A MEAN VALUE THEOREM
ON BOUNDED SYMMETRIC DOMAINS

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Abstract. Let Ω be a Cartan domain of rank r and genus p and $B_\nu$, $\nu > p - 1$, the Berezin transform on Ω; the number $B_\nu f(z)$ can be interpreted as a certain invariant-mean-value of a function $f$ around $z$. We show that a Lebesgue integrable function satisfying $f = B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f$, $\nu \geq p$, must be $\mathcal{M}$-harmonic. In a sense, this result is reminiscent of Delsarte's two-radius mean-value theorem for ordinary harmonic functions on the complex $n$-space $\mathbb{C}^n$, but with the role of radius $r$ played by the quantity $1/\nu$.

Let $\Omega = G/K$ be an irreducible bounded symmetric (Cartan) domain in its Harish-Chandra realization (i.e. a circular convex domain in $\mathbb{C}^d$ centered at the origin), $dm$ the Lebesgue measure on $\Omega$ normalized so that $m(\Omega) = 1$, and denote by $K(z, w)$ the Bergman kernel of $\Omega$ with respect to $dm$ and by $p, r$ its genus and rank, respectively. For $\nu \in \mathbb{R}$, it is known [FK1] that the integral $c(\nu)^{-1} := \int_\Omega K(z, z)^{-\nu/p} dm(z)$ is finite if and only if $\nu > p - 1$; in that case, one can consider the weighted Bergman spaces $A^2_\nu(\Omega)$ of functions analytic on $\Omega$ and square-integrable against the probability measure

$$d\mu_\nu := c(\nu)K(z, z)^{1-\nu/p} dm(z).$$

It can be shown that the point evaluations are continuous linear functionals on $A^2_\nu$ and the corresponding reproducing kernels are [FK1]

$$K_\nu(z, w) = K(z, w)^\nu/p.$$ (1)

The Berezin transform $B_\nu$ on $\Omega$ is the integral operator defined by

$$B_\nu f(w) := \int_{\Omega} f(z) \frac{|K_\nu(z, w)|^2}{K_\nu(w, w)} \, d\mu_\nu(z)$$

$$= \int_{\Omega} f(z) \left[ \frac{|K(z, w)|^2}{K(z, z)K(w, w)} \right]^{\nu/p} \cdot c(\nu)K(z, z) \, dm(z).$$ (2)

The integral is easily seen to converge, for instance, for any $f \in L^\infty(\Omega, dm)$ and $\nu > p - 1$; we will see below that it also converges for any $f \in L^1(\Omega, d\mu_\nu)$ if $\nu > p - 1$ and for any $f \in L^1(\Omega, dm)$ if $\nu \geq p$. Let $\phi$ be any element of the group

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\( \text{Aut}(\Omega) = G \) of holomorphic automorphisms of \( \Omega \) and write \( J_\phi \) for its Jacobian. From the transformation property of the Bergman kernel
\[
K(z, w) = J_\phi(z) \cdot K(\phi(z), \phi(w)) \cdot J_\phi(w)
\]
it follows that \( (B_\nu f) \circ \phi = B_\nu(f \circ \phi) \), i.e. the operator \( B_\nu \) is \( G \)-invariant. In particular, if \( \phi \) is any holomorphic automorphism sending 0 into \( w \), then
\[
B_\nu f(w) = \frac{1}{\Omega} \int_{\Omega} f(\phi(z)) \, d\mu_\nu(z)
\]
since \( K_\nu(z, 0) \equiv 1 \) owing to the circularity of \( \Omega \). Thus one can think of \( B_\nu f(w) \) as a certain “mean value of \( f \) around \( w \)”, and of functions \( f \) satisfying \( B_\nu f = f \) as having a certain invariant mean-value property.

