

ON MODELS OF $U_q(sl(2))$ AND q -APPELL FUNCTIONS USING A q -INTEGRAL TRANSFORMATION

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ABSTRACT. We discuss few models of the quantum universal enveloping algebra of $sl(2)$ from the special function point of view. Two sets of such models are given, one acting on the space of ${}_1\phi_0$ functions while the other on the space of q -Appell functions. These models are closely related through a q -integral transformation. Some interesting identities are obtained.

1. INTRODUCTION

Quantum groups have generated considerable interest among mathematicians and physicists alike. The theory of q -special functions has found a group theoretic interpretation using the techniques of quantum groups and quantum algebras. In this direction major contributions have come from the works of Floreanini and Vinet [6]–[12], Kalnins and Miller [18], to name a few.

In a recent paper [29], the author has constructed models of irreps of classical $sl(2)$ in terms of difference-differential operators using an integral transformation which is inspired by the well known Euler integral representation for the hypergeometric functions ${}_2F_1$ with extension to some other functions as well. The models in [29] have culminated in various recurrence relations and identities involving ordinary hypergeometric series. To have an extension of this to q -series, we introduce a q -integral transformation, which is a generalization of and is motivated by Thomae's integral representation of ${}_2\phi_1$, given in [13], to construct models of $U_q(sl(2))$, the quantum universal enveloping algebra of $sl(2)$, in terms of difference-dilation operators. These models act on a space of basis functions involving q -Appell functions and give rise to identities involving certain generalized q -hypergeometric series in two and three variables. A q -difference equation satisfied by q -Appell functions is also obtained. Section-wise treatment is as follows.

In Section 2, we list various definitions and results needed for our discussion. In Section 3, we discuss models of $U_q(sl(2))$ and give its two variable models in terms of dilation operators in which ${}_1\phi_0$ functions appear as the basis functions. In Section 4, a q -integral transformation is defined and transforms of certain expressions are obtained. Next, we reproduce two theorems which help us in upgrading the models of Section 3 into new models of $U_q(sl(2))$ in terms of difference-dilation operators

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in which the basis functions involve $\Phi^{(2)}$, the q -Appell functions. These models, in Section 5, are exploited for identities, which are believed to be new. Finally in Section 6, we outline the corresponding transformed models for a m -fold q -integral transformation.

2. PRELIMINARIES

The generalized basic (or q -) hypergeometric series is defined by [13]

$$(1) \quad {}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} x^n,$$

where

$$(2) \quad (a_1, \dots, a_r; q)_n = \prod_{i=1}^r (a_i; q)_n$$

and q -shifted factorial $(a; q)_n$ is defined by

$$(3) \quad (a; q)_n = \begin{cases} (1-a)(1-aq) \cdots (1-aq^{n-1}), & n = 1, 2, \dots, \\ 1, & n = 0, \\ [(1-aq^{-1}) \cdots (1-aq^{-n})]^{-1}, & n = -1, -2, \dots, \end{cases}$$

$$(4) \quad (a; q)_{\infty} = \prod_{r=0}^{\infty} (1-aq^r).$$

The series ${}_r\phi_s$ terminates if one of the numerator parameter is of the form q^{-m} , $m = 0, 1, 2, \dots$ and $q \neq 0$. When $0 < |q| < 1$, the series ${}_r\phi_s$ converges absolutely for all z if $r \leq s$, and for $|z| < 1$ if $r = s + 1$. If $|q| > 1$ and $|z| < |b_1 \cdots b_s|/|a_1 \cdots a_r|$, then also ${}_r\phi_s$ converges absolutely. The series ${}_r\phi_s$ diverges for $z \neq 0$ when $0 < |q| < 1$ and $r > s + 1$, and when $|q| > 1$ and $|z| > |b_1 \cdots b_s|/|a_1 \cdots a_r|$, unless it terminates.

We now list q -analogues of some important functions which will be needed in the discussion. The q -analogues of the exponential function are

$$(5) \quad e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_{\infty}}, \quad |x| < 1,$$

and

$$(6) \quad E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{(q; q)_n} = (-x; q)_{\infty}.$$

$E_q(x)$ converges for all x . Note that $e_q(x)E_q(-x) = 1$. The q -analogue of the binomial function is

$$(7) \quad {}_1\phi_0 \left(\begin{matrix} a \\ - \end{matrix}; x \right) = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad |x| < 1, |q| < 1.$$

The gamma function has the following q -analogue:

$$(8) \quad \Gamma_q(\alpha) = \frac{e_q(q^{\alpha})}{e_q(q)} (1-q)^{1-\alpha}, \quad \alpha \neq 0, -1, -2, \dots,$$

and satisfies the equation

$$(9) \quad \Gamma_q(\alpha + 1) = \frac{1-q^{\alpha}}{1-q} \Gamma_q(\alpha).$$

The q -derivative operator is defined as

$$(10) \quad \Delta_z f(z) = \frac{(1 - T_z) f(z)}{(1 - q)z}$$

where the q -dilation operator T_z is defined by $T_z f(z) = f(qz)$. Indeed, as $q \rightarrow 1$, the q -derivative operator becomes an ordinary differential operator, provided f is differentiable at z .

