ITERATION OF A CLASS OF HYPERBOLIC MEROMORPHIC FUNCTIONS

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Abstract. We look at the class $B_n$ which contains those transcendental meromorphic functions $f$ for which the finite singularities of $f^{-n}$ are in a bounded set and prove that, if $f$ belongs to $B_n$, then there are no components of the set of normality in which $(f^n)(z) \to \infty$ as $m \to \infty$. We then consider the class $\hat{B}$ which contains those functions $f$ in $B_1$ for which the forward orbits of the singularities of $f^{-1}$ stay away from the Julia set and show (a) that there is a bounded set containing the finite singularities of all the functions $f^{-n}$ and (b) that, for points in the Julia set of $f$, the derivatives $(f^n)'$ have exponential-type growth. This justifies the assertion that $\hat{B}$ is a class of hyperbolic functions.

1. Introduction

Let $f$ be a meromorphic function which is not rational of degree less than two, and denote by $f^n$, $n \in \mathbb{N}$, the $n$-th iterate of $f$. The set of normality, $N(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that $(f^n)_{n \in \mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of $z$. The complement of $N(f)$ is called the Julia set, $J(f)$, of $f$. An introduction to the properties of these sets can be found in, for example, [3].

We will use the following notation concerning singularities:

\[
S(f) = \{z \in \mathbb{C} : z \text{ is a singularity of } f^{-1}\},
\]

\[
P(f) = \{z \in \mathbb{C} : z \text{ is a singularity of } f^{-n}, \text{ for some } n \in \mathbb{N}\}.
\]

It was shown by Herring [7, Theorem 7.1.2] that

\[
\{z \in \mathbb{C} : z \text{ is a singularity of } f^{-n}\} \subseteq S_n(f) = \bigcup_{j=0}^{n-1} f^j(S(f) \setminus A_j(f)),
\]

where

\[
A_j(f) = \{z \in \mathbb{C} : f^j \text{ is not analytic at } z\},
\]

and that

\[
P(f) = \bigcup_{n=0}^{\infty} S_n(f).
\]
Eremenko and Lyubich [6] investigated the properties of entire functions in the class
\[ B = \{ f : f \text{ is a transcendental meromorphic function with } S(f) \text{ bounded} \}. \]

In Section 2 we look at the properties of functions in the class
\[ B_n = \{ f : f \text{ is a transcendental meromorphic function with } S_n(f) \text{ bounded} \}. \]
(Note that \( B_1 \) is equal to \( B \).) We prove the following result.

**Theorem A.** If \( f \in B_n \), then there is no component of \( N(f) \) in which \( f^{mn}(z) \to \infty \) as \( m \to \infty \).

**Remarks.** Our proof is based on ideas of Eremenko and Lyubich [6, Theorem 1] who proved this result in the case when \( f \) is entire and \( n = 1 \). The proof of Theorem A given by Bergweiler [3, Theorem 16] uses [3, Lemma 8] which asserts that, if \( f \in B \), \( p \geq 1 \) and \( 0 \) is not a pre-image of \( \infty \), then there exist a positive constant \( R \) and a curve \( \Gamma \) connecting \( 0 \) to \( \infty \) such that \( |f^p(z)| \leq R \) for \( z \in \Gamma \). Unfortunately, this lemma is not correct, as shown by the counterexample \( f(z) = \tan \frac{z}{z+\pi} \). Although \( f \in B \), \( f^2 \) is unbounded on each path to \( \infty \). The rest of the proof of [3, Theorem 16] is correct and the reference to [3, Lemma 8] can be successfully replaced by a reference to Lemma 2.1 of this paper.

It follows from Theorem A that, if \( f \in B_n \), then there can be no periodic cycle \( \{N_0, \ldots, N_{n-1}\} \) of components of \( N(f) \) with \( f^{mn}(z) \to \infty \) as \( m \to \infty \) in one of the components—such a cycle is known as a cycle of Baker domains or essentially parabolic domains. Thus we have the following Corollary to Theorem A.