For functions on the complex space \( \mathbb{C}^n \), it is well understood that the ordinary mean-value property is equivalent to harmonicity: if \( f \) is a harmonic function (i.e. \( f \in C^\infty \) and \( \Delta f = 0 \)), then \( f(w) \) is equal to the mean value of \( f \) over any sphere or ball centered at \( w \), for each \( w \) (Gauss’s theorem); conversely, any locally integrable function \( f \) with this property must be harmonic (Koebe). The converse part can be strengthened considerably: it is enough that \( f \) have the mean value property only over sufficiently small spheres (balls) around each \( w \), or even only for a sequence of spheres (balls) around \( w \) whose radii tend to zero. A beautiful result of Delsarte says that it even suffices that there exist two radii \( r_1, r_2 > 0 \), such that \( f \) has the mean value property over the two spheres \( S(w, r_1) \) and \( S(w, r_2) \) (balls \( B(w, r_1) \) and \( B(w, r_2) \)) around each \( w \) and \( r_1/r_2 \) is not a quotient of two zeroes of a certain Bessel function; see Zalcman’s paper [Za] or the recent survey of Netuka and Vesely [NV]. There is also a local version of the converse: if \( f \in C^3 \) in a neighbourhood of a point \( w \) and has the mean-value property over spheres \( S(w, r_n) \) (or balls \( B(w, r_n) \)) where \( r_n \to 0 \), then \( \Delta f(w) = 0 \).

The counterpart on bounded symmetric domains of harmonic functions on \( \mathbb{C}^n \) are the \( \mathcal{M} \)-harmonic functions (the term is due to Rudin [R]). Denote by \( \text{Diff}_G(\Omega) \) the ring of all \( G \)-invariant linear differential operators on \( \Omega \). We say that a function \( f \) is \( \mathcal{M} \)-harmonic if it is annihilated by all operators in \( \text{Diff}_G(\Omega) \) that annihilate the constants:
\[
Lf = 0 \quad \text{if} \quad L \in \text{Diff}_G(\Omega) \quad \text{and} \quad L1 = 0.
\]
Clearly \( f \) is \( \mathcal{M} \)-harmonic if and only if \( f \circ \phi \) is, for any \( \phi \in G \). An analogue of Gauss and Koebe theorems for \( \mathcal{M} \)-harmonic functions is due to Godement: \( f \in C^\infty(\Omega) \) is \( \mathcal{M} \)-harmonic if and only if
\[
f(\phi(0)) = \int_K f(\phi(k(z))) \, dk \quad \forall \phi \in G, \ \forall z \in \Omega = G/K,
\]
where the integration is carried against the Haar probability measure on the compact subgroup \( K \), the stabilizer of the origin in \( G \).

From the Bruhat decomposition \( G = KA^+K \) and Fubini’s theorem, it is easy to see that if \( f \) satisfies (5), then it even satisfies
\[
f(\phi(0)) = \int_\Omega f(\phi(x)) \, d\lambda(x)
\]
for any \( K \)-invariant probability measure \( \lambda \) on \( \Omega \); in particular \( f = B_\nu f \) for every \( \nu \). It follows that every \( \mathcal{M} \)-harmonic function \( f \) satisfies \( B_\nu f = f \), that is, has the “invariant mean-value property” mentioned above, for all \( \nu \). A natural question
is, does the converse also hold, i.e. does $B_\nu f = f$ for some $\nu$ imply that $f$ is $\mathcal{M}$-harmonic?

This problem was first considered by Fürstenberg [Fu] who proved that for $f \in L^\infty(\Omega, dm)$, the answer is “yes” (for any $\nu$); more generally, it even suffices that $f$ satisfies (6) for any absolutely continuous $K$-invariant probability measure $\lambda$ on $\Omega$ and all $\phi \in G$. For the special case of the unit disc $\Omega = \mathbb{D} \subset \mathbb{C}$ ($p = 2, r = 1$) and $\nu = 2$, a “Fourier-transform” proof has been given (in oblivion of Fürstenberg’s work) in [E1], while some partial cases were also resolved by Axler and Cuckovic [AC]. For unbounded functions $f$, however, the answer turns out to be “no”: Ahern, Flores and Rudin [AFR] showed that for $\Omega$ the unit ball in $\mathbb{C}^d$ and $f \in L^1(\Omega, dm)$, $B_\nu f = f$ implies that $f$ is $\mathcal{M}$-harmonic if $d \leq 11$, but not if $d \geq 12$. Later, Arazy and Zhang [AZ] gave a proof that for any Cartan domain $\Omega$ of rank $r \geq 2$, any $q \in [1, \infty)$ and $\nu > p - 1$, there exists a spherical function $f$ in $L^q(\Omega, dm)$ which satisfies $B_\nu f = f$ but is not $\mathcal{M}$-harmonic. (The Cartan domains of rank $r = 1$ are precisely the unit balls in $\mathbb{C}^d$, $d = 1, 2, \ldots$)

In this paper we prove the following result in the positive direction, which is reminiscent of Delsarte’s two-radius theorem above but with the role of the radius $r$ played by the quantity $1/\nu$.