The q -integral is defined as

$$(11) \quad \int_0^1 f(t) d_q t = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n.$$

If f is continuous on $[0, 1]$, then the q -integral reduces to the ordinary integral.

Also we need the q -analogue of Appell function F_2 , [33], as

$$(12) \quad \Phi^{(2)} \left(\begin{matrix} a, b, b' \\ c, c' \end{matrix}; x, y \right) = \sum_{m,n=0}^{\infty} \frac{(a; q)_{m+n} (b; q)_m (b'; q)_n}{(c, q; q)_m (c', q; q)_n} x^m y^n$$

and the q -analogue of generalized Kampé de Fériet function $F_{E:F;G;H}^{A:B;C;D}$, as

$$(13) \quad \phi_{E:F;G;H}^{A:B;C;D} \left(\begin{matrix} (a_i); (b_j); (c_k); (d_l) \\ (e_m); (f_n); (g_u); (h_v) \end{matrix}; x, y, z \right) \\ = \sum_{p,r,t=0}^{\infty} \frac{((a_i); q)_{p+r+t} ((b_j); q)_p ((c_k); q)_r ((d_l); q)_t}{((e_m); q)_{p+r+t} ((f_n), q; q)_p ((g_u), q; q)_r ((h_v), q; q)_t} x^p y^r z^t$$

where (a_i) abbreviates parameters a_1, \dots, a_A , and where $((a_i); q)_{p+r+t}$ denotes $(a_1, \dots, a_A; q)_{p+r+t}$, etc. The arguments and the complex parameters in (12) and (13) are so constrained that the multiple series converges.

3. MODELS OF $U_q(sl(2))$

The commutation relations among the generators e, f, h

$$(14) \quad [h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = \frac{q^{h/2} - q^{-h/2}}{q^{1/2} - q^{-1/2}}$$

define an associative algebra [14], which may be regarded as the q -deformation of the universal enveloping algebra of $sl(2)$, referred to as $U_q(sl(2))$ in this paper. The Casimir element which belongs to the center of $U_q(sl(2))$ is given by

$$(15) \quad \mathbb{C} = \frac{k^2 q^{1/2} + k^{-2} q^{-1/2} - 2}{q^{1/2} - q^{-1/2}} + fe.$$

As the parameter $q \rightarrow 1^-$, these reduce to familiar commutator rules for $sl(2)$.

Let V_q be a complex vector space consisting of q -special functions with a basis $\{\phi_\lambda : \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q \rightarrow 1} \phi_\lambda : \lambda \in S\}$ form a basis for the vector space, say V . Let $A(V_q)$ be the associative algebra of all linear operators on V_q over a complex field. Then a representation ρ_q of $sl(2)$ on V_q is defined as a mapping $\rho_q : sl(2) \rightarrow A(V_q)$ satisfying

(i) $\rho_q(ax + by) = a\rho_q(x) + b\rho_q(y)$.

(ii) There exists a Lie algebra representation ρ of $sl(2)$ on V such that

$$\lim_{q \rightarrow 1} \rho_q(x)\phi_\lambda = \rho(x)f_\lambda,$$

for all $x, y \in sl(2)$ and $a, b \in C$.

This representation ρ_q of $sl(2)$ is said to be irreducible if there is no proper subspace W_q of V_q which is invariant under ρ_q . Define

$$(16) \quad J_q^+ = \rho_q(\mathcal{J}^+), \quad J_q^- = \rho_q(\mathcal{J}^-), \quad J_q^0 = \rho_q(\mathcal{J}^0)$$

where $J_q^+, J_q^-, J_q^0 \in A(V_q)$ and $\{\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}^0\}$ form a basis of $sl(2)$. On his work on irreps of $sl(2)$, Manocha [25] has defined the following commutator rules satisfied by J_q -operators:

$$(17) \quad \begin{aligned} J_q^0 J_q^+ - q J_q^+ J_q^0 &= J_q^+, \\ q J_q^0 J_q^- - J_q^- J_q^0 &= -J_q^-, \\ q J_q^+ J_q^- - J_q^- J_q^+ &= 2q^{2u} J_q^0 - (1-q)q^{2u} J_q^0 J_q^0, \quad u \in C. \end{aligned}$$

