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# A LIOUVILLE TYPE THEOREM FOR THE SCHRÖDINGER OPERATOR

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ABSTRACT. In this paper we prove that the equation $\Delta u(x)+h(x)u(x)=0$ on a complete Riemannian manifold of dimension $n$ without boundary and with nonnegative Ricci curvature admits no positive solution provided that $h$ is a $C^2$ function satisfying $\limsup_{r\to\infty} r^{-2}\inf_{x\in B_p(r)} h(x)\geq -b_n a^2$ and $\Delta h(x)\geq -c_n a^2$ where $0\leq a<\sup_{x\in M} h(x)$, and $b_n, c_n$ are constants depending only on the dimension, thus generalizing similar results in P. Li and S. T. Yau (Acta Math. **156** (1986), 153–201), J. Li (J. Funct. Anal. **100** (1991), 233–256) and E. R. Negrin (J. Funct. Anal. **127** (1995), 198–203) in all of which $h$ is assumed to be subharmonic. We also give a generalization in case the Ricci curvature of $M$ is not necessarily positive but its negative part has quadratic decay under the additional assumption that $h$ is unbounded from above.

## 1. Introduction

We will be concerned with the following differential equation,

$$\Delta u(x)+h(x)u(x)=0, \tag{1.1}$$

on a complete Riemannian manifold without boundary and with nonnegative Ricci curvature. P. Li and S. T. Yau (cf. [5, Corollary 1.1]) proved the following Liouville type theorem.

**Theorem 1.1.** *Let $M$ be a complete Riemannian manifold without boundary. Suppose that the Ricci curvature of $M$ is nonnegative, and that $h$ is a $C^2$ function defined on $M$ such that $\Delta h(x)\geq 0$ for all $x\in M$, there exists a point $x_0\in M$ with $h(x_0)>0$ and*

$$\lim_{r\to\infty} r^{-1}\cdot \sup_{x\in B_p(r)} |\nabla h|(x)=0, \tag{1.2}$$

*where $B_p(r)$ denotes the geodesic ball of radius $r$ centered at some fixed point $p\in M$. Then the equation (1.1) does not have a positive smooth solution on $M$.*

Later J. Li (cf. Theorem A of [4] and the remark following that theorem) proved that Theorem 1.1 holds if condition (1.2) is replaced by

$$\forall x\in M,\quad h(x)\geq 0, \tag{1.3}$$

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and more recently E. R. Negrin (cf. Theorem 2.1 of [6]) replaced (1.3) by

$$\lim_{r\to\infty} r^{-2} \cdot \inf_{x\in B_p(r)} h(x) = 0 \tag{1.4}$$

which is weaker than (1.2) and (1.3) as explained in the remark following Theorem 2.1 of [6]. However all these theorems assume that $h$ is subharmonic.

In this paper we will prove the following

**Theorem 1.2.** *Let $M$ be a complete Riemannian manifold without boundary of dimension $n$. Suppose that the Ricci curvature of $M$ is nonnegative, and that $h$ is a $C^2$ function defined on $M$ such that there exists a point $x_0 \in M$ with $h(x_0) > 0$,*

$$\limsup_{r\to\infty} r^{-2} \cdot \inf_{x\in B_p(r)} h(x) \geq -b_n a^2 \tag{1.5}$$

*where $B_p(r)$ denotes the geodesic ball of radius $r$ centered at some fixed point $p \in M$ and*

$$\Delta h(x) \geq -c_n a^2 \tag{1.6}$$

*for all $x \in M$ where $0 \leq a < \sup_{x\in M} h(x)$, and $b_n, c_n$ are positive constants depending only on the dimension. Then the equation $\Delta u(x) + h(x)u(x) = 0$ does not have a positive smooth solution on $M$.*