**Theorem 1.** Suppose that $\Omega$ is an irreducible Cartan domain of rank $r$ and genus $p$, $\nu > p - 1$, and $f \in L^1(\Omega, d\mu_\nu)$ is a function satisfying $B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f = f$. Then $f$ is $\mathcal{M}$-harmonic.

Note that $L^1(d\mu_\nu) \subset L^1(d\mu_{\nu+\alpha})$ for any $\alpha \geq 0$; in particular, $L^1(d\mu_\nu) \subset L^1(d\mu_\nu) = L^1(\Omega, dm)$ if $\nu \geq p$.

So, for instance, on the ball $B_\nu f = B_{\nu+1} f = f$ implies that $f$ is $\mathcal{M}$-harmonic; this has already been noted by [AFR] (Remark 2.4).

We do not know whether the subscripts $\nu, \nu+1, \ldots, \nu+r$ can in fact be replaced by any $r+1$ distinct values $\nu > p - 1$, or whether there exist some “exceptional” $(r+1)$-tuples like the quotients of roots of the Bessel functions in Delsarte’s theorem. However, as the case of the ball shows, the number of $\nu$’s cannot be reduced to $r$.

For completeness, we state also the following local version of the result. Again, it is reminiscent of the above-mentioned “local version” of the converse to the Gauss theorem, with a sequence of indices $\nu_j \to \infty$ in the place of the sequence of radii $r_k \to 0$.

**Theorem 2.** Let $w$ be a point in $\Omega$ and $f \in L^1(\Omega, d\mu_\alpha)$ a function which is $C^\infty$ in a neighbourhood of $w$ and satisfies $B_{\nu_1} f(w) = B_{\nu_2} f(w) = B_{\nu_3} f(w) = \cdots$ for some sequence $\{\nu_j\}_{j=1}^{\infty}, \nu_j \geq \alpha, \nu_j \to \infty$. Then $Lf(w) = 0$ for any $L \in \text{Diff}_G(\Omega)$ such that $L1 = 0$.

The proofs rely on the results from [E2] and a simple algebraic argument involving symmetric polynomials. Let us recall the salient facts from the harmonic analysis and about the Helgason-Fourier transform on $\Omega$. Our main reference is [E2] which concentrates the necessary material from [He], [UU], [Dix], [FK1], [FK2], [AZ] and [Wa]. See also [Ka].

Employing the standard notation, let $G = NAK$ be the Iwasawa decomposition of $G$, $a$ the Lie algebra of $A$, $\rho = (\rho_1, \ldots, \rho_r) \in (a^*)^C \simeq C^r$ the half-sum of positive roots, $W$ the Weyl group, and $M$ the centralizer of $a$ in $K$; the coset space $B := K/M = G/MA$ can be viewed as the boundary (in the sense of symmetric
spaces) of \( \Omega = G/K \). Introduce the “plane waves”

\[
e_{\lambda,b}(x) := e^{(-i\lambda + \rho)(A(x,b))}, \quad x \in \Omega, b \in B,
\]

where \( A(x,b) \) is the unique element of \( a \) satisfying, if \( b = kM \) and \( x = gK \),

\[
k^{-1}g \in N \exp A(x,b) K
\]

under the Iwasawa decomposition \( G = NAK \). The Helgason-Fourier transform of \( f \in C_0^\infty(\Omega) \) is a function on \( a^* \times B (\simeq \mathbb{R}^r \times K/M) \) given by

\[
\hat{f}(\lambda, b) := \int_\Omega f(x)e_{\lambda,b}(x) \, d\mu(x)
\]