**Corollary.** If \( f \in B_n \), then \( f \) has no Baker domains of period \( n \).

Many authors have considered functions in the class
\[ S = \{ f : f \text{ is a transcendental meromorphic function with } S(f) \text{ finite} \}. \]

It is easy to see that, if \( f \in S \), then \( f \in \bigcap_{n=1}^{\infty} B_n \) and so a special case of the above Corollary is that functions in \( S \) have no Baker domains.

In Sections 3 and 4 we consider the iteration of functions in the class
\[ \hat{B} = \{ f : f \in B \text{ and } \bar{P}(f) \cap J(f) = \emptyset \}, \]
where \( \bar{P} \) denotes closure with respect to the plane. In Section 3 we use Theorem A to prove the following result.

**Theorem B.** If \( f \in \hat{B} \), then \( P(f) \) is bounded.

In Section 4, we use Theorem B together with the results of Section 2 to prove the following result for meromorphic functions which has applications to estimating the Hausdorff dimension of \( J(f) \) when \( f \in \hat{B} \); see [8].

**Theorem C.** If \( f \in \hat{B} \), then there exist \( K > 1 \) and \( c > 0 \) such that
\[ |(f^n)'(z)| > cK^n \frac{|f^n(z)| + 1}{|z| + 1}, \]
for each \( z \in J(f) \setminus A_n(f), n \in \mathbb{N} \).

If \( f \) is rational, then the following conditions are equivalent—see [2, Section 9.7] and [4, Section 5.2] and note that, for rational functions, \( P \) denotes closure in the sphere:
- \( \bar{P}(f) \cap J(f) = \emptyset \);
• \( \hat{P}(f) \) is a compact subset of \( N(f) \);
• \( f \) is expanding, in the sense that there exist \( K > 1 \) and \( c > 0 \) such that 
  \[ |(f^n)'(z)| > cK^n \]
  for each \( z \in J(f), n \in \mathbb{N} \).

A rational function with these properties is said to be hyperbolic. For transcendental meromorphic functions, these conditions are no longer equivalent and so it is not clear what the definition of a hyperbolic transcendental meromorphic function should be. In view of Theorems B and C, however, it does seem natural to say that the functions in \( \hat{B} \) are hyperbolic.

2. Properties of functions in the class \( B_n \)

We use the following notation:

\[
B(z, r) = \{ w : |w - z| < r \}, \\
D_R = \{ z \in \mathbb{C} : |z| > R \} \cup \{ \infty \}.
\]

The following lemma is probably ‘well known’; we include a proof for the sake of completeness.

**Lemma 2.1.** If \( f \in B_n \) and \( S_n(f) \subseteq B(0, R) \), then each component of \( f^{-n}(D_R) \) is simply connected in \( \mathbb{C} \).

**Proof.** Let \( V \) be a component of \( f^{-n}(D_R) \), let \( g \) denote a branch of \( f^{-n} \) which maps a point of \( D_R \) into \( V \) and let \( h \) denote all analytic continuations of \( g(e^t) \) to \( H = \{ t : \text{Re } t > \log R \} \). Then, by the monodromy theorem, \( h \) is analytic in \( H \) and maps \( H \) onto \( V \). There are now two cases to consider.

**Case A.** The function \( h \) is univalent in \( H \) and hence \( h(H) = V \) is simply connected.

**Case B.** The function \( h \) is \( 2m\pi i \)-periodic in \( H \), for some minimal positive integer \( m \).

Indeed, if \( h \) is not univalent in \( H \), then there is some minimal positive integer \( m \) for which \( h(t_m) = h(t_m + 2m\pi i) \) for some \( t_m \in H \) and, if \( t \) is close to \( t_m \), then it follows from the open mapping theorem that there exists \( t' \) close to \( t_m + 2m\pi i \) with \( h(t) = h(t') \) and hence \( t' = t + 2m\pi i \). Thus \( h \) has period \( 2m\pi i \).

