})$$

and we denote respectively by $\pi_n, \gamma_n$ the canonical maps from $A_{n-1}$ onto $A_n$ and from $B_{n-1}$ onto $B_n$. Let $I_n = \ker(\pi_1 \circ \cdots \circ \pi_n), J_n = \ker(\gamma_1 \circ \cdots \circ \gamma_n)$ and note that $J_n = T(I_n)$. Define a linear map $T_n : A_n \to B_n$ by $T_n(\pi_1) = T_1(\pi_1) = T_n(\pi_1)$. Then $T_n$ is spectrum preserving and by using Lemma 1.3 it is easy to see that for every $a \in I_n$, we have $Ta^2 = (Ta)^2$. If $\omega$ is the first ordinal number, we define

$$I_\omega = kh \left( \bigcup_{n \geq 1} I_n \right), \quad J_\omega = kh \left( \bigcup_{n \geq 1} J_n \right)$$

and also notice that $T((\bigcup_{n \geq 1} I_n)) = (\bigcup_{n \geq 1} J_n)$. Define the linear mapping

$$T'_\omega : A/\left( \bigcup_{n \geq 1} I_n \right) \to B/\left( \bigcup_{n \geq 1} J_n \right) \text{ by } T'_\omega(\pi_1) = T_\omega(\pi_1).$$

By using Theorem 1.2 it follows that $T'_\omega$ is spectrum preserving and we have $J_\omega = T(I_\omega)$. Let $A_\omega = A/I_\omega, B_\omega = B/J_\omega$ and define a linear operator $T_\omega : A_\omega \to B_\omega$ by $T_\omega(\pi_1) = T_\omega(\pi_1)$. Notice that $T_\omega$ is spectrum preserving and that $A_\omega$ and $B_\omega$ are semisimple. We claim that $Ta^2 = (Ta)^2$ for every $a \in I_\omega$. Let $a \in I_\omega$ and suppose that $u = Ta^2 - (Ta)^2 \neq 0$. By Lemma 1.3 we conclude that $\gamma_1 \circ \cdots \circ \gamma_1(u) \neq 0$ for $n = 1, 2, \ldots, u \neq 0$ and by Theorem 1.1 $(\gamma_1 \circ \cdots \circ \gamma_1(u))y = 0$ for every $y \in \text{Soc} B_n$. Since $u \neq 0$ and $B$ is semisimple, there exists $b \in B$ such that $\text{Sp}(ub) \neq \{0\}$ and since $ub \in J_\omega$, once more by Theorem 1.2 we have

$$\bigcap_{y \in \bigcup_{n \geq 1} J_n} \text{Sp}(ub + y) = 0.$$
We now prove that there exists an integer \( n \) such that \( \text{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \text{Sp}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub)) \), where \( B_0 = B \) and \( \gamma_0 \) is the identity map on \( B \). Suppose the contrary: then \( \text{Sp}(ub) = \text{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \subset \text{Sp}(ub + y) \) for every \( n \geq 1 \) and \( y \in J_n \), consequently, by continuity of the spectrum, \( \text{Sp}(ub) \subset \text{Sp}(ub + y) \) for every \( y \in \bigcup_{n \geq 1} J_n \). Now by (**) we have \( \text{Sp}(ub) = \{0\} \), which is a contradiction. Hence suppose \( \text{Sp}_{B_n}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \text{Sp}_{B_{n+1}}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub)) \) for some \( n \geq 0 \) and let \( x = (\gamma_n \circ \cdots \circ \gamma_0(ub)) \). Then there exists an isolated point \( \lambda \neq 0 \) of \( \text{Sp}_{B_{n+1}}(x) \) such that \( \lambda \not\in \text{Sp}_{B_{n+1}}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \). If we denote by \( p \) the spectral projection associated to \( \lambda \), we have \( p \in \text{Soc}_{B_0}(\gamma_n \circ \cdots \circ \gamma_0(b)) \), which contradicts the fact that \( (\gamma_n \circ \cdots \circ \gamma_0(b))y = 0 \) for every \( y \in \text{Soc}_{B_n} \). So finally we have proved that for every \( a \in I_\omega : Ta^2 - (Ta)^2 = 0 \). Now, continuing by transfinite induction, there exists an ordinal \( \beta \) in the first class of ordinals such that \( A = I_\beta \) ([3]). By the previous arguments it is easy to see that \( B = I_\beta \) and \( Ta^2 = (Ta)^2 \) for every \( a \in A \). 

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References


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