where \( d\mu(x) := K(x,x) \, dm(x) \) is the invariant measure on \( \Omega \). (In general, for a symmetric space \( \Omega = G/K \) of noncompact type, \( d\mu \) will be the image of the Haar measure on \( G \) under the projection map \( g \mapsto g0 \) of \( G \) onto \( G/K \); here and below we denote by \( 0 \) the coset \( eK = K \) of the unit element \( e \) of \( G \) in \( G/K \), which in our case of the Cartan domain coincides with the origin \( 0 \in \Omega \subset \mathbb{C}^d \).) For any \( f \in C_0^\infty(\Omega) \) we have the Fourier inversion formula

\[
f(x) = \int_{a^*} \int_B \hat{f}(\lambda, b)e_{-\lambda,b}(x)|c(\lambda)|^{-2} \, db \, d\lambda
\]

and the Plancherel theorem

\[
\int_\Omega |f(x)|^2 \, d\mu(x) = \int_{a^*} \int_B |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} \, db \, d\lambda.
\]

Here \( db \) is the unique \( K \)-invariant probability measure on \( K/M \) and \( d\lambda \) is a suitably normalized Lebesgue measure on \( a^* \simeq \mathbb{R}^r \); \( c(\lambda) \) is a certain meromorphic function on \( (a^*)^C \simeq \mathbb{C}^r \) (Harish-Chandra \( c \)-function). From the Plancherel theorem it can be deduced, in particular, that \( f \mapsto \hat{f} \) extends to a Hilbert space isomorphism of \( L^2(\Omega, d\mu) \) into \( L^2(a^* \times B, |c(\lambda)|^{-2} \, d\lambda \, db) \) whose image consists of functions \( \hat{F}(\lambda, b) \) which satisfy the symmetry condition \( \hat{F}(g, \lambda) = \hat{F}(g, s\lambda) \), for all \( s \) in the Weyl group \( W \), where

\[
\hat{F}(g, \lambda) := \int_B F(\lambda, g^{-1}(b))e_{\lambda,b}(g0) \, db, \quad g \in G, \lambda \in a^*.
\]

If \( \phi \) is an arbitrary measurable function on \( a^* \) invariant under \( W \), we associate to it the (possibly unbounded) linear operator \( M_\phi \) (the non-euclidean Fourier multiplier) on \( L^2(\Omega, d\mu) \) given by

\[
M_\phi f(\lambda, b) = \phi(\lambda)\hat{f}(\lambda, b).
\]

The domain of \( M_\phi \) consists of those functions \( f \in L^2(\Omega, d\mu) \) for which

\[
\int_{a^*} \int_B |\phi(\lambda)|^2 |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} \, db \, d\lambda < +\infty.
\]

The operator \( M_\phi \) is always closed and normal, is bounded iff \( \phi \) is essentially bounded, and selfadjoint (possibly unbounded) if \( \phi \) is real-valued. The set

\[
\mathfrak{M} := \{ M_\phi | \phi \in L^\infty(a^*/W) \}
\]

is precisely the set of all bounded linear operators on \( L^2(\Omega, d\mu) \) which are invariant under \( G \) (i.e. \( T(f \circ \phi) = (Tf) \circ \phi \), for all \( \phi \in G \)). Now it is immediate from the
definition that $B_\nu$ is a bounded operator on $L^\infty(\Omega, d\mu)$:

$$|B_\nu f(w)| \leq \|f\|_\infty \cdot \int_{\Omega} \frac{|K_\nu(z, w)|^2}{K_\nu(w, w)} \, d\mu_\nu(z)$$

$$= \|f\|_\infty \cdot \frac{\langle K_\nu(\cdot, w), K_\nu(\cdot, w) \rangle_{A^2}}{K_\nu(w, w)} = \|f\|_\infty.$$  