The conditions of the above theorem are much weaker than the conditions in the previous theorems. One can take any function $h^*$ satisfying the conditions of Theorem 2.1 of [6] and add to it appropriate smooth cut-off functions supported in the set $\{x \in M\colon\ |h^*(x)| > 1\}$ to produce a nonsubharmonic function $h$ satisfying the conditions of Theorem 1.2. Also condition (1.5) allows the negative part of $h$ to have some quadratic growth as $r \to \infty$. However some restriction on the constants $b_n$ and $c_n$ in (1.5) and (1.6) is necessary as is seen by taking $M$ to be $\mathbf{R}^n$ with the Euclidean metric, $u(x) = e^{-|x|^2} > 0$ and $h(x) = -e^{|x|^2}\Delta e^{-|x|^2} = 2n - 4|x|^2$. Then it is easy to see that this $h$ satisfies all the conditions of Theorem 1.2 with $b_n$ and $c_n$ replaced by numbers slightly larger than $1/n^2$ and $2/n$, respectively, without of course satisfying its conclusion. Moreover by taking $n = 1$,

$$u(x) = \exp\left(-\frac{x^2}{\log(x^2+2)}\right) \tag{1.7}$$

and $h = -u^{-1} \cdot u''$, one can check that $h$ satisfies the stronger condition (1.4) and condition (1.6) with $c_n$ replaced by another constant but still does not satisfy the conclusion of Theorem 1.2.

Theorem 1.2 has the following

**Corollary 1.3.** *Let $M$ be a complete Riemannian manifold without boundary. Suppose that the Ricci curvature of $M$ is nonnegative, and that $h$ is a $C^2$ function defined on $M$ that is unbounded from above and has the property that the functions $\Delta h(x)$ and $(1 + d(x,p))^{-2}h(x)$ (where $d(x,p)$ denotes the geodesic distance from $x$ to a fixed point $p \in M$) are bounded from below on $M$. Then equation* (1.1) *does not have a positive smooth solution.*

*Proof.* Since $\sup_{x\in M} h(x) = +\infty$ we can take $a$ in Theorem 1.2 to be sufficiently large so that both (1.5) and (1.6) are satisfied, which is possible since the quantities appearing there are by assumption bounded from below. □

The proof of Theorem 1.2 uses the maximum principle as originated first in Yau [7] and Cheng and Yau [2].

## 2. A gradient estimate implying the Liouville property

We will start with the following lemma which provides a gradient type estimate. Here $C_1, C_2, \ldots,$ will denote universal constants (independent of the dimension).

**Lemma 2.1.** *Let $M$ be a complete Riemannian manifold of dimension $n$. Suppose that $p \in M$ and $R > 0$ are such that $B_p(2R)$ does not meet $\partial M$ (if nonempty) and that $-K(2R)$ is a lower bound of the Ricci curvature on $B_p(2R)$ with $K(2R) \geq 0$. Also let $\varepsilon, \delta$ be positive numbers satisfying*

$$\varepsilon + 2\delta < \frac{2}{n}. \tag{2.1}$$

*Suppose that $a \geq 0$ and $h$ is a $C^2$ function defined on $M$ such that*

$$0 \leq a < \max_{x \in B_p(R)} h(x). \tag{2.2}$$

*Suppose also that $h$ satisfies*

$$-\min_{x \in B_p(2R)} h(x) < C_1 R^2 \delta^2 a^2 \tag{2.3}$$

*and*

$$\Delta h(x) \geq -\left(\frac{2}{n} - \varepsilon - 2\delta\right) a^2, \tag{2.4}$$

*for all $x \in B_p(2R)$.*

*If $u(x)$ is a smooth positive function defined on $M$ and satisfying the equation*

$$\Delta u(x) + h(x)u(x) = 0 \tag{2.5}$$

*on $M$, then we have*

$$\frac{|\nabla u|^2}{u^2} + h \leq \min\left\{\frac{C_4 n(1 + \sqrt{K(2R)}R)}{\delta R^2}, \frac{C_3 \delta \sqrt{K(2R)} R a}{\sqrt{\varepsilon}} + \frac{2}{\varepsilon} K(2R)\right\} \tag{2.6}$$

*on $B_p(R)$, provided that $R$ satisfies*

$$R \geq \max\left(\frac{C_4}{\delta\sqrt{a}}, 1\right). \tag{2.7}$$

*Proof.* Define $f(x) = \log u(x)$ and

$$F(x) = |\nabla f(x)|^2 + h(x), \tag{2.8}$$

on $M$. We have $\Delta f = -F$, and by the well known Bochner–Lichnerowicz formula ([3]) we conclude that

$$\Delta F \geq \frac{2}{n} F^2 - 2\nabla f \cdot \nabla F - 2K(2R)|\nabla f|^2 + \Delta h, \tag{2.9}$$

on $B_p(2R)$, because of the bound on the Ricci curvature.