If we define an operator C_q on V_q by

$$(18) \quad C_q = q J_q^+ J_q^- + q^{2u} J_q^0 J_q^0 - q^{2u} J_q^0,$$

then it can be verified that

$$(19) \quad \begin{aligned} q J_q^+ C_q &= C_q J_q^+, \\ J_q^- C_q &= q C_q J_q^-, \\ J_q^0 C_q &= C_q J_q^0. \end{aligned}$$

As will be seen, the operator C_q will be used to derive a q -difference equation satisfied by q -Appell function. Further, as $q \rightarrow 1$, the operators J_q^+, J_q^-, J_q^0 reduce to J^+, J^-, J^0 which satisfy the commutation relations obeyed by $sl(2)$ and the operator C_q reduces to the Casimir operator C and the relations (19) exhibit that C commutes with J^+, J^- and J^0 . Hence, as $q \rightarrow 1$, ρ_q reduces to a Lie algebra representation ρ of $sl(2)$.

Notice that the commutation relations given by eqs. (17) are equivalent to that given by (14) in the sense that if the operators e, f, h satisfy (14), then J_q -operators defined by

$$(20) \quad J_q^+ = e, \quad J_q^- = q^{2u + \frac{h-1}{2}} f, \quad J_q^0 = \frac{1 - q^{h/2}}{1 - q}$$

satisfy (17).

Following the analysis as in Manocha [25] and Miller [27], we have the following theorem:

Theorem 3.1. *Every representation ρ_q of $sl(2)$ is isomorphic to a representation in the following list:*

- (i) *The representation $D_q(\alpha, u)$, $\alpha \in C \setminus \{0\}$, $0 \leq \operatorname{Re} \alpha < 1$, such that $\alpha - 2u$ is not an integer. $S = \{\alpha + n : n = 0, \pm 1, \pm 2, \dots\}$.*
- (ii) *The representation $\uparrow_q(u)$, $u \in C$ and $2u$ is not a non-negative integer. $S = \{0, 1, 2, \dots\}$.*
- (iii) *The representation $D_q(2u)$ where $2u$ is a non-negative integer. $S = \{0, 1, 2, \dots, 2u\}$.*

For each of these representations there is a basis of V_q consisting of vectors $\{f_\lambda\}$, defined for each $\lambda \in S$ such that

$$\begin{aligned}
 J_q^0 f_\lambda &= \frac{1 - q^{\lambda-u}}{1 - q} f_\lambda, \\
 J_q^+ f_\lambda &= \frac{q^{2u} - q^\lambda}{1 - q} f_{\lambda+1}, \\
 J_q^- f_\lambda &= -\frac{1 - q^\lambda}{1 - q} f_{\lambda-1}.
 \end{aligned}
 \tag{21}$$

Guided by the above theorem, we give below two-variable models of representation $D_q(\alpha)$ in the particular case when $u = 0$.

Model I

$$\begin{aligned}
 J_q^+ &= t(zT_t\Delta_z + t\Delta_t), \\
 J_q^- &= t^{-1}T_z^{-1}(z(1 - zT_t)\Delta_z - t\Delta_t), \\
 J_q^0 &= t\Delta_t,
 \end{aligned}
 \tag{22}$$

$$f_\lambda = \frac{(q^\lambda z; q)_\infty}{(z; q)_\infty} t^\lambda = {}_1\phi_0 \left(\begin{matrix} q^\lambda \\ - \end{matrix}; z \right) t^\lambda, \quad \lambda \in S.
 \tag{23}$$

Model II

$$\begin{aligned}
 J_q^+ &= t(z(1 - z)T_t\Delta_z + t\Delta_t), \\
 J_q^- &= t^{-1}T_z^{-1}(z\Delta_z - t\Delta_t), \\
 J_q^0 &= t\Delta_t,
 \end{aligned}
 \tag{24}$$

$$f_\lambda = \frac{(z; q)_\infty}{(q^\lambda z; q)_\infty} t^\lambda = {}_1\phi_0 \left(\begin{matrix} q^{-\lambda} \\ - \end{matrix}; q^\lambda z \right) t^\lambda, \quad \lambda \in S,
 \tag{25}$$

where $S = \{\alpha + n : \alpha \in C \setminus \{0\}, 0 \leq \text{Re } \alpha < 1, n = 0, \pm 1, \pm 2, \dots\}$. The dilation models given above satisfy (17) as well as (21) with $u = 0$.

A model of representation \uparrow_q is same as Model II above with $S = \{0, 1, 2, \dots\}$. However, we concentrate on the model of representation $D_q(\alpha)$ and will return to the representation \uparrow_q later in Section 5, where we obtain identities based on a model of representation \uparrow_q .