In Case B,

\[
h(t) = \varphi(e^{t/m}), \quad \text{for } t \in H,
\]
where \( \varphi(s) = a_1s + a_0 + a_1s^{-1} + \cdots \) is univalent in \( \{ s : |s| > R^{1/m} \} \), and so

\[
f^n(z) = (\varphi^{-1}(z))^m, \quad \text{for } z \in \varphi(\{ s : |s| > R^{1/m} \}).
\]

Now, if \( a_1 \neq 0 \), then \( \varphi(\{ s : |s| > R^{1/m} \}) \) includes a neighbourhood of \( \infty \), so

(2.1)

\[
f^n(z) \approx a_1^{-m}z^m \quad \text{as } z \to \infty.
\]

But (2.1) is impossible because \( \infty \) is an essential singularity of \( f^n \) and not a pole. Thus \( a_1 = 0 \) and \( \varphi \) maps \( \{ s : |s| > R^{1/m} \} \cup \{ \infty \} \) onto a simply connected region in \( \mathbb{C} \) containing \( a_0 \), and this region is \( V \).

We now use Lemma 2.1 to prove the following result.

**Lemma 2.2.** Let \( f \) be a transcendental meromorphic function. There exists \( R_f \) such that, if \( R > R_f \), \( S_n(f) \subseteq B(0, R) \) and \( |z|, |f^n(z)| > R^2 \), then

\[
|(f^n)'(z)| > \frac{|f^n(z)| \log |f^n(z)|}{16\pi |z|}.
\]
follows from Lemma 2.1 that there exists a branch $B_c$.

Then, taking $f$ the component of $p$ exist.

The lemma follows by using (2.2) where $\Phi(z) = 2\log z$. Hence $\phi(z) = 2\log z$.

Recall that Theorem A states that, if $f \in B_n$, then there is no component of $N(f)$ in which $f^{mn}(z) \to \infty$ as $m \to \infty$. We are now in a position to give a proof of this result.

**Proof of Theorem A.** If $f \in B_n$, then there exists $R > \max(e^{16\pi}, R_f)$ with $S_n(f) \subseteq B(0, R)$. If $N(f)$ has a component $U$ in which $f^{mn}(z) \to \infty$ as $m \to \infty$, then there exist $p \in N(f)$ and $r > 0$ such that $B(w, r) \subseteq f^{mn}(U)$ and $|f^{mn}(z)| > R^2$, for each $z \in B(w, r)$, $m = 0, 1, 2, \ldots.$

Now let $V_m$ be the component of $f^{−n}(D_R)$ in which $U_m = f^{mn}(B(w, r))$ lies. Then, taking $c$ to be the same periodic point as in the proof of Lemma 2.2, it follows from Lemma 2.1 that there exists a branch $L_m$ of log for which $L_m(z - c)$ is analytic in $V_m$. Next, put $T_m = L_m(U_m, c)$ and $F_n(t) = L_m(f^n(e^t + c) - c)$, so that $T_{m+1} = F_{m+1}(T_m)$. It follows from (2.2) that, if $t \in T_m$, then

$$|F'_n(t)| = \left|\frac{(f^n)'(e^t + c)e^t}{f^n(e^t + c) - c}\right| = \frac{(f^n)'(z)(z - c)}{f^n(z) - c} \geq \frac{(f^n)'(z)(z - c)}{2f^n(z)} \geq \frac{B}{2\pi} (\log |f^n(z)| - \log R) \geq \frac{B \log R}{2\pi} \geq 2,$$

and so

$$|(F_m \circ \cdots \circ F_1)'(t)| \geq 2^m, \quad \text{for } t \in T_1.$$
Thus, by Bloch’s Theorem, $T_m$ contains a disc of radius $r_m$, where $r_m \to \infty$ as $m \to \infty$. This, however, is impossible since $T_m \subseteq L_m(V_m - c)$ which contains no disc of radius greater than $\pi$.

