

SPECTRUM PRESERVING LINEAR MAPPINGS FOR SCATTERED JORDAN-BANACH ALGEBRAS

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ABSTRACT. Given two semisimple complex Jordan-Banach algebras with identity A and B , we say that T is a spectrum preserving linear mapping from A to B if T is surjective and we have $\text{Sp}(Tx) = \text{Sp}(x)$, for all $x \in A$. We prove that if B is a scattered Jordan-Banach algebra, then T is a Jordan isomorphism.

The aim of this note is to provide a detailed proof of a question raised in [2], saying that surjective linear and spectrum preserving mappings on separable scattered Jordan-Banach algebras are Jordan isomorphisms. This is in fact an extension of Theorem 3.7 of [4] from the associative case to the more general situation of Jordan-Banach algebras. In the proof of our result we will use the extension of Harte's theorem obtained by the author in [6, 7] and the structure theorem for scattered Jordan-Banach algebras of Aupetit-Baribeau [3]. In [2] Aupetit generalized some results previously obtained in [4], from the associative setting to the Jordan case.

The next theorem contains Theorem 2.1 and Corollary 2.4 of [2].

Theorem 1.1. *Let T be a spectrum preserving linear mapping from A onto B . Then $Tx^2 - (Tx)^2 \in \text{Ann}(\text{Soc } B)$, for every $x \in A$. Moreover T is a Jordan isomorphism from $kh(\text{Soc } A)$ onto $kh(\text{Soc } B)$.*

We recall that a complex Jordan algebra A is non-associative and the product satisfies the identities $ab = ba$ and $(ab)a^2 = a(ba^2)$, for all a, b in A . A unital Jordan-Banach algebra is a Jordan algebra with a complete norm satisfying $\|xy\| \leq \|x\| \|y\|$, for $x, y \in A$, and $\|1\| = 1$. An element $a \in A$ is said to be invertible if there exists $b \in A$ such that $ab = 1$ and $a^2b = a$. For an element x of a Jordan-Banach algebra A , the spectrum of x is by definition the set of $\lambda \in \mathbb{C}$ for which $\lambda - x$ is not invertible. It is a non-empty subset of \mathbb{C} . Moreover $x \mapsto \text{Sp}(x)$ is upper semicontinuous on A . As in the associative case, the set $\Omega(A)$ of invertible elements is open, but unfortunately it is not anymore a multiplicative group (see [6, 7]). We also denote by $\Omega_1(A)$ the connected component of $\Omega(A)$, which contains the identity. At this stage it is appropriate to notice that if A is a Jordan-Banach algebra and we define $\exp(A) = \{e^x : x \in A\}$, then $\exp(A) \subset \Omega_1(A)$. It is clear that e^{-x} is the inverse of e^x for $x \in A$. It is also possible to give a concise notion of exponential spectrum of an element x , that we denote $\varepsilon(x)$, as the compact set defined by $\lambda \notin \varepsilon(x)$ if and

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only if $\lambda - x \in \Omega_1(A)$. As in the associative case we have $\text{Sp}(x) \subset \varepsilon(x) \subset \sigma(x)$, where $\sigma(x)$ denotes the full spectrum of x . To see the first inclusion, suppose $\lambda \notin \varepsilon(x)$; then $\lambda - x \in \Omega_1(A)$, which means $\lambda - x$ is invertible, so $\lambda \notin \text{Sp}(x)$. For the second one, if $\lambda \notin \sigma(x)$, then by the Holomorphic Functional Calculus $\lambda - x = e^y$ for y in the closed associative subalgebra $C(1, x)$ generated by 1 and x . Then $x(t) = e^{ty}$ defines a continuous path of invertible elements joining $\lambda - x$ and 1. Hence $\lambda - x \in \Omega_1(A)$ and $\lambda \notin \varepsilon(x)$.

In [6] we proved the following result for which another application is given in this note.