The models of irreducible representation of $U_q(sl(2))$ can be found in the literature too. For example, in [12], Floreanini and Vinet have considered two such models. Also, models of $U_q(su_2)$ are discussed in detail by Kalnins *et al.* [17, 19, 20, 21] and by Rideau and Winternitz [28]. The metaplectic representation of $su_q(1, 1)$ are presented in [8] and are connected with q -Gegenbauer polynomials. Models of q -harmonic oscillators given by Biedenharn [1] and Mcfarlane [26] are closely related to the representation of $U_q(sl(2))$. Work on these representations has been carried out by Drinfeld [3], Faddeev [5], Kulish and Reshetikhin [23], Sklyanin [31, 32] and Woronowich [34, 35] among others. Some models of q -representations of the Lie algebra $\mathcal{G}(a, b)$ are contained in Sahai [30]. For more details, see Chari and Pressley [2], Jantzen [15], Kassel [22] and Majid [24].

4. A q -INTEGRAL TRANSFORMATION AND TRANSFORMED MODELS

In this section, we introduce a two-fold q -integral transformation, based on Thomae's integral representation of ${}_2\phi_1$ [13, 4], viz.

$$(26) \quad {}_2\phi_1 \left(\begin{matrix} q^\alpha, q^\beta \\ q^\gamma \end{matrix}; q, z \right) = \Gamma_q \left(\begin{matrix} \gamma \\ \beta, \gamma - \beta \end{matrix} \right) \int_0^1 t^{\beta-1} \frac{(tq; q)_\infty}{(tq^{\gamma-\beta}; q)_\infty} {}_1\phi_0 \left(\begin{matrix} q^\alpha \\ - \end{matrix}; q, tz \right) d_q t$$

and then compute transforms of certain operator expressions. These transforms are later used to obtain new models from the ones given in Section 3 in which the basis functions will be in terms of q -Appell function.

Define

$$(27) \quad h(\beta, \beta', \gamma, \gamma') = \mathcal{I}[f(u, v)] = \Gamma_q \left(\begin{matrix} \gamma, \gamma' \\ \beta, \beta', \gamma - \beta, \gamma' - \beta' \end{matrix} \right) \\ \times \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} \frac{(qu, qv; q)_\infty}{(q^{\gamma-\beta}u, q^{\gamma'-\beta'}v; q)_\infty} f(u, v) d_q u d_q v,$$

$$\operatorname{Re} \gamma > \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma' > \operatorname{Re} \beta' > 0.$$

Defining the operators

$$(28) \quad E_\beta(h(\beta, \beta', \gamma, \gamma')) = h(\beta + 1, \beta', \gamma, \gamma'), \\ L_\beta(h(\beta, \beta', \gamma, \gamma')) = h(\beta - 1, \beta', \gamma, \gamma'),$$

and denoting $E_{\beta\gamma} = E_\beta E_\gamma$, etc. we have the following transforms of operator expressions needed for our discussion:

$$(29) \quad \begin{aligned} \mathcal{I}[u\Delta_u f] &= x\Delta_x h, \\ \mathcal{I}[v\Delta_v f] &= y\Delta_y h, \\ \mathcal{I}[u^2\Delta_u f] &= \frac{1-q^\beta}{1-q^\gamma} x\Delta_x E_{\beta\gamma} h, \\ \mathcal{I}[v^2\Delta_v f] &= \frac{1-q^{\beta'}}{1-q^{\gamma'}} y\Delta_y E_{\beta'\gamma'} h, \\ \mathcal{I}[\Delta_u f] &= \frac{1-q^{\gamma-1}}{1-q^{\beta-1}} x\Delta_x L_{\beta\gamma} h, \\ \mathcal{I}[\Delta_v f] &= \frac{1-q^{\gamma'-1}}{1-q^{\beta'-1}} y\Delta_y L_{\beta'\gamma'} h, \\ \mathcal{I}[uf] &= \frac{1-q^\beta}{1-q^\gamma} E_{\beta\gamma} h, \\ \mathcal{I}[vf] &= \frac{1-q^{\beta'}}{1-q^{\gamma'}} E_{\beta'\gamma'} h. \end{aligned}$$

To obtain models of $U_q(sl(2))$ in terms of difference-dilation operators (difference in $\beta, \beta', \gamma, \gamma'$ and dilation in x and y) in which the basis functions are in terms of q -Appell functions, we need the following theorems.

Theorem 4.1 ([25]). *Let ρ_q be a representation of $sl(2)$ in terms of $\{J_q^+, J_q^-, J_q^0\}$ with basis functions $\{f_\lambda : \lambda \in S\}$. Then ρ_q is also a representation of $sl(2)$ in terms of $\{K_q^+, K_q^-, K_q^0\}$ with basis functions $\{h_\lambda : \lambda \in S\}$ where*

$$(30) \quad K_q^+ = \mathcal{I}J_q^+ \mathcal{I}^{-1}, \quad K_q^- = \mathcal{I}J_q^- \mathcal{I}^{-1}, \quad K_q^0 = \mathcal{I}J_q^0 \mathcal{I}^{-1}, \quad h_\lambda = \mathcal{I}f_\lambda.$$

Theorem 4.2 ([13]). *If a and b are indeterminates such that $ab = qba$, q commutes with a and b and the associative law holds, then*

$$(31) \quad (a + b)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q b^k a^{n-k},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q -binomial coefficient.