3. Proof of Theorem B

Recall that Theorem B states that, if $f \in \hat{B}$, then $P(f)$ is bounded. Let $f \in \hat{B}$. Since $S(f) \subseteq P(f)$ and $P(f) \cap J(f) = \varnothing$, it follows that $S(f) \subseteq N(f)$ and so, since $S(f)$ is bounded, we deduce that $f \in \bigcap_{n=0}^{\infty} B_n$. The fact that $S(f) \subseteq N(f)$ also implies that

$$P(f) = \bigcup_{j=0}^{\infty} f^j(S(f)).$$

Since $S(f)$ is bounded and contained in $N(f)$, there exist $r > 0$ and a finite number of points $w_1, \ldots, w_M \in S(f)$ such that

$$S(f) \subseteq \bigcup_{i=1}^{M} B(w_i, r) \subseteq N(f).$$

It follows from (3.1) and (3.2) that

$$P(f) \subseteq \bigcup_{i=1}^{M} \bigcup_{j=0}^{\infty} f^j(B(w_i, r)).$$

Therefore, for $1 \leq i \leq M$, we let $U_i$ denote the component of $N(f)$ which contains $w_i$ and consider the possible forward orbits of $U_i$.

We first show that $U_i$ cannot be a wandering domain. If $U_i$ is a wandering domain, that is, $f^n(U_i) \cap f^m(U_i) = \varnothing$ when $n \neq m$, then there cannot exist a non-constant limit function of $\{f^n|_{U_i}\}$; see, for example, [1, Lemma 2.1]. Since $f \in B_1$, it follows from Theorem A that there exist a sequence $\{n_k\}$ and a finite value $a \in \mathbb{C}$ such that $f^{n_k}(z) \to a$ in $U_i$ as $n_k \to \infty$. Since $w_i \in S(f) \cap U_i$, it follows that $a \in \hat{P}(f)$ and, since $f \in \hat{B}$, this implies that $a \in N(f)$. This, however, is impossible if $U_i$ is a wandering domain.

Thus, $U_i$ eventually lands in a periodic cycle $\{N_0, \ldots, N_{n-1}\}$ of components of $N(f)$. Since $P(f) \cap J(f) = \varnothing$, there are no Siegel discs or Hermann rings and so, for $0 \leq p \leq n - 1$, there exists $z_p \in N_p$ with $f^{mn}(z) \to z_p$ locally uniformly in $N_p$. Since $f \in \bigcap_{n=0}^{\infty} B_n$, it follows from Theorem A that $z_p \neq \infty$, for $0 \leq p \leq n - 1$, and so $\bigcup_{j=0}^{\infty} P(B(w_i, r))$ is bounded. The result now follows from (3.3).

4. Proof of Theorem C

The proof of Theorem C uses results from earlier sections and the following two well known results. The first is Koebe’s one-quarter theorem; see for example, [5].

**Lemma 4.1.** If $f$ is univalent in $B(z, r)$, then

$$f(B(z, r)) \supset B(f(z), |f'(z)|r/4).$$

The other result we need is a basic property of Julia sets. Let

$$O^-(w) = \{z : f^n(z) = w \text{ for some } n \in \mathbb{N}\},$$

$$E(f) = \{w : O^-(w) \text{ is finite}\}.$$
If $f$ is meromorphic, then $E(f)$ contains at most two points and we have the following result; see, for example, [3, Section 2].

**Lemma 4.2.** If $U$ is compact, $U \cap E(f) = \emptyset$, $z \in J(f)$ and $V$ is an open neighbourhood of $z$, then there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have

$$f^n(V) \supset U.$$ 

Theorem C states that, if $f \in \hat{B}$, then there exist $K > 1$ and $c > 0$ such that

$$|(f^n)'(z)| > cK^n \frac{|f^n(z)| + 1}{|z| + 1},$$

for each $z \in J(f) \setminus A_n(f), n \in \mathbb{N}$.