Theorem 1.2 (Extension of Harte’s Theorem [6, 7]). *Let T be a continuous morphism from a Jordan-Banach algebra A onto a Jordan-Banach algebra B . Then $T(\Omega_1(A)) = \Omega_1(B)$. Moreover,*

$$\varepsilon(Tx) = \bigcap_{y \in \ker T} \varepsilon(x + y)$$

and

$$\text{Sp}(Tx) \subset \bigcap_{y \in \ker T} \text{Sp}(x + y) \subset \sigma(Tx).$$

We recall that for a semisimple Jordan-Banach algebra A , the socle $\text{Soc } A$ is the sum of all the quadratic minimal ideals of A . It is known that $\text{Soc } A$ is an ideal which is the sum of simple ideals generated by minimal projections. The theory of the socle has been extensively studied in recent years; for more details and references see [3, 7]. The notion of annihilator of a set of a Jordan algebra has been introduced by E. Zelmanov [8]; in particular the annihilator of an ideal is also an ideal. It is shown in [5] that $\text{Ann}(\text{Soc } A) = \{a \in A : a \cdot \text{Soc } A = \{0\}\}$.

In the proof of the following important lemma, we use implicitly that a projection which is in $kh(\text{Soc } B)$ is actually in $\text{Soc } B$ and if u is in $\text{Soc } B$, then its non-zero spectrum consists of isolated points. These two assertions follow from Corollaries 2.5, 2.6 and 2.7 of [6].

Lemma 1.3. *The ideal $I = kh(\text{Soc } B) \cap \text{Ann}(\text{Soc } B) = \{0\}$.*

Proof. First we prove that $u \in I$ implies $\rho(u) = 0$ where ρ denotes the spectral radius. Suppose $\rho(u) \neq 0$; then there exists $\alpha \neq 0$, $\alpha \in \text{Sp } u$. Now, the non-zero Riesz projection p associated to α and u is in the socle of B and we have $p = \frac{u}{2\pi i} \int_{\Gamma} \frac{1}{\lambda} (\lambda - u)^{-1} d\lambda$, where Γ is a small circle centred at α . Since $u \in \text{Ann}(\text{Soc } B)$, which is an ideal, we deduce that $p \in \text{Ann}(\text{Soc } B)$, so $p \in \text{Soc}(B) \cap \text{Ann}(\text{Soc } B) = \{0\}$, and this is absurd. Consequently, the ideal I contains only quasi-nilpotent elements, so $I \subset \text{Rad } B = \{0\}$. □

We now prove the main result of this note for scattered Jordan-Banach algebras, that is, Jordan-Banach algebras for which the spectrum of every element is finite or countable. By Aupetit-Baribeau’s theorem their socle is non-empty and they have a very particular algebraic structure [3].

Theorem 1.4 (Aupetit-Baribeau). *Let J be a complex Jordan-Banach algebra with identity such that the spectrum of every element is at most countable. Moreover, suppose that J is separable. Then there exists an ordinal α_0 of the first or second class and a sequence $(I_\alpha)_{\alpha \leq \alpha_0}$ of closed ideals of J such that $I_0 = \text{Rad } J$, $I_{\alpha_0} = J$ and $I_{\alpha+1}/I_\alpha$ is a modular annihilator for $\alpha \leq \alpha_0$.*

We are ready now to state and prove the main result of this note, which is an adaptation to the Jordan case of the proof of Theorem 3.7 of [4].