To upgrade the two-variable models to three-variable ones, we substitute $z = ux + vy$. This gives

$$(32) \quad z\Delta_z = u\Delta_u + v\Delta_v T_u,$$

where Δ_z is defined by (10) and

$$(33) \quad T_z f(z) = f(qz) = T_u T_v f(ux + vy).$$

Putting $z = ux + vy$, the Model I takes the following shape:

$$(34) \quad \begin{aligned} J_q^+ &= t[(u\Delta_u + v\Delta_v T_u) T_t + t\Delta_t], \\ J_q^- &= t^{-1} T_u^{-1} T_v^{-1} [(1 - T_t(ux + vy))(u\Delta_u + v\Delta_v T_u) - t\Delta_t], \\ J_q^0 &= t\Delta_t, \end{aligned}$$

with the basis functions as

$$(35) \quad f_\lambda = {}_1\phi_0 \left(\begin{matrix} q^\lambda \\ - \end{matrix}; ux + vy \right) t^\lambda, \quad \lambda \in S.$$

The J -model (34) along with (35) is now ready to be transformed into a K -model involving the q -Appell function $\Phi^{(2)}$ as the basis functions with the help of Theorems 4.1 and 4.2. We have

Model IA.

$$(36) \quad \begin{aligned} K_q^+ &= \mathcal{I} J_q^+ \mathcal{I}^{-1} = t[(x\Delta_x + y\Delta_y T_x) T_t + t\Delta_t], \\ K_q^- &= \mathcal{I} J_q^- \mathcal{I}^{-1} \\ &= t^{-1} T_x^{-1} T_y^{-1} \left[x\Delta_x + y\Delta_y T_x - xT_t \left(\frac{1 - q^\beta}{1 - q^\gamma} x\Delta_x E_{\beta\gamma} + \frac{1 - q^\beta}{1 - q^\gamma} y\Delta_y T_x E_{\beta\gamma} \right) \right. \\ &\quad \left. - yT_t \left(\frac{1 - q^{\beta'}}{1 - q^{\gamma'}} x\Delta_x E_{\beta'\gamma'} + \frac{1 - q^{\beta'}}{1 - q^{\gamma'}} y\Delta_y T_x E_{\beta'\gamma'} \right) - t\Delta_t \right], \\ K_q^0 &= \mathcal{I} J_q^0 \mathcal{I}^{-1} = t\Delta_t, \end{aligned}$$

with the basis functions as

$$(37) \quad h_\lambda(x, y, t) = \Phi^{(2)} \left(\begin{matrix} q^\lambda; b, b' \\ c, c' \end{matrix}; x, y \right) t^\lambda, \quad \lambda \in S,$$

where $b = q^\beta, b' = q^{\beta'}, c = q^\gamma, c' = q^{\gamma'}$. Note that Theorem 4.2 is utilized in obtaining $\mathcal{I}({}_1\phi_0) = \Phi^{(2)}$.

Likewise Model II is transformed to the following

Model IIA.

(38)

$$K_q^+ = t \left[(x\Delta_x + y\Delta_y T_x) T_t - xT_t \left(\frac{1-q^\beta}{1-q^\gamma} x\Delta_x E_{\beta\gamma} + \frac{1-q^\beta}{1-q^\gamma} y\Delta_y T_x E_{\beta\gamma} \right) \right. \\ \left. - yT_t \left(\frac{1-q^{\beta'}}{1-q^{\gamma'}} x\Delta_x E_{\beta'\gamma'} + \frac{1-q^{\beta'}}{1-q^{\gamma'}} y\Delta_y T_x E_{\beta'\gamma'} \right) + t\Delta_t \right], \\ K_q^- = t^{-1} T_x^{-1} T_y^{-1} (x\Delta_x + y\Delta_y T_x - t\Delta_t), \\ K_q^0 = t\Delta_t,$$

with the basis functions as

$$(39) \quad h_\lambda(x, y, t) = \Phi^{(2)} \left(q^{-\lambda}; b, b', c, c'; q^\lambda x, q^\lambda y \right) t^\lambda, \quad \lambda \in S.$$

The K^- -operators given above satisfy the following relations:

$$(40) \quad K_q^0 h_\lambda = \frac{1-q^\lambda}{1-q} h_\lambda, \\ K_q^+ h_\lambda = \frac{1-q^\lambda}{1-q} h_{\lambda+1}, \\ K_q^- h_\lambda = -\frac{1-q^\lambda}{1-q} h_{\lambda-1}, \\ C_q h_\lambda = 0 \text{ where } C_q = qK_q^+ K_q^- + K_q^0 K_q^0 - K_q^0$$

as well as the commutation relations

$$(41) \quad K_q^0 K_q^+ - qK_q^+ K_q^0 = K_q^+, \\ qK_q^0 K_q^- - K_q^- K_q^0 = -K_q^-, \\ qK_q^+ K_q^- - K_q^- K_q^+ = 2K_q^0 - (1-q)K_q^0 K_q^0$$

and as such give difference-dilation models of $U_q(sl(2))$.