We also fix $\eta$ to be a $C^2$ function on $[0, \infty)$ satisfying $\eta(t) = 1$ for $0 \leq t \leq 1$, $\eta(t) = 0$ for $t \geq 2$ and $0 \leq \eta(t) \leq 1$, $-C_5 \eta^{1/2}(t) \leq \eta'(t) \leq 0$ and $\eta''(t) \geq -C_5$ for all $t \geq 0$. Let $d(x,p)$ denote the geodesic distance between $p$ and $x$, and define $\phi(x) = \eta(d(x,p)/R)$. Using the well known argument of Calabi [1] we can assume that the point $x_1 \in B_p(2R)$ where the function $\phi(x)F(x)$ attains its maximum is not in the cut locus of $p$ and therefore $\phi$ is $C^2$ near $x_1$. Hence at $x_1$ we have

$$\nabla(\phi F) = 0, \quad \Delta(\phi F) \leq 0, \quad \phi F \geq \max_{x \in B_p(R)} h(x) > a, \tag{2.10}$$

the last inequality holding since $\phi$ is identically equal to 1 on $B_p(R)$ and $F(x) \geq h(x)$ for every $x \in M$. We also have [1]

$$|\nabla\phi|^2 \leq \frac{C_6}{R^2}\phi, \qquad \Delta\phi \geq -\frac{C_6}{R^2}n(1+\sqrt{K(2R)}/R) \tag{2.11}$$

and since from (2.10), $\nabla F = -F\nabla\phi/\phi$ at $x_1$, we have by (2.9)

$$\begin{aligned} 0 &\geq \phi\Delta(\phi F) = \phi^2\Delta F - 2F|\nabla\phi|^2 + \phi F\Delta\phi \\ &\geq \frac{2}{n}(\phi F)^2 - 2\phi F|\nabla f|\,|\nabla\phi| + \phi^2\Delta h - 2K(2R)\phi^2|\nabla f|^2 - 2F|\nabla\phi|^2 + \phi F\Delta\phi \\ &\geq \frac{2}{n}(\phi F)^2 + \phi^2\Delta h - \frac{C_7 n(1+\sqrt{K(2R)}R)}{R^2}\phi F - \frac{C_7}{R}(\phi F)\sqrt{\phi(F-h)} \\ &\quad - 2K(2R)\phi^2(F-h) \end{aligned}$$

at $x_1$ because $|\nabla f|^2 = F - h$, and so since $\phi \leq 1$ we have at $x_1$,

$$\begin{aligned} \frac{2}{n}(\phi F)^2 + \phi^2\Delta h \leq{}& \frac{C_7 n(1+\sqrt{K(2R)}R)}{R^2}\phi F \\ &+ \frac{C_7}{R}(\phi F)\sqrt{\phi(F-h)} + 2K(2R)\phi(F-h). \end{aligned} \tag{2.12}$$

But since $x_1 \in B_p(2R)$ conditions (2.3) and (2.7) with (2.10) imply that

$$\begin{aligned} \phi(F-h) &\leq \phi F - \min_{x\in B_p(2R)} h(x) \\ &< \phi F + C_1R^2\delta^2a^2 \leq \phi F(1 + C_1R^2\delta^2 a) \\ &\leq \frac{\delta^2 a}{C_7^2}R^2\phi F \leq \frac{\delta^2}{C_7^2}R^2(\phi F)^2 \end{aligned} \tag{2.13}$$

at $x_1$, provided we have chosen $C_1$ in (2.3) sufficiently small and $C_4$ in (2.7) sufficiently large. Also by (2.4) we have

$$\Delta h \geq -\left(\frac{2}{n} - \varepsilon - 2\delta\right)a^2 > -\left(\frac{2}{n} - \varepsilon - 2\delta\right)(\phi F)^2 \tag{2.14}$$