Let $f \in \hat{B}$. We know that $\hat{P}(f) \cap J(f) = \emptyset$ and, from Theorem B, that $\hat{P}(f)$ is bounded. Thus there exist $C > 1$ and an open set $G$ containing $\hat{P}(f)$, such that

$$(4.1) \quad B\left(z, \frac{|z| + 1}{C}\right) \cap G = \emptyset,$$

for each $z \in J(f)$.

Since $\hat{P}(f)$ is bounded, it follows from Lemma 2.2 that there exists $R > 0$ such that

$$(4.2) \quad |(f^n)'(z)| > 16C\frac{|f^n(z)| + 1}{|z| + 1}, \quad \text{for } n \in \mathbb{N}, \ |z| > R, \ |f^n(z)| > R.$$ 

We now claim that there exists $N_1 \in \mathbb{N}$ such that

$$(4.3) \quad |(f^n)'(z)| > 16C\frac{|f^n(z)| + 1}{|z| + 1}, \quad \text{for } n \geq N_1, \ z \in (J(f) \setminus A_n(f)) \cap \hat{B}(0, R).$$

Otherwise, there exists a sequence of points $z_{n_k} \in (J(f) \setminus A_{n_k}(f)) \cap \hat{B}(0, R)$ such that

$$|(f^{n_k})'(z_{n_k})| \leq 16C\frac{|f^{n_k}(z_{n_k})| + 1}{|z_{n_k}| + 1},$$

with $z_{n_k} \rightarrow \alpha \in J(f) \cap \hat{B}(0, R)$ as $n_k \rightarrow \infty$.

It follows from (4.1) and Lemma 4.1 that, if $g$ is the branch of $f^{-n_k}$ that maps $f^{n_k}(z_{n_k})$ to $z_{n_k}$, then

$$g\left(B\left(f^{n_k}(z_{n_k}), \frac{|f^{n_k}(z_{n_k})| + 1}{C}\right)\right) \supseteq B\left(z_{n_k}, \frac{|z_{n_k}| + 1}{64C^2}\right).$$

Thus, for large $n_k$,

$$f^{n_k}\left(B\left(\alpha, \frac{1}{100C^2}\right)\right) \subseteq f^{n_k}\left(B\left(z_{n_k}, \frac{|z_{n_k}| + 1}{64C^2}\right)\right) \subseteq B\left(f^{n_k}(z_{n_k}), \frac{|f^{n_k}(z_{n_k})| + 1}{C}\right)$$

and so, by (4.1),

$$f^n\left(B\left(\alpha, \frac{1}{100C^2}\right)\right) \cap G = \emptyset,$$

for arbitrarily large values of $n$. Since $\alpha \in J(f)$, this contradicts Lemma 4.2, and hence (4.3) is true.
Thus, by (4.1),

\[(f^n)'(z) > \frac{1}{8C} \frac{|f^n(z)| + 1}{|z| + 1}.\]

Otherwise, there exists a sequence of points \(z_n \in J(f) \setminus A_n(f)\) such that

\[
|(f^n)'(z_n)| \leq \frac{1}{8C} \frac{|f^n(z_n)| + 1}{|z_n| + 1},
\]

with \(z_n \to \alpha \in J(f)\) or \(z_n \to \infty\) as \(n \to \infty\).

It follows from (4.1) and Lemma 4.1 that, if \(g\) is the branch of \(f^{-n_k}\) that maps \(f^{n_k}(z_n)\) to \(z_n\), then

\[
g \left( B \left( f^{n_k}(z_n), \frac{|f^{n_k}(z_n)| + 1}{C} \right) \right) \supseteq B(z_n, 2(|z_n| + 1)) \supseteq B(0, |z_n| + 1).
\]