Note that the equation $C_q h_\lambda = 0$ in (40) can be used to obtain a q -difference equation satisfied by $\Phi^{(2)}$. For example, if h_λ is given by (37), then we get the q -difference equation

$$(42) \quad \left\{ (1-aq^{-1}) + aq^{-1} (xD_x^+ + yD_y^+ T_x) \right\} \\ \times \left\{ a (xD_x^- + yD_y^- T_x) \left(\frac{1-b}{1-c} xT_{bc} + \frac{1-b'}{1-c'} yT_{b'c'} - 1 \right) - (1-a) \right\} \\ + q(1-a)(1-aq^{-1}) \Phi^{(2)} \left(a; b, b', c, c'; x, y \right) = 0$$

where $D_x^+ = x^{-1}(1-T_x)$ and $D_x^- = x^{-1}(1-T_x^{-1})$ are q -difference operators [11]. Observe that $\frac{1}{1-q} D_x^+ \rightarrow \frac{\partial}{\partial x}$ and $\frac{1}{1-q^{-1}} D_x^- \rightarrow \frac{\partial}{\partial x}$ as $q \rightarrow 1$.

Further, we have

$$(43) \quad qK_q^+ C_q = C_q K_q^+, \\ K_q^- C_q = qC_q K_q^-, \\ K_q^0 C_q = C_q K_q^0.$$

The eqs. (43) will become instrumental in obtaining identities.

5. IDENTITIES

5.1. **Based on Model IA.** We have shown in eq. (40) that

$$(44) \quad h_\lambda(x, y, t) = \Phi^{(2)} \left(\begin{matrix} q^\lambda; b, b' \\ c, c' \end{matrix}; x, y \right) t^\lambda, \quad \lambda \in S,$$

is a solution of $C_q h_\lambda(x, y, t) = 0$ where $C_q = qK_q^+ K_q^- + K_q^0 K_q^0 - K_q^0$. This suggests that

$$(45) \quad \begin{aligned} u(x, y, t) &= \phi_{-:1;1;1}^{1:1;1;2} \left(\begin{matrix} a : b; b', b'' \\ - : c; c', c'' \end{matrix}; x, y, t \right) t^\alpha \\ &= \sum_{n=0}^{\infty} \frac{(a, b'', b'''; q)_n}{(c'', q; q)_n} \Phi^{(2)} \left(\begin{matrix} aq^n; b, b' \\ c, c' \end{matrix}; x, y \right) t^{\alpha+n} \end{aligned}$$

is a solution of $C_q u(x, y, t) = 0$. Using the fact that $qK_q^+ C_q = C_q K_q^+$, we have

$$(46) \quad C_q [e_q(sK_q^+)u](x, y, t) = 0,$$

where

$$(47) \quad \begin{aligned} [e_q(sK_q^+)u](x, y, t) &= \sum_{n=0}^{\infty} \frac{s^n}{(q; q)_n} K_q^{+n} u(x, y, t) \\ &= \frac{(\frac{ast}{1-q}; q)_\infty}{(\frac{st}{1-q}; q)_\infty} \phi_{1:1;1;1}^{1:1;1;2} \left(\begin{matrix} a : b; b', b'' \\ \frac{ast}{1-q} : c; c', c'' \end{matrix}; x, y, t \right) t^\alpha. \end{aligned}$$

Now we have, using the Weisner's expansion, as explained in [25, 16]

$$(48) \quad [e_q(sK_q^+)u](x, y, t) = \sum_{n=0}^{\infty} A_n h_{\alpha+n}(x, y, t),$$

where A_n is found by putting $x = y = 0$. Rescaling suitably, we eventually arrive at

$$(49) \quad \begin{aligned} &\frac{(at; q)_\infty}{(t; q)_\infty} \phi_{1:1;1;1}^{1:1;1;2} \left(\begin{matrix} a : b; b', b'' \\ at : c; c', c'' \end{matrix}; x, y, \omega t \right) \\ &= \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \Phi^{(2)} \left(\begin{matrix} aq^n; b, b' \\ c, c' \end{matrix}; x, y \right) {}_3\phi_1 \left(\begin{matrix} q^{-n}, b'', b''' \\ c'' \end{matrix}; q, q^n \omega \right) t^n, \end{aligned}$$

where $\omega = \frac{s}{1-q}$.