at $x_1$, and now from (2.12), (2.13) and (2.14) we have

$$(\varepsilon+\delta)(\phi F)^2 < 2K(2R)\phi(F-h) + \frac{C_7 n(1+\sqrt{K(2R)}R)}{R^2}\phi F \tag{2.15}$$

at $x_1$, and hence either

$$\delta \max_{x\in B_p(2R)} \phi(x)F(x) \leq \frac{C_7 n(1+\sqrt{K(2R)}R)}{R^2} \tag{2.16}$$

or

$$\varepsilon(\phi F)^2 < 2K(2R)\phi(F-h) \tag{2.17}$$

at $x_1$, which by (2.4) and (2.10) gives

$$\begin{aligned} \max_{x\in B_p(2R)} \phi(x)F(x) &< \frac{K(2R)}{\varepsilon} + \sqrt{\frac{K(2R)^2}{\varepsilon^2} - \frac{h(x_1)}{\varepsilon}} \\ &\leq \frac{2K(2R)}{\varepsilon} + \frac{\sqrt{C_1}\delta\sqrt{K(2R)}Ra}{\sqrt{\varepsilon}}. \end{aligned} \tag{2.18}$$

Now by taking $x \in B_p(R)$, in (2.16) and (2.18) we get (2.6) and this proves the lemma. □

From the above lemma we can prove Theorem 1.2 as follows.

*Proof of Theorem 1.2.* We suppose that $b_n$ and $c_n$ are positive numbers satisfying

$$c_n < \frac{2}{n}, \qquad b_n < \frac{C_1}{8}\left(\frac{1}{n} - \frac{c_n}{2}\right)^2 \tag{2.19}$$

where $C_1$ is as in (2.3) and then we choose $\varepsilon > 0$ and $\delta > 0$ to satisfy (2.1) and

$$\frac{2}{n} - \varepsilon - 2\delta = c_n, \qquad C_1\delta^2 > 8b_n. \tag{2.20}$$

We will use Lemma 2.1 now with $a$ as in Theorem 1.2, $K(2R) = 0$ and $R$ sufficiently large to satisfy (2.7), (2.2) (possible since $a < \sup_{x\in M} h(x)$ by assumption) and

$$(2R)^{-2} \cdot \inf_{x\in B_p(2R)} h(x) \geq -2b_n a^2, \tag{2.21}$$

which is possible since (1.5) implies that there are arbitrarily large $R$ satisfying (2.21). Then all the assumptions of the lemma are satisfied and therefore (2.6) gives

$$\max_{x\in B_p(R_j)} h(x) \leq \max_{x\in B_p(R_j)} \left(\frac{|\nabla u(x)|^2}{u^2(x)} + h(x)\right) \leq \frac{C_3 n}{R_j^2} \tag{2.22}$$

for a sequence of $R_j$ with $R_j \to +\infty$, which obviously contradicts (2.2) and hence proves the theorem. $\square$

*Remark* 1. The above proof produces a contradiction at a finite stage, that is, for some sufficiently large $R$. Therefore the proof can be modified to show the nonexistence of positive smooth solutions of (1.1) in the case when $M$ has nonnegative Ricci curvature outside a compact set $X$, provided that the other conditions of Theorem 2.1 are satisfied (with a smaller $b_n$) and that there exists a sequence of points $p_j \in M$ such that $d(p_j, p) \to \infty$ and $\limsup h(p_j) > a$. The value of $b_n$ must be adjusted because we will be using the lemma with base point $p_j$ for some large $j$ and $R$ such that $B_{p_j}(2R)$ does not meet $X$ and to produce (2.3) from (1.5) (which is based at $p$) we must apply (1.5) on $B_p(10R)$ for $R$ sufficiently large to ensure $B_{p_j}(2R) \subset B_p(10R)$. This situation is in contrast with the case of positive harmonic functions which might exist in such manifolds. We will push this contrast even more in Theorem 2.2 below.

*Remark* 2. As is seen from the examples in the introduction the inequality (2.17) for $c_n$ is in a sense best possible. Also the value of $b_n$ depends on how close to $2/n$ we take $c_n$, and is (for $c_n = 1/n$) of the order of $1/n^2$.

From the proof of Lemma 2.1 we have the following generalization of Corollary 1.3.