A similar computation shows that $B_\nu$ is bounded as an operator on $L^1(\Omega, d\mu)$; by interpolation, $B_\nu$ is a bounded operator on $L^2(\Omega, d\mu)$. Moreover, in view of (2), $B_\nu$ is also selfadjoint. It thus follows from the invariance of $B_\nu$ that $B_\nu \in \mathcal{D}$, i.e., $B_\nu = M_{\psi_\nu}$ for some real-valued function $\psi_\nu$ in $L^\infty(a^*/W)$. The function $\psi_\nu$ has been computed explicitly [UU]:

$$\psi_\nu(\lambda) = \prod_{j=1}^r \frac{\Gamma(-i\lambda_j + \nu - \frac{\nu-1}{2})\Gamma(i\lambda_j + \nu - \frac{\nu-1}{2})}{\Gamma(-p_j + \nu - \frac{\nu-1}{2})\Gamma(p_j + \nu - \frac{\nu-1}{2})}.$$  

For any $L \in \mathcal{D}_G(\Omega)$, an invariant differential operator on $\Omega$, it is known that the “plane waves” (7) are eigenfunctions of $L$:

$$Le_{-\lambda,b} = \hat{L}(\lambda)e_{-\lambda,b}$$  

where $\hat{L}(\lambda)$ is a polynomial in $r$ variables\(^1\); that is, $L = M_{\hat{L}}$. The correspondence $L \mapsto \hat{L}$ sets up an isomorphism between the ring $\mathcal{D}_G(\Omega)$ and the ring $\mathcal{P}^W(C^r)$ of all polynomials on $C^r \simeq a^*$ invariant under the Weyl group $W$; moreover, $L$ annihilates the constants if and only if $\hat{L}(\lambda)$ vanishes at the point $\lambda = i\rho$. In our case of the irreducible Cartan domain $\Omega$, the Weyl group consists simply of permutations and sign changes $(\varepsilon_j \mapsto \varepsilon_j x_{\sigma(j)}, \varepsilon_j = \pm 1)$, so $\mathcal{P}^W$ is just the ring of all symmetric polynomials in $\lambda_1, \lambda_2^2, \ldots, \lambda_r^2$. By a standard fact from algebra, the elementary symmetric polynomials

$$s_0(z_1, \ldots, z_r) = 1,$$

$$s_1(z_1, \ldots, z_r) = z_1 + z_2 + \cdots + z_r,$$

$$s_2(z_1, \ldots, z_r) = z_1 z_2 + z_1 z_3 + \cdots + z_{r-1} z_r,$$

$$\vdots$$

$$s_r(z_1, \ldots, z_r) = z_1 z_2 \cdots z_r$$

are a set of free generators of the ring of all symmetric polynomials in variables $z_1, z_2, \ldots, z_r$. Thus the polynomials $\hat{D}_0, \hat{D}_1, \ldots, \hat{D}_r$, where

$$\hat{D}_0 := 1, \quad \hat{D}_k := [s_k(\lambda_1^2, \ldots, \lambda_r^2) - s_k(-\rho_1^2, \ldots, -\rho_r^2)], \quad k = 1, 2, \ldots, r,$$

form a set of generators of $\mathcal{P}^W$, and therefore the corresponding invariant differential operators

$$D_0 = I \ (\text{the identity}), \quad D_1, D_2, \ldots, D_r,$$

$$(D_k = M_{\hat{D}_k})$$

form a set of generators for the ring $\mathcal{D}_G(\Omega)$. Note that we have chosen $D_k$ so that they vanish on constant functions for $k > 0$; consequently, a function $f$ on $\Omega$ will be $\mathcal{M}$-harmonic if and only if it is annihilated by the operators $D_1, D_2, \ldots, D_r.$

\(^1\)The function $\hat{L}(\lambda)$ was denoted $\hat{L}(i\lambda)$ in [E2].
Finally, we need also a few basics from the Jordan algebra theory of Cartan domains; our references here are [Lo] or [FK2] (condensed summaries can be found e.g. in §2 of [AZ], Chapter 1.5 of [Up], or §2 of [UU]). There can be chosen $R$-linearly independent vectors $e_1, \ldots, e_r$ in $C^d$ (a Jordan frame) such that each $z \in C^d$ has a polar decomposition

$$z = k \sum_{j=1}^r a_j e_j, \quad k \in K, \; a_1 \geq a_2 \geq \cdots \geq a_r \geq 0,$$