Since \(z_n \to \alpha \in J(f)\) or \(z_n \to \infty\) as \(n \to \infty\), there exist \(\beta \in J(f), r > 0\) such that, for large values of \(n_k\),

\[
B(\beta, r) \subseteq B(0, |z_n| + 1),
\]

and hence

\[
f^{n_k}(B(\beta, r)) \subseteq f^{n_k}(B(0, |z_n| + 1)) \subseteq B \left( f^{n_k}(z_n), \frac{|f^{n_k}(z_n)| + 1}{C} \right).
\]

Thus, by (4.1),

\[
f^n(B(\beta, r)) \cap G = \emptyset,
\]

for arbitrarily large values of \(n\). Since \(\beta \in J(f)\), this contradicts Lemma 4.2, and hence (4.4) is true.

We now put \(N = \max(N_1, N_2)\). If \(z \in J(f) \setminus A_{2N+p}(f)\), then it follows from (4.2) and (4.3) that

\[
|(f^{2N+p})'(z)| > 16C \frac{|f^{2N+p}(z)| + 1}{|z| + 1},
\]

for each \(p \in \mathbb{N} \cup \{0\}\), provided that either \(|z| \leq R\) or \(|z|, |f^{2N+p}(z)| > R\).

If \(z \in J(f) \setminus A_{2N+p}(f), |z| > R\) and \(|f^{2N+p}(z)| \leq R\), then either \(|f^N(z)| \leq R\) in which case, by (4.3) and (4.4),

\[
|(f^{2N+p})'(z)| > \frac{1}{8C} \frac{|f^N(z)| + 1}{|z| + 1} 16C \frac{|f^{2N+p}(z)| + 1}{|f^N(z)| + 1} = 2 \frac{|f^{2N+p}(z)| + 1}{|z| + 1},
\]

or \(|f^N(z)| > R\) in which case, by (4.2) and (4.4),

\[
|(f^{2N+p})'(z)| > 16C \frac{|f^N(z)| + 1}{|z| + 1} \frac{1}{8C} \frac{|f^{2N+p}(z)| + 1}{|f^N(z)| + 1} = 2 \frac{|f^{2N+p}(z)| + 1}{|z| + 1}.
\]

It follows from (4.5), (4.6) and (4.7) that, for each \(z \in J(f) \setminus A_{2N+p}(f), p \in \mathbb{N} \cup \{0\}\), we have

\[
|(f^{2N+p})'(z)| > 2 \frac{|f^{2N+p}(z)| + 1}{|z| + 1}.
\]
If \( n \geq 2N \), then there exist \( m \in \mathbb{N} \), \( 0 \leq p < 2N \) such that \( n = m2N + p \) and so, if \( z \in J(f) \setminus A_n(f) \) and \( n \geq 4N \), then it follows from (4.8) that

\[
| (f^n)'(z) | > 2^m | f^n(z) | + 1 \quad > (2^{1/2})^n | f^n(z) | + 1.
\]

To complete the proof of Theorem C we need to show that there exist \( c_n > 0 \), for \( n = 1, 2, \ldots, 4N - 1 \), such that

\[
| (f^n)'(z) | > c_n \left( \frac{| f^n(z) | + 1}{| z | + 1} \right),
\]

for \( z \in J(f) \setminus A_n(f) \).

If this is not true, then there exist \( m \in \mathbb{N} \) and a sequence of points \( z_k \in J(f) \setminus A_m(f) \) such that

\[
\varepsilon_k = \left( \frac{| (f^m)'(z_k) | (| z_k | + 1)}{| f^m(z_k) | + 1} \right) \quad \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]

An argument similar to the proof of (4.4) with \( \varepsilon_k \) instead of \( 1/(8C) \) and \( m \) instead of \( n_k \) now leads to the fact that, for large \( k \),

\[
f^m \left( B \left( 0, \frac{| z_k | + 1}{8C \varepsilon_k} \right) \right) \cap G = \emptyset.
\]

Thus \( f^m(C) \cap G = \emptyset \), which is a contradiction, and so the proof of Theorem C is now complete.

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REFERENCES


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