**Theorem 2.2.** *Let $M$ be a complete Riemannian manifold without boundary. Suppose that the Ricci curvature of $M$ is bounded from below by $-Cd(x,p)^{-2}$ for $d(x,p) \geq 1$ where $C$ is some positive constant and $d(x,p)$ is the geodesic distance from $x$ to a fixed point $p \in M$, and that $h$ is a $C^2$ function defined on $M$ that is unbounded from above and has the property that the functions $\Delta h(x)$ and $(1 + d(x,p))^{-2}h(x)$ are bounded from below on $M$. Then the equation $\Delta u(x) + h(x)u(x) = 0$ does not have a positive smooth solution.*

*Proof.* Since the functions $\Delta h(x)$ and $(1+d(x,p))^{-2}h(x)$ are bounded from below we can choose $a>0$ sufficiently large so that both conditions (2.3) and (2.4) in Lemma 2.1 are satisfied (fixing some positive $\varepsilon$ and $\delta$ satisfying (2.1)). Next we choose $R$ sufficiently large so that (2.2) and (2.7) are satisfied which is possible since $\sup_{x\in M} h(x)=+\infty$. Note also that $K(2R)$ is bounded, say by $K$ which is independent of $R$. Then from Lemma 2.1 the gradient estimate (2.6) holds. What is important here is that by examining the proof of the lemma, $K(2R)$ in the *second* argument of min in (2.6) can be replaced by $K(x_1)$ which is a lower bound for the Ricci curvature *at the point* $x_1$ where the function $\phi F$ assumes its maximum. The reason for this is that $K(2R)$ there comes from the last term $2K(2R)\phi(F-h)$ in (2.12) which in turn comes from (2.9) so from the Bochner-Lichnerovicz formula *evaluated at* $x_1$. Hence by the assumption on the Ricci curvature, if $d(x_1,p)\geq 1$ we have

$$h \leq \frac{|\nabla u|^2}{u^2}+h \leq \min\left\{\frac{C_4 n(1+\sqrt{K}R)}{\delta R^2}, \frac{C_3\delta\sqrt{C}Ra}{\sqrt{\varepsilon}d(x_1,p)}+\frac{2C}{\varepsilon d(x_1,p)}\right\} \tag{2.23}$$

on $B_p(R)$. Hence if $R/2\leq d(x_1,p)$, (2.23) gives

$$\max_{x\in B_p(R)} h(x) \leq \min\left\{\frac{C_4 n(1+\sqrt{K}R)}{\delta R^2}, \frac{C_3\delta\sqrt{2C}a}{\sqrt{\varepsilon}}+\frac{8C}{\varepsilon R^2}\right\}, \tag{2.24}$$

which gives a contradiction for $R$ sufficiently large since $\max_{x\in B_p(R)} h(x)\to+\infty$ as $R\to+\infty$. Therefore there exists a positive number $R_0$ such that for every $R\geq 2R_0$ the function

$$G_R(x)=\phi_R(x)F(x)=\eta\left(\frac{d(x,p)}{R}\right)\cdot\left(\frac{|\nabla u(x)|^2}{u^2(x)}+h(x)\right) \tag{2.25}$$

assumes its maximum value only at points in $B_p(\frac{R}{2})$. But since $\phi_R(x)=1$ on $B_p(R)$ this implies that

$$\max_{x\in B_p(R)\setminus B_p(\frac{R}{2})} F(x) < \max_{x\in B_p(\frac{R}{2})} F(x), \tag{2.26}$$

for all $R\geq 2R_0$. Hence for all $R\geq 2R_0$ we have

$$\max_{x\in B_p(R)} F(x) = \max_{x\in B_p(\frac{R}{2})} F(x) \tag{2.27}$$

which by an easy induction gives

$$\max_{x\in B_p(2^k R_0)} F(x) = \max_{x\in B_p(R_0)} F(x), \tag{2.28}$$

for all integers $k\geq 1$. Hence we have

$$h(x) \leq \frac{|\nabla u(x)|^2}{u^2(x)}+h(x) \leq \max_{x\in B_p(R_0)} F(x)=C_0, \tag{2.29}$$

for all $x\in M$ which is a contradiction since $h$ is unbounded from above and this proves the theorem. $\square$

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