with $z \in \Omega$ iff $a_1 < 1$; the $r$-tuple $(a_1, a_2, \ldots, a_r)$ is uniquely determined by $z$ (but $k$ need not be). The Jordan triple determinant $h(z, w)$ is the polynomial in $z, \overline{w}$ on $C^d \times C^d$ such that for $z = w$ with the polar decomposition (9),

$$h(z, z) = \prod_{j=1}^r (1 - a_j^2).$$

This condition determines $h(z, w)$ uniquely, and $h(z, w)$ is independent of the choice of the Jordan frame $e_1, \ldots, e_r$. For any $k \in K$, $h(k(z), k(w)) = h(z, w)$ ($K$-invariance), and for $0 \leq a_r \leq \cdots \leq a_1 < 1$, $0 \leq b_r \leq \cdots \leq b_1 < 1$, $h(\sum_j a_j e_j, \sum_j b_j e_j) = \prod_j (1 - a_j b_j)$; in particular,

$$h(z, 0) = 1, \quad \text{for all } z \in \Omega.$$

In terms of $h(z, w)$, the Bergman kernel can be expressed by

$$K(z, w) = h(z, w)^{-1}.$$

From the transformation law (3) we thus have

$$h(\phi(z), \phi(w)) = J_{\phi}(z)^{1/p} h(z, w) J_{\phi}(w)^{1/p}, \quad \phi \in G, \; z, w \in \Omega.$$

Setting, in particular, $w = 0$ and $z = w = 0$ yields

$$|J_{\phi}(z)| = |h(\phi(z), \phi(0))|^p \cdot |h(\phi(0), \phi(0))|^{-p/2}$$

and as $h(z, w)$ is a polynomial in $z, \overline{w}$, it follows that

$$J_{\phi}$$

is bounded on $\Omega$, for each $\phi \in G$.

From (13), (11) and (1), we also obtain the transformation formula for $d \mu_{\nu}$:

$$d \mu_{\nu}(\phi(z)) = |h(z, a)|^{-2\nu} h(a, a)^\nu d \mu_{\nu}(z), \quad \phi \in G, \; a = \phi(0),$$

which shows that the quotients $d \mu_{\nu}(z)/d \mu_{\nu}(\phi(z))$ and (upon replacing $\phi$ by $\phi^{-1}$) $d \mu_{\nu}(\phi(z))/d \mu_{\nu}(z)$ are bounded. In particular, $f \circ \phi \in L^1(\Omega, d \mu_{\nu})$ whenever $f \in L^1(\Omega, d \mu_{\nu})$, so $B_{\nu} f$ is well-defined by (4) for $f \in L^1(\Omega, d \mu_{\nu}) \supset L^1(\Omega, dm)$ if $\nu \geq p$.

**Proposition 3.** For any $\nu > p - 1$, $B_{\nu}$ is a continuous operator from $L^1(\Omega, d \mu_{\nu})$ into $C^\infty(\Omega)$ endowed with the topology of uniform convergence of all derivatives on compact subsets.

**Proof.** In view of (11) the formula (2) defining $B_{\nu}$ can be rewritten as

$$B_{\nu} f(w) = \int_{\Omega} f(z) \frac{h(w, w)^{\nu}}{|h(z, w)|^{2\nu}} d \mu_{\nu}(z).$$
By the theorem on differentiation under the integral sign, it therefore suffices to show that for each compact set \( K \subset \Omega \) and multiindices \((m_1, \ldots, m_d), (n_1, \ldots, n_d)\),

\[
\sup_{z \in \Omega, w \in K} \left[ \left| \frac{\partial^{\vert m \vert}}{\partial w_1^{m_1} \cdots \partial w_d^{m_d}} \frac{\partial^{\vert n \vert}}{\partial \bar{w}_1^{n_1} \cdots \partial \bar{w}_d^{n_d}} \frac{h(w, w)'^{\nu}}{h(z, w)^{\nu} h(w, z)^{\nu}} \right| \right] < +\infty.
\]