5.2. **Based on Model IIA.** We follow exactly the same approach as given in section 5.1 with the change that we use the other q -exponential E_q instead of e_q .

We have shown that $C_q h_\lambda = 0$, where h_λ is given by eq.(39). It therefore follows that

$$(50) \quad u(x, y, t) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q, c''; q)_n} \Phi^{(2)} \left(\begin{matrix} a^{-1}q^{-n}; b, b' \\ c, c'; aq^n x, aq^n y \end{matrix} \right) t^{\alpha+n}$$

is a solution of $C_q u(x, y, t) = 0$. Considering that $K_q^- C_q = qC_q K_q^-$, we have

$$(51) \quad C_q [E_q(sK_q^-)u](x, y, t) = 0,$$

where

(52)

$$\begin{aligned} [E_q(sK_q^-)u](x, y, t) &= \sum_{n=0}^{\infty} \frac{q^{(2)}s^n}{(q; q)_n} K_q^{-n} u(x, y, t) \\ &= \frac{(\frac{\omega}{t}; q)_{\infty}}{(\frac{a\omega}{t}; q)_{\infty}} \sum_{k, n=0}^{\infty} \frac{(a, \frac{a\omega}{t}; q)_n (a^{-1}q^{-n}, b; q)_k}{(c'', q; q)_n (\frac{q^{1-n}t}{a\omega}, c, q; q)_k} \\ &\quad \times {}_2\phi_2 \left(\begin{matrix} a^{-1}q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{a\omega}, c' \end{matrix}; q, \frac{-q^{k+1}yt}{\omega} \right) q^{(k)} \left(\frac{-qxt}{\omega} \right)^k t^{\alpha+n}. \end{aligned}$$

By Weisner's expansion, we have

$$(53) \quad [E_q(sK_q^-)u](x, y, t) = \sum_{n=-\infty}^{\infty} B_n h_{\alpha+n}(x, y, t),$$

where B_n is obtained by putting $x = y = 0$. This leads to

$$\begin{aligned} (54) \quad &\frac{(\frac{\omega}{t}; q)_{\infty}}{(\frac{a\omega}{t}; q)_{\infty}} \sum_{k, n=0}^{\infty} \frac{(a, \frac{a\omega}{t}; q)_n (a^{-1}q^{-n}, b; q)_k}{(c'', q; q)_n (\frac{q^{1-n}t}{a\omega}, c, q; q)_k} \\ &\times {}_2\phi_2 \left(\begin{matrix} a^{-1}q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{a\omega}, c' \end{matrix}; q, \frac{-q^{k+1}yt}{\omega} \right) \left(\frac{-q^{\frac{k+1}{2}}xt}{\omega} \right)^k t^n \\ &= \sum_{n=-\infty}^{\infty} \frac{\Gamma_q(\alpha+n)\Gamma_q(\gamma'')}{\Gamma_q(\alpha)\Gamma_q(\gamma''+n)\Gamma_q(q+1)} {}_2\phi_2 \left(\begin{matrix} aq^n, aq^{n+1} \\ q^{n+1}, c''q^n \end{matrix}; q, \omega \right) \\ &\times \Phi^{(2)} \left(\begin{matrix} a^{-1}q^{-n}; b, b' \\ c, c'; aq^n x, aq^n y \end{matrix} \right) t^n, \end{aligned}$$

where $\omega = \frac{s}{1-q}$.

We have discussed above identities corresponding to models of representation $D_q(\alpha)$. We now consider the Model IIA for the representation \uparrow_q (i.e. eqs. (38) and (39) with $\lambda \in S = \{n = 0, 1, 2, \dots\}$). It can be verified that

$$(55) \quad u(x, y, t) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \Phi^{(2)} \left(\begin{matrix} q^{-n}; b, b' \\ c, c'; q^n x, q^n y \end{matrix} \right) t^n$$

satisfies $C_q u(x, y, t) = 0$. Using the relation $\frac{1}{q}K_q^- C_q = C_q K_q^-$, we have

$$(56) \quad C_q [E_q(sK_q^-)u](x, y, t) = 0,$$

where

(57)

$$\begin{aligned} [E_q(sK_q^-)u](x, y, t) &= \sum_{n=0}^{\infty} \frac{q^{(2)}s^n}{(q; q)_n} K_q^{-n} u(x, y, t) \\ &= \sum_{k, n=0}^{\infty} \frac{(a, \frac{\omega}{t}; q)_n (q^{-n}, b; q)_k}{(q; q)_n (\frac{q^{1-n}t}{\omega}, c, q; q)_k} \\ &\quad \times {}_2\phi_2 \left(\begin{matrix} q^{-n+k}, b' \\ \frac{q^{1-n+k}t}{\omega}, c' \end{matrix}; q, \frac{-q^{k+1}yt}{\omega} \right) q^{(k)} \left(\frac{-qxt}{\omega} \right)^k t^n. \end{aligned}$$