Theorem 1.5. *Let T be a spectrum preserving linear mapping from a Jordan-Banach algebra A onto a separable scattered Jordan-Banach algebra B . Then T is a Jordan isomorphism (i.e., $Tx^2 = (Tx)^2$ for all $x \in A$).*

Proof. Because $\text{Sp}(Tx) = \text{Sp}(x)$ for every $x \in A$ and B is a scattered Jordan-Banach algebra, it is clear that A is also a scattered Jordan-Banach algebra. Now let $I_1 = kh(\text{Soc } A)$ and $J_1 = kh(\text{Soc } B)$. By Theorem 1.1 we have $J_1 = T(I_1)$. Define semisimple Jordan-Banach algebras A_1 and B_1 by $A_1 = A/I_1, B_1 = B/J_1$. Let $\pi_1 : A \rightarrow A_1$ and $\gamma_1 : B \rightarrow B_1$ be the corresponding canonical maps. Define a linear map $T_1 : A_1 \rightarrow B_1$ by $T_1(\bar{a}) = \overline{Ta}$. By Theorem 1.2 and since the spectrum of every element of A and B is at most countable, we have $\text{Sp}(x) = \sigma(x)$ and $\text{Sp}(Tx) = \sigma(Tx)$. Then

$$\text{Sp}(\bar{a}) = \bigcap_{x \in I_1} \text{Sp}(a+x) = \bigcap_{y \in J_1} \text{Sp}(Ta+y) = \text{Sp}(T_1\bar{a}),$$

and hence T_1 is spectrum preserving. Furthermore if $a \in I_1$, then $Ta^2 - (Ta)^2 \in J_1 = kh(\text{Soc } B)$ and also $Ta^2 - (Ta)^2 \in \text{Ann}(\text{Soc } B)$. Since $\text{Soc } B \subset kh(\text{Soc } B)$ implies $\text{Ann}(kh(\text{Soc } B)) \subset \text{Ann}(\text{Soc } B)$ we conclude that $Ta^2 - (Ta)^2 \in kh(\text{Soc } B) \cap \text{Ann}(\text{Soc } B)$ which is zero by Lemma 1.3. Then $Ta^2 = (Ta)^2$ for every $a \in I_1$. Continuing inductively, we define

$$A_n = A_{n-1}/kh(\text{Soc } A_{n-1}), \quad B_n = B_{n-1}/kh(\text{Soc } B_{n-1})$$

and we denote respectively by π_n, γ_n the canonical maps from A_{n-1} onto A_n and from B_{n-1} onto B_n . Let $I_n = \ker(\pi_1 \circ \dots \circ \pi_n)$, $J_n = \ker(\gamma_n \circ \dots \circ \gamma_1)$ and note that $J_n = T(I_n)$. Define a linear map $T_n : A_n \rightarrow B_n$ by $T_n(\bar{a}) = \overline{T_{n-1}(a)}$. Then T_n is spectrum preserving and by using Lemma 1.3 it is easy to see that for every $a \in I_n$, we have $Ta^2 = (Ta)^2$. If ω is the first ordinal number, we define

$$I_\omega = kh \left(\bigcup_{n \geq 1} I_n \right), \quad J_\omega = kh \left(\bigcup_{n \geq 1} J_n \right)$$

and also notice that $T(\bigcup_{n \geq 1} I_n) = \bigcup_{n \geq 1} J_n$. Define the linear mapping

$$T'_\omega : A / \left(\bigcup_{n \geq 1} I_n \right) \rightarrow B / \left(\bigcup_{n \geq 1} J_n \right) \quad \text{by } T'_\omega \bar{a} = \overline{Ta}.$$

By using Theorem 1.2 it follows that T'_ω is spectrum preserving and we have $J_\omega = T(I_\omega)$. Let $A_\omega = A/I_\omega, B_\omega = B/J_\omega$ and define a linear operator $T_\omega : A_\omega \rightarrow B_\omega$ by $T_\omega(\bar{a}) = \overline{Ta}$. Notice that T_ω is spectrum preserving and that A_ω and B_ω are semisimple. We claim that $Ta^2 = (Ta)^2$ for every $a \in I_\omega$. Let $a \in I_\omega$ and suppose that $u = Ta^2 - (Ta)^2 \neq 0$. By Lemma 1.3 we conclude that $\gamma_n \circ \dots \circ \gamma_1(u) \neq 0$ for $n = 1, 2, \dots$, and by Theorem 1.1 $(\gamma_n \circ \dots \circ \gamma_1(u))y = 0$ for every $y \in \text{Soc } B_n$. Since $u \neq 0$ and B is semisimple, there exists $b \in B$ such that $\text{Sp}(ub) \neq \{0\}$ and since $ub \in J_\omega$, once more by Theorem 1.2 we have