An easy induction argument shows that the derivative in the square brackets is equal to

\[ h(w, w)'^{\nu} \cdot \frac{\text{a polynomial in } h(w, w), h(z, w), h(w, z) \text{ and their derivatives}}{|h(z, w)|^{2(\nu + \vert n \vert + \vert m \vert)} \cdot h(w, w)^{\vert n \vert + \vert m \vert}}. \]

Since \( h(z, w) \) is a polynomial in \( z \) and \( w \), it is bounded on (the closure of) \( \Omega \times \Omega \), and so are its derivatives of all orders; hence, the same is true for the numerator in the last formula. It therefore suffices to show that

\[
\inf_{z \in \Omega, w \in K} |h(z, w)| > 0.
\]

Assume, to the contrary, that there are \( z_n \in \Omega, w_n \in K \) with \( h(z_n, w_n) \to 0 \). Passing to a subsequence if necessary, we may assume that \( w_n \to w \in K \); by continuity (recall that \( h \) is a polynomial!), \( h(z_n, w) \to 0 \). Pick \( \phi \in G \) with \( \phi(w) = 0 \); from (12) and (14) it then follows that

\[
h(\phi(z_n), 0) \to 0,
\]

contradicting (10). This completes the proof. \( \square \)

It is known that the numbers \( \rho_j \) on any Cartan domain are given by

\[
\rho_j = \frac{(j - 1)a + b + 1}{2}, \quad j = 1, \ldots, r,
\]

where the \textit{characteristic multiplicities} \( a \) and \( b \) are nonnegative integers satisfying

\[ \frac{1}{2} r(r - 1)a + rb + r = d, \quad (r - 1)a + b + 2 = p. \]

In particular, \( \rho_j \leq \frac{p-1}{2} \quad \forall j = 1, \ldots, r \); hence, the number

\[
d(\nu) := \prod_{j=1}^{r} \left( \left( \nu - \frac{p - 1}{2} \right)^2 - \rho_j^2 \right)^{-1}
\]

is well-defined and positive for all \( \nu > p - 1 \).

**Proposition 4.** Let \( P_\nu \in \text{Diff}_G(\Omega) \) be the differential operator

\[
P_\nu = I + d(\nu) \sum_{k=1}^{r} \left( \nu - \frac{p - 1}{2} \right)^{r-k} D_k.
\]

Then

\[
P_\nu B_{\nu+1} f = B_{\nu+1} f
\]

for any \( \nu > p - 1 \) and \( f \in L^1(\Omega, d\mu_\nu) \).
Proof. Consider first $f \in L^2(\Omega, d\mu)$. Then we know that $\tilde{B}_{\nu}f = \psi_{\nu} \tilde{f}$ with $\psi_{\nu}$ given by (8). On the other hand, $P_{\nu}f = \tilde{P}_{\nu}f$ where

$$\tilde{P}_{\nu} = 1 + d(\nu) \sum_{k=1}^{r} \left( \nu - \frac{p-1}{2} \right)^{-k} \hat{D}_k$$

$$= 1 + d(\nu) \sum_{k=0}^{r} \left( \nu - \frac{p-1}{2} \right)^{-k} [s_k(\lambda_1^2, \ldots, \lambda_r^2) - s_k(-\rho_1^2, \ldots, -\rho_r^2)]$$

$$= d(\nu) \prod_{j=1}^{r} \left( \nu - \frac{p-1}{2} \right)^2 + \lambda_j^2$$

$$= \frac{\psi_{\nu+1}}{\psi_{\nu}},$$

since $\sum_{k=0}^{r} z^{-k}s_k(z_1, \ldots, z_r) = \prod_{j=1}^{r} (z + z_j)$. Thus, $\tilde{P}_{\nu} \tilde{B}_{\nu}f = \tilde{B}_{\nu+1}f$, i.e. (15) holds for $f \in L^2(\Omega, d\mu)$.