Using Weisner's expansion and proceeding as in the above two cases, we get the following identity:

$$\begin{aligned}
 (58) \quad & \sum_{k,n=0}^{\infty} \frac{(a, \frac{\omega}{t}; q)_n (q^{-n}, b; q)_k}{(q; q)_n (\frac{q^{1-n}t}{\omega}, c, q; q)_k} {}_2\phi_2 \left(\frac{q^{-n+k}, b'}{\omega}, c'; q, \frac{-q^{k+1}yt}{\omega} \right) \left(\frac{-q^{\frac{k+1}{2}}xt}{\omega} \right)^k t^n \\
 &= \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} \Phi^{(2)} \left(\begin{matrix} q^{-n}; b, b' \\ c, c'; q^n x, q^n y \end{matrix} \right) {}_1\phi_0 \left(\begin{matrix} aq^n \\ - \end{matrix}; \omega \right) t^n,
 \end{aligned}$$

where $\omega = \frac{s}{1-q}$.

6. CONCLUSION

The models and identities obtained in above sections can be generalized. For this, we need the m -fold q -integral transformation, defined as

$$\begin{aligned}
 (59) \quad & h(\beta_i, \gamma_i) = \mathcal{I}[f(u_1, \dots, u_m)] \\
 &= \Gamma_q \left(\begin{matrix} \gamma_1, \dots, \gamma_m \\ \beta_1, \dots, \beta_m, \gamma_1 - \beta_1, \dots, \gamma_m - \beta_m \end{matrix} \right) \\
 &\times \underbrace{\int_0^1 \dots \int_0^1}_{m\text{-fold}} \prod_{i=1}^m u_i^{\beta_i-1} \frac{(qu_1, \dots, qu_m; q)_{\infty}}{(q^{\gamma_1-\beta_1}u_1, \dots, q^{\gamma_m-\beta_m}u_m; q)_{\infty}} \\
 &\times f(u_1, \dots, u_m) d_q u_1 \dots d_q u_m, \\
 &\text{Re } \gamma_i > \text{Re } \beta_i > 0, \quad i = 1, 2, \dots, m.
 \end{aligned}$$

Putting $z = \sum_{i=1}^m u_i x_i$ in Model I (*i.e.* eqs. (22) and (23)), and using

$$(60) \quad z\Delta_z = u_1\Delta_{u_1} + u_2\Delta_{u_2}T_{u_1} + \dots + u_m\Delta_{u_m}T_{u_1} \dots T_{u_{m-1}},$$

$$(61) \quad T_z f(z) = T_{u_1} \dots T_{u_m} f\left(\sum_{i=1}^m u_i x_i\right),$$

we get the transformed $(m + 1)$ -variable model as

$$\begin{aligned}
 (62) \quad & K_q^+ = t[(x_1\Delta_{x_1} + \dots + x_m\Delta_{x_m}T_{x_1} \dots T_{x_{m-1}})T_t + t\Delta_t], \\
 & K_q^- = t^{-1}T_{x_1}^{-1} \dots T_{x_m}^{-1} \\
 &\times \left[\left(1 - x_1T_t \frac{1 - q^{\beta_1}}{1 - q^{\gamma_1}} E_{\beta_1\gamma_1} - \dots - x_mT_t \frac{1 - q^{\beta_m}}{1 - q^{\gamma_m}} E_{\beta_m\gamma_m} \right) \right. \\
 &\times (x_1\Delta_{x_1} + \dots + x_m\Delta_{x_m}T_{x_1} \dots T_{x_{m-1}}) - t\Delta_t \left. \right], \\
 & K_q^0 = t\Delta_t,
 \end{aligned}$$

with the basis functions as

$$\begin{aligned}
 (63) \quad & h_{\lambda} = \phi_{-1; \dots; 1}^{1; 1; \dots; 1} \left(\begin{matrix} q^{\lambda} : b_1; \dots; b_m; x_1, \dots, x_m \\ - : c_1; \dots; c_m \end{matrix} \right) t^{\lambda} \\
 &= \sum_{k_1, \dots, k_m=0}^{\infty} \frac{(q^{\lambda}; q)_{k_1+\dots+k_m} (b_1; q)_{k_1} \dots (b_m; q)_{k_m}}{(c_1, q; q)_{k_1} \dots (c_m, q; q)_{k_m}} x_1^{k_1} \dots x_m^{k_m} t^{\lambda}, \quad \lambda \in S,
 \end{aligned}$$

where $b_i = q^{\beta_i}$, $c_i = q^{\gamma_i}$.

Using the same techniques as given in section 5, the above model, and similar other $(m + 1)$ -variable models, can be exploited for obtaining identities.

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