$$(**) \quad \bigcap_{y \in \bigcup_{n \geq 1} I_n} \text{Sp}(ub+y) = 0.$$

We now prove that there exists an integer n such that $\text{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \text{Sp}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub))$, where $B_0 = B$ and γ_0 is the identity map on B . Suppose the contrary: then $\text{Sp}(ub) = \text{Sp}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \subset \text{Sp}(ub + y)$ for every $n \geq 1$ and $y \in J_n$, consequently, by continuity of the spectrum, $\text{Sp}(ub) \subset \text{Sp}(ub + y)$ for every $y \in \overline{\bigcup_{n \geq 1} J_n}$. Now by (**) we have $\text{Sp}(ub) = \{0\}$, which is a contradiction. Hence suppose $\text{Sp}_{B_n}(\gamma_n \circ \cdots \circ \gamma_0(ub)) \neq \text{Sp}_{B_{n+1}}(\gamma_{n+1} \circ \cdots \circ \gamma_0(ub))$ for some $n \geq 0$ and let $x = (\gamma_n \circ \cdots \circ \gamma_0)(ub)$. Then there exists an isolated point $\lambda \neq 0$ of $\text{Sp}_{B_{n+1}}(x)$ such that $\lambda \notin \text{Sp}_{B_{n+1}}(\gamma_{n+1}(x))$. If we denote by p the spectral projection associated to p and λ we have $p \in \text{Soc } B_0$ and $0 \neq xp = (\gamma_n \circ \cdots \circ \gamma_0(u))(\gamma_n \circ \cdots \circ \gamma_0(b))p$, which contradicts the fact that $(\gamma_n \circ \cdots \circ \gamma_0(u))y = 0$ for every $y \in \text{Soc } B_n$. So finally we have proved that for every $a \in I_\omega : Ta^2 - (Ta)^2 = 0$. Now, continuing by transfinite induction, there exists an ordinal β in the first class of ordinals such that $A = I_\beta$ ([3]). By the previous arguments it is easy to see that $B = I_\beta$ and $Ta^2 = (Ta)^2$ for every $a \in A$. \square

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REFERENCES

1. B. Aupetit, *A Primer on Spectral Theory*, Springer Universitext, 1991. MR **92c**:46001
2. B. Aupetit, *Trace and spectrum preserving linear mappings in Jordan-Banach algebras*, Monatsh. Math. **125** (1998), 179–187. CMP 98:11
3. B. Aupetit and L. Baribeau, *Sur le socle dans les algèbres de Jordan-Banach*, Can. J. Math. **41** (1989), 1090–1100. MR **90m**:46085
4. B. Aupetit and H. du Toit Mouton, *Spectrum preserving linear mappings in Banach algebras*, Studia Math. **109** (1) (1994), 91–100. MR **95c**:46070
5. A. Fernández López, *On annihilators in Jordan algebras*, Publ. Mat. **36** (1992), 569–589. MR **94c**:17047
6. A. Maouche, *Extension d'un Théorème de Harte et applications aux algèbres de Jordan-Banach*, Monatsh. Math. **122** (1996), 205–213. MR **97h**:46082
7. A. Maouche, *Caractérisations spectrales du radical et du socle d'une paire de Jordan-Banach*, Can. Math. Bull. **40** (1997), 488–497. MR **99a**:46093
8. A. Maouche, *Théorie spectrale dans les systèmes de Jordan-Banach*, Ph. D. Thesis, Université Laval, Québec, 1994.
9. E. Zelmanov, *Goldie Theorems for Jordan algebras*, Siberian Math. J. **28** (1987), 44–52. MR **89f**:17034

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