For $z \in \Omega$, let $g_z \in G$ be the geodesic symmetry interchanging $z$ and 0. For any invariant differential operator $L \in \text{Diff}_C(\Omega)$, we can write

$$Lf(z) = (Lf)(g_z(0)) = (L(f \circ g_z))(0).$$

As the mapping $z \mapsto g_z$ from $\Omega$ into $G$ is $C^\infty$-smooth, it follows that any such $L$ is a differential operator with $C^\infty$ coefficients, and, hence, is a continuous linear map from $C^\infty(\Omega)$ into itself. In particular, this is true for $P_{\nu}$, and since $L^2(\Omega, d\mu)$ is dense in $L^1(\Omega, d\mu)$, the validity of (15) for all $f$ in the latter space follows by the preceding proposition.

Proof of Theorem 1. By Proposition 4, $B_{\nu}f = B_{\nu+1}f = f$ implies that $P_{\nu}f = f$; similarly for $\nu + 1, \ldots, \nu + r - 1$ in the place of $\nu$. Thus

$$f \in \text{Ker}(P_{\nu+k} - I), \quad k = 0, 1, \ldots, r - 1,$$

or, by the definition of $P_{\nu}$ (recall that $d(\nu) > 0$ for all $\nu$),

$$\sum_{j=1}^{r} \left( \nu + k - \frac{p-1}{2} \right)^{-j} D_j f(x) = 0, \quad k = 0, 1, \ldots, r - 1, \ x \in \Omega.$$

Thus for each $x \in \Omega$, the polynomial

$$Q_x(t) := \sum_{j=1}^{r} \left( t + \nu - \frac{p-1}{2} \right)^{-j} D_j f(x)$$

of degree $\leq r - 1$ has $r$ distinct roots $t = 0, 1, \ldots, r - 1$. It follows that $Q_x$ vanishes identically; that is, $D_j f(x) = 0$ for all $x \in \Omega$ and $j = 1, 2, \ldots, r$. Thus $f$ is $\mathcal{M}$-harmonic.

Proof of Theorem 2. By the standard Laplace method (see e.g. [F], Theorem II.4.1), if the integral (2) defining $B_{\nu}f(w)$ converges absolutely for some $\nu = \nu_0$, then it converges also for all $\nu > \nu_0$ and as $\nu \to +\infty$,

$$B_{\nu}f(w) \approx \sum_{k=0}^{\infty} Q_k f(w) \cdot \nu^{-k}$$

for any function $f$ sufficiently smooth at $w$. Here $Q_k$ are certain differential operators which depend only on the kernel $K(z, w)$, i.e. only on the domain $\Omega$. Owing
to the $G$-invariance of $B_\nu$, the operators $Q_k$ are also $G$-invariant. The general formulas for $Q_k$ in [F] are quite involved; fortunately, it is possible to describe the operators $Q_k$ fairly explicitly using the Helgason-Fourier transform and the Unterberger-Upmeier formula (8). In particular, it was shown in [E2] that

$$Q_1, Q_3, \ldots, Q_{2r-1} \text{ and } Q_0 = I$$

are free generators of the ring $\text{Diff}_G(\Omega)$. Now from $B_\nu f(w) = B_{\nu+1} f(w) = \ldots$, $\nu_k \to +\infty$, it follows that

$$Q_k f(w) = 0 \quad \text{for all } k > 1.$$  

On the other hand, setting $f = 1$ in (16) shows that $Q_0 1 = 0 \forall k > 1$, i.e. $Q_1, Q_2, \ldots$ annihilate the constants. Thus $Q f(w) = 0$ whenever $Q \in \text{Diff}_G(\Omega)$ and $Q 1 = 0$, which is what we wanted to prove.

Remark. By (16) we may write $\lim_{\nu \to \infty} B_\nu =: B_\infty = I$; the hypothesis of Theorem 1 can thus be rewritten, formally, as

$$B_\nu f = B_{\nu+1} f = \cdots = B_{\nu+r} f = B_\infty f.$$

This makes it conceivable that perhaps even

$$B_{\nu_1} f = B_{\nu_2} f = \cdots = B_{\nu_{2r+1}} f = B_{\nu_{2r+2}} f \Rightarrow f \text{ is } M\text{-harmonic}$$

for any $f \in L^1(\Omega, d\mu_\alpha)$ and mutually distinct parameters $\nu_1, \nu_2, \ldots, \nu_{2r+2} \geq \alpha$. However, we are not going to pursue this idea any further.

References


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